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On the denominator values and barycentric weights of rational interpolants

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Abstract

We improve upon the method of Zhu and Zhu [A method for directly finding the denominator values of rational interpolants, J. Comput. Appl. Math. 148 (2002) 341–348] for finding the denominator values of rational interpolants, reducing considerably the number of arithmetical operations required for their computation. In a second stage, we determine the points (if existent) which can be *discarded* from the rational interpolation problem. Furthermore, when the interpolant has a linear denominator, we obtain a formula for the barycentric weights which is simpler than the one found by Berrut and Mittelmann [Matrices for the direct determination of the barycentric weights of rational interpolation, J. Comput. Appl. Math. 78 (1997) 355–370]. Subsequently, we give a necessary and sufficient condition for the rational interpolant to have a pole.

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1. Introduction

Let $\mathbb{P}_d$ be the space of the polynomials with degree at most d, and $\mathscr{R}_{m,n}$ be the set of all rational functions $r = p/q$, where $p \in \mathbb{P}_m$ and $q \in \mathbb{P}_n$. Also, let $z_0, z_1, \dots, z_N$ be $N+1$ distinct nodes in $\mathbb{R}$ and $f_0, f_1, \dots, f_N$ the corresponding values in $\mathbb{R}(\mathbb{C})$. Then, consider the rational interpolation problem of finding $r = p/q \in \mathscr{R}_{m,n}$ such that $r(z_k) = f_k$, $k = 0(1)N$. First we give some basic assumptions and well-known results (see e.g., [2,10]):

(a) One may suppose without loss of generality that $m \geqslant n$.
(b) If one takes $m + n = N$, then if the rational interpolant r exists, it is unique (up to possible factors common to p and q).

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(c) Every rational interpolant $r = p/q \in \mathscr{R}_{N,N}$ can be written in its barycentric form, given by

$$r(z) = \sum_{k=0}^{N} \frac{b_k}{z - z_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{b_k}{z - z_k}, \tag{1.1}$$

with the barycentric weights b_k such that

$$b_k = q(z_k) w_k, \quad k = 0(1)N,$$

where

$$w_k = 1 \Bigg/ \prod_{j=0, j\neq k}^{N} (z_k - z_j), \quad k = 0(1)N$$

are the barycentric weights of polynomial interpolation.

(d) This representation is very useful, since it has many advantages in comparison with the canonical one: (1) this form of the rational function admits a very simple formula for its derivatives. Usually, formulas for the derivatives of a rational function are based on partial fraction decomposition, which requires knowledge of the zeros of the denominator. (2) It provides information concerning the existence and location of poles of r, as well as about the existence of possible unattainable points. (A point (z_k, f_k) is said to be unattainable if r solves the linearized problem, but $r(z_k) \neq f_k$.) (3) Even if the barycentric weights are perturbed, r still satisfies the interpolation conditions, as long as the perturbed weights are all nonzero.

In [10], Schneider and Werner give an algorithm for computing the weights b_k by finding the vector $\mathbf{v}=(v_0, v_1, \ldots, v_n)^{\mathrm{T}}$ of the Newton form of q, namely, $q(z) = \sum_{i=0}^{n} v_i \prod_{j=0}^{i-1}(z - z_j)$, and then transforming it into a vector of barycentric weights $\mathbf{b} = (b_0, b_1, \ldots, b_N)^{\mathrm{T}}$ by an algorithm of Werner [14]. In contrast, in [2] the authors determine directly a vector $\mathbf{b}$ by finding a $N \times (N + 1)$-matrix whose kernel is the space spanned by the $\mathbf{b}$'s.

On the other hand, Zhu and Zhu determine in [15] the vector $\mathbf{q} = (q_0, q_1, \ldots, q_N)^{\mathrm{T}}$ of the denominator values of $r = p/q$ at the nodes, by considering r in its Newtonian form.

Here, we modify the method given in [15], so that a considerable number of operations are spared. Subsequently, we find δ with a slightly simpler version of the method of Schneider and Werner [10] for finding the unique interpolant $\tilde{r}$ with minimal denominator degree $\delta \leqslant n$. Thus, we are able to find $n - \delta$ points which can be *discarded* without affecting neither the rational interpolant for the attainable points, nor any information concerning poles and unattainable points. Then, if $\delta < n$, we determine $\tilde{r}$ by deleting those $n - \delta$ points and, after a possible reordering of the nodes, we consider the equivalent interpolation problem of finding $\tilde{r} \in \mathscr{R}_{m,\delta}$ such that $\tilde{r}(z_k) = f_k$, $k = 0(1)m + \delta$. This is an alternative procedure to the method introduced by Berrut and Mittelmann in [2, p. 366], which is also used by Zhu and Zhu in [15, p. 347]. Indeed, if the vector $\mathbf{b}$ (resp. $\mathbf{q}$) is not unique (up to a constant factor), they decrease n to n^* and *increase* m to m^* such that $m^* + n^* = m + n$, until the uniqueness of these vectors is achieved. In this case, one has necessarily $n^* = \delta$.

2. Some preliminary results

In this section we provide a short proof of the fundamental result given in [2]. For this purpose, we first recall a well-known result concerning divided differences.

The unique polynomial of degree at most d that interpolates the points $(z_i, g(z_i))$, $i = 0(1)d$, where $z_i \neq z_j$ for $i \neq j$, can be written in Newtonian form as

$$G(z) = g(z_0) + \sum_{k=1}^{d} g[z_0, \ldots, z_k] \prod_{j=0}^{k-1} (z - z_j). \tag{2.1}$$

On the other hand, it can be also written in Lagrangian form as

$$G(z) = \sum_{i=0}^{d} \frac{P(z)}{(z - z_i) P'(z_i)} g(z_i), \tag{2.2}$$

where $P(z) = \prod_{j=0}^{d} (z - z_j)$.

Hence, equating the leading coefficients of G in (2.1) and in (2.2), one readily obtains the very well-known formula (see e.g., [7,12]):

$$g[z_0, \ldots, z_d] = \sum_{i=0}^{d} g(z_i)/P'(z_i). \tag{2.3}$$

Remark 2.1. One therefore concludes that if $g(z)$ is a polynomial of degree at most d, i.e., $g(z) = \sum_{j=0}^{d} a_j z^j$, then $\sum_{i=0}^{d} g(z_i)/P'(z_i) = a_d$.

Theorem 2.1 (*Berrut and Mittelmann [2]*). *Let $r = p/q \in \mathscr{R}_{N,N}$ be a rational function given in its barycentric form* (1.1), *with*

$$p(z) = P(z) \sum_{k=0}^{N} \frac{b_k}{z - z_k} f_k \quad \textit{and} \quad q(z) = P(z) \sum_{k=0}^{N} \frac{b_k}{z - z_k},$$

where $P(z) = \prod_{j=0}^{N} (z - z_j)$ and $z_i \neq z_j$ for $i \neq j$. Then,

$$\begin{aligned} &(1) \quad \deg(q) \leqslant n \quad \Leftrightarrow \quad \textstyle\sum_{k=0}^{N} z_k^i b_k = 0, \quad i = 0(1)N - (n+1), \\ &(2) \quad \deg(p) \leqslant m \quad \Leftrightarrow \quad \textstyle\sum_{k=0}^{N} z_k^i f_k b_k = 0, \quad i = 0(1)N - (m+1). \end{aligned} \tag{2.4}$$

Proof. We see that $p(z_k) w_k = b_k f_k$ and $q(z_k) w_k = b_k$ for $k = 0(1)N$, where $1/w_k = P'(z_k) \neq 0$. Let $q(z) = \sum_{j=0}^{n} c_j z^j$. Then, by Remark 2.1 we obtain $\sum_{k=0}^{N} z_k^i b_k = 0$, $i = 0(1)N - (n+1)$. Conversely, consider $q(z) = \sum_{j=0}^{N} c_j z^j$ and suppose that $\sum_{k=0}^{N} z_k^i b_k = 0$, $i = 0(1)N - (n+1)$. This yields, by Remark 2.1, the following homogeneous linear system

$$\begin{cases} 0 = c_N, \\ 0 = c_N (\sum_{k=0}^{N} z_k^{N+1} w_k) + c_{N-1}, \\ 0 = c_N (\sum_{k=0}^{N} z_k^{N+2} w_k) + c_{N-1} (\sum_{k=0}^{N} z_k^{N+1} w_k) + c_{N-2}, \\ \quad \vdots \\ 0 = c_N (\sum_{k=0}^{N} z_k^{2N-(n+1)} w_k) + \cdots + c_{n+2} (\sum_{k=0}^{N} z_k^{N+1} w_k) + c_{n+1}, \end{cases}$$

which has only the trivial solution $c_N = c_{N-1} = \cdots = c_{n+1} = 0$. This proves part 1 of (2.4). To prove part 2 of (2.4), we simply replace b_k by $b_k f_k$. This completes the proof of the theorem. □

Hence, by choosing m and n such that $m + n = N$, one sees that the space spanned by the vectors $\mathbf{b} = (b_0, b_1, \ldots, b_N)^{\mathrm{T}}$ of the weights of the rational function $r \in \mathscr{R}_{m,n}$ in its barycentric form (1.1) is the kernel of the $N \times (N+1)$-matrix

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_N \\ \vdots & \vdots & & \vdots \\ z_0^{m-1} & z_1^{m-1} & \cdots & z_N^{m-1} \\ f_0 & f_1 & \cdots & f_N \\ z_0 f_0 & z_1 f_1 & \cdots & z_N f_N \\ \vdots & \vdots & & \vdots \\ z_0^{n-1} f_0 & z_1^{n-1} f_1 & \cdots & z_N^{n-1} f_N \end{pmatrix}.$$

3. Simplifying the method of directly finding the denominator values

Applying (2.3) with k instead of d yields that the Newtonian form of the interpolating polynomial which satisfies the interpolation conditions $q(z_i)=q_i$, $i=0(1)N$, can be written as (cf. [15, p. 343]):

$$q(z)=q_0+\sum_{k=1}^{N}\left\{\sum_{j=0}^{k}d_k^{(j)}q_j\right\}\omega_k(z),$$

where

$$\begin{cases}\omega_0(z)=1,\\ \omega_j(z)=\omega_{j-1}(z)(z-z_{j-1}), \quad j=1(1)N\end{cases} \tag{3.1}$$

and the numbers $d_k^{(j)}=1/\omega'_{k+1}(z_j)$ can be determined by the following algorithm (see [3]):

$$\begin{array}{l} d_0^{(0)}=1,\\ \text{for } k=1(1)N \text{ do}\\ \left[\begin{array}{l} d_k^{(j)}=\dfrac{d_{k-1}^{(j)}}{(z_j-z_k)}, \quad j=0,1,\ldots,k-1,\\ d_k^{(k)}=1/\omega_k(z_k). \end{array}\right. \end{array} \tag{3.2}$$

Remark 3.1. This algorithm is frequently used for the calculation of the barycentric weights of polynomial interpolation

$$d_N^{(i)}=w_i=1\bigg/\prod_{\substack{j=0\\ j\neq i}}^{N}(z_i-z_j), \quad i=0(1)N.$$

Although it is well known that $d_k^{(k)}=-\sum_{j=0}^{k-1}d_k^{(j)}$ (see e.g., [5,14]), this sum is subject to significant cancellations whenever $\max_{0\leqslant j\leqslant k-1}|d_k^{(j)}|$ is much larger than $|d_k^{(k)}|$, which is often the case.

So, denoting $q_k:=q(z_k)$ and $p_k:=f_kq_k$, $k=0(1)N$, the authors of [15] construct the interpolant $r=p/q\in\mathscr{R}_{m,n}$ such that p and q are in Newtonian form. In other words,

$$q(z)=q_0+\sum_{k=1}^{N}\left\{\sum_{j=0}^{k}d_k^{(j)}q_j\right\}\omega_k(z), \quad p(z)=f_0q_0+\sum_{k=1}^{N}\left\{\sum_{j=0}^{k}d_k^{(j)}f_jq_j\right\}\omega_k(z).$$

Now, assume that $m+n=N$. Then, since the conditions $\deg(p)\leqslant m$ and $\deg(q)\leqslant n$ are desirable, one obtains the following linear system:

$$\begin{cases}\displaystyle\sum_{j=0}^{k}d_k^{(j)}q_j=0, & k=n+1(1)n+m,\\ \displaystyle\sum_{j=0}^{k}d_k^{(j)}f_jq_j=0, & k=m+1(1)m+n.\end{cases} \tag{3.3}$$

Therefore, the authors of [15] obtain the following theorem which determines directly the denominator values q_k, $k=0(1)m+n$.

Theorem 3.1 (*Zhu and Zhu [15]*). *$(q_0, q_1, \ldots, q_{m+n})^{\mathrm{T}} \in S$, where S is the kernel of the matrix M given by*

$$\begin{pmatrix} d_{n+1}^{(0)} & d_{n+1}^{(1)} & \cdots & d_{n+1}^{(n+1)} & 0 & \cdots & 0 \\ d_{n+2}^{(0)} & d_{n+2}^{(1)} & \cdots & d_{n+2}^{(n+1)} & d_{n+2}^{(n+2)} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & 0 \\ d_{n+m}^{(0)} & d_{n+m}^{(1)} & \cdots & d_{n+m}^{(n+1)} & d_{n+m}^{(n+2)} & \cdots & d_{n+m}^{(n+m)} \\ f_0 d_{m+1}^{(0)} & f_1 d_{m+1}^{(1)} & \cdots & f_{m+1} d_{m+1}^{(m+1)} & 0 & \cdots & 0 \\ f_0 d_{m+2}^{(0)} & f_1 d_{m+2}^{(1)} & \cdots & f_{m+2} d_{m+2}^{(m+1)} & f_{m+2} d_{m+2}^{(m+2)} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & 0 \\ f_0 d_{m+n}^{(0)} & f_1 d_{m+n}^{(1)} & \cdots & f_{m+1} d_{m+n}^{(m+1)} & f_{m+2} d_{m+n}^{(m+2)} & \cdots & f_{m+n} d_{m+n}^{(m+n)} \end{pmatrix}.$$

Moreover, if $q_k \neq 0$, $k = 0(1)m+n$, we get $r(z_k) = f_k$, $k = 0(1)m+n$, where

$$r(z) = \frac{p(z)}{q(z)} = \frac{f_0 q_0 + \sum_{k=1}^{m} \left\{ \sum_{j=0}^{k} d_k^{(j)} f_j q_j \right\} \omega_k(z)}{q_0 + \sum_{k=1}^{n} \left\{ \sum_{j=0}^{k} d_k^{(j)} q_j \right\} \omega_k(z)}.$$

We claim that the matrix M can be simplified considerably. Indeed, we now give the following modification of Theorem 3.1:

Theorem 3.2. *$(q_0, q_1, \ldots, q_{m+n})^{\mathrm{T}} \in S$, where S is the kernel of the matrix $M^\star$ given by*

$$\begin{pmatrix} \frac{f_0 d_m^{(0)}}{(z_0 - z_{m+1})} & \cdots & \frac{f_m d_m^{(m)}}{(z_m - z_{m+1})} & \frac{f_{m+1}}{\omega_{m+1}(z_{m+1})} & 0 & \cdots & 0 \\ \frac{f_0 d_m^{(0)}}{(z_0 - z_{m+2})} & \cdots & \frac{f_m d_m^{(m)}}{(z_m - z_{m+2})} & 0 & \frac{f_{m+2}}{\omega_{m+1}(z_{m+2})} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{f_0 d_m^{(0)}}{(z_0 - z_{m+n})} & \cdots & \frac{f_m d_m^{(m)}}{(z_m - z_{m+n})} & 0 & 0 & \cdots & \frac{f_{m+n}}{\omega_{m+1}(z_{m+n})} \\ \frac{d_n^{(0)}}{(z_0 - z_{n+1})} & \cdots & \frac{d_n^{(n)}}{(z_n - z_{n+1})} & \frac{1}{\omega_{n+1}(z_{n+1})} & 0 & \cdots & 0 \\ \frac{d_n^{(0)}}{(z_0 - z_{n+2})} & \cdots & \frac{d_n^{(n)}}{(z_n - z_{n+2})} & 0 & \frac{1}{\omega_{n+1}(z_{n+2})} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{d_n^{(0)}}{(z_0 - z_{n+m})} & \cdots & \frac{d_n^{(n)}}{(z_n - z_{n+m})} & 0 & 0 & \cdots & \frac{1}{\omega_{n+1}(z_{n+m})} \end{pmatrix}.$$

Moreover, if $q_k \neq 0$, $k = 0(1)m+n$, we get $r(z_k) = f_k$, $k = 0(1)m+n$, where

$$r(z) = \frac{p(z)}{q(z)} = \frac{f_0 q_0 + \sum_{k=1}^{m} \left\{ \sum_{j=0}^{k} d_k^{(j)} f_j q_j \right\} \omega_k(z)}{q_0 + \sum_{k=1}^{n} \left\{ \sum_{j=0}^{k} d_k^{(j)} q_j \right\} \omega_k(z)}. \tag{3.4}$$

For the proof of Theorem 3.2 we shall need two lemmas concerning divided differences. The first is a basic fact, which can be found in [10]:

Lemma 3.1. *Let $j \in \{0, \ldots, l\}$ and $g_j(z) := z - z_j$. Then, $(f g_j)[z_0, \ldots, z_l] = f[z_0, \ldots, z_{j-1}, z_{j+1}, \ldots, z_l]$.*

Lemma 3.2. *$f[z_0,\ldots,z_j]=0$, $j=m+1(1)m+n$ if, and only if $f[z_0,\ldots,z_m,z_j]=0$, $j=m+1(1)m+n$.*

Proof. Consider the functions λ_j, $j=m+1(1)m+n$, given by

$$\lambda_j(z)=\prod_{\substack{i=m+1\\ i\neq j}}^{m+n}(z-z_i).$$

By Lemma 3.1 we have that $f[z_0,\ldots,z_m,z_j]=0$, $j=m+1(1)m+n$ if, and only if $(f\lambda_j)[z_0,\ldots,z_{m+n}]=0$, $j=m+1(1)m+n$. Since the functions λ_j, $j=m+1(1)m+n$ form a basis of $\mathbb{P}_{n-1}$, this last equality is equivalent to $(fQ)[z_0,\ldots,z_{m+n}]=0$, for all $Q\in\mathbb{P}_{n-1}$, which is in turn equivalent to $(fQ_j)[z_0,\ldots,z_{m+n}]=0$, $j=m+1(1)m+n$, where $Q_j(z)=\prod_{i=j+1}^{m+n}(z-z_i)$. Again, by Lemma 3.1 this last equality is equivalent to $f[z_0,\ldots,z_j]=0$, $j=m+1(1)m+n$. □

Proof of Theorem 3.2. By Lemma 3.2, the linear system (3.3) is equivalent to the following one:

$$\begin{cases}\displaystyle\sum_{j=0}^{m}\frac{d_m^{(j)}f_j}{(z_j-z_k)}q_j+\frac{f_k}{\omega_{m+1}(z_k)}q_k=0, & k=m+1(1)m+n,\\ \displaystyle\sum_{j=0}^{n}\frac{d_n^{(j)}}{(z_j-z_k)}q_j+\frac{1}{\omega_{n+1}(z_k)}q_k=0, & k=n+1(1)n+m.\end{cases}\tag{3.5}$$

In its turn, (3.5) can be written as $M^\star(q_0,q_1,\ldots,q_{m+n})^{\mathrm{T}}=0$. □

At this point, one can detect the following advantages in using matrix $M^\star$ instead of matrix M:

(1) In M, one must evaluate $d_k^{(i)}$, $k=n+1(1)n+m$, $i=0(1)k$, whereas in $M^\star$ one only has to evaluate $d_n^{(i)}$, $i=0(1)n$ and $d_m^{(i)}$, $i=0(1)m$. Furthermore, the numbers $\omega_{n+1}(z_k)$, $k=n+1(1)n+m$ and $\omega_{m+1}(z_k)$, $k=m+1(1)m+n$ can be easily computed by the recurrence relation (3.1).

(2) The large number of zero entries of matrix $M^\star$, together with its structure, permit us to triangulate it with a minimal number of arithmetical operations.

Remark 3.2. Once one has obtained the denominator values q_k, $k=0(1)N$ by triangulation of matrix $M^\star$, the determination of the barycentric weights $b_k=w_kq_k$, $k=0(1)N$ is immediate. In fact, since $w_k=d_N^{(k)}$, $k=0(1)N$, one only has to compute the numbers $d_N^{(k)}$ by the recurrence relation (3.2).

The barycentric representation (1.1) and the Newtonian representation (3.4) of the rational function r have many advantages over the canonical one. For example, they provide information concerning the existence of the interpolant, as well as on the location of poles and the presence of possible unattainable points (see [2,10,15]). Furthermore, they complement each other on two fundamental aspects:

(α) The former guarantees that the rational function r satisfies the interpolation conditions, as long as the barycentric weights are all nonzero. However, it may not guarantee the correct numerator and denominator degrees, which can be as large as N. This can occur because of the pitfalls of the finite precision arithmetics used when computing the barycentric weights.

Another drawback is the fact that its evaluation at a node z requires about $2N$ flops and N additions, twice as many as the canonical one. To avoid this (partially), Berrut developed in [1] an algorithm in which only $M+1<N+1$ nodes z_l, $l=0(1)M$, are considered in the barycentric representation of r, such that merely $M+1$ interpolation conditions are guaranteed by the barycentric formula. The remaining interpolation conditions at the other $N-M$ nodes are then given by imposing $r(z_l)=f_l$, $l=M+1(1)N$ in (1.1), with M instead of N. This method then leads to a structured $M\times(M+1)$-matrix, made of two (modified) Vandermonde and one Löwner, whose kernel is the set of weights of r.

(β) In the latter, the correct numerator and denominator degrees are guaranteed by the representation itself. However, for the same reason as in (α), the interpolation conditions are not necessarily satisfied.

We shall now show that matrix $M^\star$ of Theorem 3.2 may further be analytically simplified. Moreover, the corresponding new matrix can be stably computed, and requires less arithmetical operations for its computation than the matrix obtained by Berrut and Mittelmann's diagonalizing method applied to Zhu and Zhu's matrix M. The latter procedure has been performed in Steffen master's thesis [11].

Theorem 3.3. $(q_0, q_1, \ldots, q_{m+n})^{\mathrm{T}} \in S$, *where S is the kernel of the matrix* $A^\star$ *given by*

$$\begin{pmatrix} d_n^{(0)} f[z_0, \mathbf{z}, z_{m+1}] & \cdots & d_n^{(n)} f[z_n, \mathbf{z}, z_{m+1}] & 0 & \cdots & 0 \\ d_n^{(0)} f[z_0, \mathbf{z}, z_{m+2}] & \cdots & d_n^{(n)} f[z_n, \mathbf{z}, z_{m+2}] & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ d_n^{(0)} f[z_0, \mathbf{z}, z_{m+n}] & \cdots & d_n^{(n)} f[z_n, \mathbf{z}, z_{m+n}] & 0 & \cdots & 0 \\ \dfrac{d_n^{(0)}}{(z_0 - z_{n+1})} & \cdots & \dfrac{d_n^{(n)}}{(z_n - z_{n+1})} & \dfrac{1}{\omega_{n+1}(z_{n+1})} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \dfrac{d_n^{(0)}}{(z_0 - z_{n+m})} & \cdots & \dfrac{d_n^{(n)}}{(z_n - z_{n+m})} & 0 & \cdots & \dfrac{1}{\omega_{n+1}(z_{n+m})} \end{pmatrix},$$

and $\mathbf{z} = (z_{n+1}, \ldots, z_m)$.

Moreover, if $q_k \neq 0$, $k = 0(1)N$, we get $r(z_k) = f_k$, $k = 0(1)N$, *where*

$$r(z) = \frac{p(z)}{q(z)} = \frac{f_0 q_0 + \sum_{k=1}^{m} \{\sum_{j=0}^{k} d_k^{(j)} f_j q_j\} \omega_k(z)}{q_0 + \sum_{k=1}^{n} \{\sum_{j=0}^{k} d_k^{(j)} q_j\} \omega_k(z)}. \tag{3.6}$$

Proof. Write $M^\star = (a_{i,j})$, where $1 \leqslant i \leqslant n + m$ and $0 \leqslant j \leqslant n + m$. In order to produce a lower triangular matrix, we need to perform the following two sequences of operations on $M^\star$:

Step 1:

- For $i = n(-1)1$
- $\gamma = a_{i,m+i}/a_{m+i,m+i}$
  - For $j = 0(1)n$
  - $a_{i,j} = a_{i,j} - \gamma\, a_{m+i,j}$
- $a_{i,m+i} = 0$

Step 2:

- For $k = m(-1)n + 1$
  - For $i = n(-1)1$
  - $\gamma = a_{i,k}/a_{k,k}$
    - For $j = 0(1)n$
    - $a_{i,j} = a_{i,j} - \gamma\, a_{k,j}$
  - $a_{i,k} = 0$

After performing the above two sequences of operations on $M^\star$, we obtain a new matrix $A^\star = (A_{i,j})$, where $A_{i,j} = a_{i,j}$ if $n + 1 \leqslant i \leqslant n + m$ and $A_{i,j} = 0$ if $1 \leqslant i \leqslant n$ and $n + 1 \leqslant j \leqslant n + m$. Otherwise, $A^\star$ is given by

$$A_{i,j} = a_{i,j} - \left(\frac{a_{i,m+i}}{a_{m+i,m+i}}\right) a_{m+i,j} - \sum_{k=1}^{m-n} \left(\frac{a_{i,n+k}}{a_{n+k,n+k}}\right) a_{n+k,j}$$

That is,

$$
\begin{aligned}
A_{i,j} &= \frac{f_j d_m^{(j)}}{(z_j - z_{m+i})} - \frac{f_{m+i}\omega_{n+1}(z_{m+i}) d_n^{(j)}}{\omega_{m+1}(z_{m+i})(z_j - z_{m+i})} - \sum_{k=1}^{m-n} \frac{f_{n+k} d_m^{(n+k)} \omega_{n+1}(z_{n+k}) d_n^{(j)}}{(z_{n+k} - z_{m+i})(z_j - z_{n+k})} \\
&= \frac{f_j d_n^{(j)}}{(z_j - z_{m+i})\prod_{k=1}^{m-n}(z_j - z_{n+k})} + \frac{f_{m+i} d_n^{(j)}}{(z_{m+i} - z_j)\prod_{k=1}^{m-n}(z_{m+i} - z_{n+k})} \\
&\quad + \sum_{k=1}^{m-n} \frac{f_{n+k} d_n^{(j)}}{(z_{n+k} - z_j)(z_{n+k} - z_{m+i})\prod_{r=1, r\neq k}^{m-n}(z_{n+k} - z_{n+r})} \\
&= d_n^{(j)} f[z_j, z_{n+1}, \ldots, z_m, z_{m+i}].
\end{aligned}
$$

Therefore, the linear system (3.5) is equivalent to the following one:

$$
\begin{cases}
\displaystyle\sum_{j=0}^{n} d_n^{(j)} f[z_j, z_{n+1}, \ldots, z_m, z_k] q_j = 0, & k = m+1(1)m+n, \\
\displaystyle\sum_{j=0}^{n} \frac{d_n^{(j)}}{(z_j - z_k)} q_j + \frac{1}{\omega_{n+1}(z_k)} q_k = 0, & k = n+1(1)n+m.
\end{cases} \tag{3.7}
$$

In its turn, (3.7) can be written as $A^{\star}(q_0, q_1, \ldots, q_{m+n})^{\mathrm{T}} = 0$. □

Remark 3.3. When $m = n$, the sub-matrix

$$
\begin{pmatrix}
d_n^{(0)} f[z_0, z_{n+1}] & \cdots & d_n^{(n)} f[z_n, z_{n+1}] \\
d_n^{(0)} f[z_0, z_{n+2}] & \cdots & d_n^{(n)} f[z_n, z_{n+2}] \\
\vdots & & \vdots \\
d_n^{(0)} f[z_0, z_N] & \cdots & d_n^{(n)} f[z_n, z_N]
\end{pmatrix}
$$

present in the top of $A^{\star}$ is simpler than the corresponding Berrut and Mittelmann's matrix (20) [2, p. 364], whose kernel yields the barycentric weights $b_0, \ldots, b_n$. Indeed, when $m = n$, that matrix is given by

$$
\begin{pmatrix}
X_0 f[z_0, z_{n+1}] & \cdots & X_n f[z_n, z_{n+1}] \\
X_0 f[z_0, z_{n+1}, z_{n+2}] & \cdots & X_n f[z_n, z_{n+1}, z_{n+2}] \\
\vdots & & \vdots \\
X_0 f[z_0, z_{n+1}, \ldots, z_N] & \cdots & X_n f[z_n, z_{n+1}, \ldots, z_N]
\end{pmatrix},
$$

where $X_k = (x_k - x_{n+1})(x_k - x_{n+2}) \cdots (x_k - x_N)$, $k = 0(1)n$.

Here, we think it is opportune to observe that $w_k X_k = d_n^{(k)}$, $k = 0(1)n$.

4. Determination of superfluous points

Consider $q \in \mathbb{P}_n$. The unique polynomial $p \in \mathbb{P}_m$ which interpolates fq at $z_0, \ldots, z_m$ is given by $(fq)(z) = p(z) + \prod_{j=0}^{m}(z - z_j)(fq)[z_0, \ldots, z_m, z]$.

The following result is a suitable restriction of Proposition 2 of [10]:

Proposition 4.1. *Assume that* $q(z) = \sum_{i=0}^{n} v_i \prod_{j=0}^{i-1}(z - z_j)$ *and that* $p \in \mathbb{P}_m$ *interpolates* fq *at* $z_0, \ldots, z_m$, *where* $m + n = N$, $m \geqslant n$. *Then, the following statements are equivalent*:

(A) p *interpolates* fq *at* $z_0, \ldots, z_N$.
(B) $\sum_{i=0}^{n} v_i f[z_i, \ldots, z_m, z_j] = 0$, $j = m+1(1)N$.

Remark 4.1. Since the homogeneous linear system (B) has n equations and $n+1$ unknowns, it admits a nontrivial solution. However, the solution may not be unique. So, the notion of minimum degree solution (see [10, p. 288]) is very useful:

Definition. A nontrivial solution $q \in \mathbb{P}_n$ of (B) is called a minimum degree solution if there is no nontrivial solution of (B) of lesser degree. The degree of a minimum degree solution is denoted by δ.

Remark 4.2. The minimum degree solution is unique up to a nonzero constant multiple (see [10]). This property justifies the following:

Notation. The minimum degree solution of (B) with leading coefficient 1 will be called q_δ. The corresponding polynomial of degree at most m that interpolates fq_δ at $z_0, \ldots, z_N$ will be denoted by p_δ.

In [10] one can also find the following important results concerning the minimum degree solution q_δ:

Proposition 4.2. (a) *If* $q \in \mathbb{P}_n$ *is any solution of* (B) *and* $p \in \mathbb{P}_m$ *is the corresponding polynomial interpolating* fq at $z_0, \ldots, z_N$, *then there exists a polynomial* $Q \in \mathbb{P}_{\deg(q)-\delta}$ *such that* $q = Qq_\delta$ *and* $p = Qp_\delta$.

(b) *If* $q_\delta(\tau) = 0$ *for some* $\tau \in \mathbb{C}$, *then either* $p_\delta(\tau) \neq 0$ *and* $\tau \neq z_i$, $i = 0(1)N$, *or* $p_\delta(\tau) = 0$ *and* $\tau = z_i$ *for some* i, $0 \leqslant i \leqslant N$.

Now we are able to give a result which allows one to compute the degree δ of the minimum degree solution q_δ of (B).

Lemma 4.3. *Let* $z_0, z_1, \ldots, z_N$ *be distinct nodes in* $\mathbb{R}$, $f_0, f_1, \ldots, f_N$ *the respective values in* $\mathbb{R}(\mathbb{C})$ *and consider the rational interpolation problem of finding* $r = p/q \in \mathscr{R}_{m,n}$, *with* $m+n=N$, $m \geqslant n \geqslant 1$, *such that* $r(z_k) = f_k$, $k = 0(1)N$. *For* $d = 0(1)n$, *let* $A_{n,d}$ *be the* $n \times (d+1)$*-matrix*

$$\begin{pmatrix} f[z_0, \ldots, z_m, z_{m+1}] & f[z_1, \ldots, z_m, z_{m+1}] & \cdots & f[z_d, \ldots, z_m, z_{m+1}] \\ f[z_0, \ldots, z_m, z_{m+2}] & f[z_1, \ldots, z_m, z_{m+2}] & \cdots & f[z_d, \ldots, z_m, z_{m+2}] \\ \vdots & \vdots & \vdots & \vdots \\ f[z_0, \ldots, z_m, z_{m+n}] & f[z_1, \ldots, z_m, z_{m+n}] & \cdots & f[z_d, \ldots, z_m, z_{m+n}] \end{pmatrix}.$$

Then, $\delta = \min(\Delta)$, *where* $\Delta = \{n\} \cup \{0 \leqslant d \leqslant n-1 / \operatorname{rank}(A_{n,d}) = d\}$.

Proof. $A_{n,n}$ is the matrix of the linear system (B). If the columns of $A_{n,d}$ are linearly independent, then there exists no nontrivial solution of the equation $A_{n,n}\gamma = 0$, where $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)^{\mathrm{T}}$, such that $\gamma_{d+1} = \cdots = \gamma_n = 0$. Hence, either δ is the minimal number d such that $\operatorname{rank}(A_{n,d}) = d$, where $0 \leqslant d \leqslant n-1$ or $\delta = n$. □

Remark 4.3. Lemma 4.3 above is a useful consequence of Lemma 9 of [10]. The key difference between them is that in order to compute δ, all one needs to do is to triangulate the matrix $A_{n,n-1}$ (not the larger matrix $A_{n,n}$). Furthermore, if $\delta < n$, it is also possible to compute q_δ: set $v_\delta := 1$, $v_{\delta+1} = \ldots = v_n = 0$ and then compute $v_0, v_1, \ldots, v_{\delta-1}$ by back substitution on the linear system $A^\star_{n,n-1}(v_0, v_1, \ldots, v_{\delta-1}, 1, 0, \ldots, 0)^{\mathrm{T}} = 0$, where $A^\star_{n,n-1}$ is the matrix $A_{n,n-1}$ after triangulation.

We now make a connection between the minimal denominator degree δ and the points that may be discarded from the rational interpolation problem. For this purpose, we make the following definition:

Definition. We say that (z_j, f_j) is a *superfluous point* for the rational interpolation problem if the rational interpolant for the attainable points, as well as the information concerning poles and unattainable points, do not depend on (z_j, f_j).

Theorem 4.1. *Let $z_0, z_1, \ldots, z_N$ be distinct nodes in $\mathbb{R}$, $f_0, f_1, \ldots, f_N$ the respective values in $\mathbb{R}(\mathbb{C})$ and consider the rational interpolation problem of finding $r = p/q \in \mathscr{R}_{m,n}$, with $m + n = N$, $m \geqslant n \geqslant 1$, such that $r(z_k) = f_k, k = 0(1)N$. If the minimal denominator degree δ is less than n, then there exist $\mu_i, i = \delta + 1(1)n$, such that $(z_{m+\mu_{\delta+1}}, f_{m+\mu_{\delta+1}}), \ldots, (z_{m+\mu_n}, f_{m+\mu_n})$, where $1 \leqslant \mu_i < \mu_j \leqslant n$ for $i < j$, are superfluous points, which can be determined by triangulation of the matrix $A_{n,n-1}$.*

Proof. Since a permutation of the rows of $A_{n,n-1}$ corresponds to a permutation of the nodes $z_{m+1}, \ldots, z_N$, one concludes the following: if $\delta < n$, then after applying Gauss elimination with partial pivoting to the matrix $A_{n,n-1}$, the last $n - \delta$ rows will have no pivot. Furthermore, by Remark 4.3, one can set the last $n - (\delta + 1)$ columns to be zero. These facts imply that the minimum degree solution q_δ does not depend on the corresponding $n - \delta$ points $(z_{m+\mu_{\delta+1}}, f_{m+\mu_{\delta+1}}), \ldots, (z_{m+\mu_n}, f_{m+\mu_n})$, where $\mu_i, i = \delta + 1(1)n$, are such that $1 \leqslant \mu_i < \mu_j \leqslant n$ for $i < j$. This yields that p_δ will have the same property, since $p_\delta(z) = \sum_{i=0}^{m} (f q_\delta)[z_0, \ldots, z_i] \prod_{j=0}^{i-1} (z - z_j)$. Therefore, the rational interpolant $r = p_\delta/q_\delta$ will not depend on such points. □

5. Numerical examples

In this section we illustrate the ideas exposed in Sections 3 and 4.

Example 1 (*Cf. Zhu and Zhu [15, Example 3]*). Let the vectors that represent the nodes and their respective values be given by $x := (-1, 0, 1, 2, 3, 4)^{\mathrm{T}}$ and $f := (-2/3, -1, 2, 5/3, 2, 17/7)^{\mathrm{T}}$, and let $m = 3$, $n = 2$. Let us first determine the minimal denominator degree δ from

$$A_{2,1} = \begin{pmatrix} f[-1, 0, 1, 2, 3] & f[0, 1, 2, 3] \\ f[-1, 0, 1, 2, 4] & f[0, 1, 2, 4] \end{pmatrix} = \begin{pmatrix} 4/9 & 2/3 \\ 20/63 & 10/21 \end{pmatrix}.$$

Triangulating $A_{2,1}$, we get the matrix $\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$. By Lemma 4.3, this means that $\delta = \mathrm{rank}(A_{2,1}) = 1$ and, by Theorem 4.1, q_1 does not depend on the point (x_5, f_5). Therefore, we only have to find $r \in \mathscr{R}_{3,1}$ such that $r(x_k) = f_k$, $k = 0(1)4$. So, we shall consider $N = 4$, $m = 3$, $n = 1$.

By the recurrence relation (3.2), we obtain $d_0^{(0)} = 1$, $d_1^{(0)} = -1$, $d_1^{(1)} = 1$, $d_2^{(0)} = \frac{1}{2}$, $d_2^{(1)} = -1$, $d_2^{(2)} = \frac{1}{2}$, $d_3^{(0)} = -\frac{1}{6}$, $d_3^{(1)} = \frac{1}{2}$, $d_3^{(2)} = -\frac{1}{2}$, $d_3^{(3)} = \frac{1}{6}$. Moreover, by (3.1), we find $\omega_2(x_2) = 2$, $\omega_2(x_3) = 6$, $\omega_2(x_4) = 12$, $\omega_4(x_4) = 24$. Also, we have $f[x_0, x_2, x_3, x_4] = \frac{2}{9}$ and $f[x_1, x_2, x_3, x_4] = \frac{2}{3}$. Therefore, we have by Theorem 3.3 that the denominator values are found by solving the linear system

$$\begin{pmatrix} -2/9 & 2/3 & 0 & 0 & 0 \\ 1/2 & -1 & 1/2 & 0 & 0 \\ 1/3 & -1/2 & 0 & 1/6 & 0 \\ 1/4 & -1/3 & 0 & 0 & 1/12 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solution is given by $S = \lambda(-3, -1, 1, 3, 5)^{\mathrm{T}}$, $\lambda \in \mathbb{R}$. Since $q_k \neq 0$, $k = 0(1)4$, the rational interpolant exists and, by Proposition 4.2, it has a pole in the interval (x_1, x_2). In fact, inserting the denominator values into (3.6), one readily gets $r(x) = (x^2 + 1)/(2x - 1)$.

Now, we can make the conversion to the barycentric representation by computing the weights $b_k = q_k w_k$, where $w_k = d_4^{(k)}$, $k = 0(1)4$. Since $d_4^{(0)} = \frac{1}{24}$, $d_4^{(1)} = -\frac{1}{6}$, $d_4^{(2)} = \frac{1}{4}$, $d_4^{(3)} = -\frac{1}{6}$, $d_4^{(4)} = \frac{1}{24}$, the vector of the barycentric weights is any multiple of $\mathbf{b} = (1, -\frac{4}{3}, -2, 4, -\frac{5}{3})^{\mathrm{T}}$. Inserting these barycentric weights into (1.1), one obtains again the rational interpolant $r(x) = (x^2 + 1)/(2x - 1)$.

Example 2 (*Cf.* *Zhu and Zhu [15, Example 2]*). Let $x:=(-2,-1,0,1,2)^{\mathrm{T}}$ and $f:=(-2,-1,0,-1,2)^{\mathrm{T}}$, and also let $m=2=n$. We have

$$A_{2,1}=\begin{pmatrix} f[-2,-1,0,1] & f[-1,0,1] \\ f[-2,-1,0,2] & f[-1,0,2] \end{pmatrix}=\begin{pmatrix} -1/3 & -1 \\ 0 & 0 \end{pmatrix}.$$

This means that $\delta=\operatorname{rank}(A_{2,1})=1$ and that q_1 does not depend on the point $(x_4,\ f_4)$. Therefore, we only have to find $r\in\mathscr{R}_{2,1}$ such that $r(x_k)=f_k$, $k=0(1)3$. So, we shall consider $N=3$, $m=2$, $n=1$.

The numbers $d_k^{(j)}$, $k=0(1)3$, $j=0(1)k$ are the same as in Example 1. Moreover, by (3.1), we find $\omega_2(x_2)=2$, $\omega_2(x_3)=6$, $\omega_3(x_3)=6$. Also, we have $f[x_0,x_2,x_3]=-\frac{2}{3}$ and $f[x_1,x_2,x_3]=-1$. Therefore, by Theorem 3.3 the denominator values are found by solving the linear system

$$\begin{pmatrix} 2/3 & -1 & 0 & 0 \\ 1/2 & -1 & 1/2 & 0 \\ 1/3 & -1/2 & 0 & 1/6 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solution is given by $S=\lambda(3,2,1,0)^{\mathrm{T}}$, $\lambda\in\mathbb{R}$. Since $q_3=0$, by Proposition 4.2 the point $(x_3,\ f_3)$ is unattainable. In fact, inserting the denominator values into (3.6), one readily gets $r(x)=x$, and therefore $r(1)=1\neq-1$.

Now, since $b_k=q_k w_k$, where $w_k=d_3^{(k)}$, $k=0(1)3$, the vector of the barycentric weights is any multiple of $\mathbf{b}=(-\frac{1}{2},1,-\frac{1}{2},0)^{\mathrm{T}}$. Inserting these barycentric weights into (1.1), one obtains again $r(x)=x$.

Example 3. In order to submit our method to a stability test, we shall take the function $f(x)=e^{(x+1.2)^{-1}}/(1+25x^2)$ and interpolate it at the Chebyshev points of the second kind on $[-1,1]$, $x_k=\cos(k\pi/N)$, $k=0(1)N$. Subsequently, we calculate the errors at $x=-0.95$ and $x=-0.05$ and compare them with the errors found by Zhu and Zhu in [15, Example 4, p. 347].

In order to cope with the numerical instability caused by the computation of the divided differences present in the top of matrix $A^\star$, we reorder the nodes via the van der Corput sequence, due to van der Corput (see e.g., [4,13]). Moreover, we consider all the $(N+1)$ distinct shifts of that sequence and then choose one which provides the smallest error.

For the sake of completeness, we record here an example of a MATLAB code segment which produces the van der Corput sequence for a prime base b, and then makes an arbitrary shift. We have chosen $b=2$, $m=32$, $n=31$ and *shift* $=-26$.

data input for $n=(N-1)/2$, $m=(N+1)/2$, $N=2^j-1$

```
m = 32
n = 31
N = m + n
for k = 1 : 1 : N + 1
   z(k) = cos((k - 1) * pi/N);
end
```

reordering of the nodes via the van der Corput sequence

```
b = 2;
    for k = 1 : 1 : N + 1
        n0 = k - 1;
        C(k) = 0;
        power = 1/b;
        while n0 > 0
```

Table 1
Comparison of the errors $E(r, f)$ and $E(\hat{r}, f)$ when interpolating $f(x) = e^{(x+1.2)^{-1}}/(1+25x^2)$ at Chebyshev points on $[-1, 1]$

m	n	Shift	$x = -0.95$		$x = -0.05$	
			$E(r, f)$	$E(\hat{r}, f)$	$E(r, f)$	$E(\hat{r}, f)$
2	1	0	2.63463	2.63463	1.80029	1.80029
4	3	0	1.74338e − 1	1.74338e − 1	4.24836e − 1	4.24836e − 1
8	7	0	1.07174e − 7	1.30251e − 7	1.16351e − 13	3.19744e − 13
16	15	−26	1.33227e − 15	1.77636e − 15	4.44089e − 16	4.84057e − 14
32	31	−26	4.44089e − 16	4.44089e − 16	7.54952e − 15	9.42890e − 12

```
        n1 = floor(n0/b);
        r = mod(n0, b);
        C(k) = C(k) + power * r;
        power = power/b;
        n0 = n1;
    end
        C(k) = C(k) * (N + 1);
    end
```

shifting the van der Corput sequence

```
shift = −26;
sh = mod(shift, N + 1);
for k = 1 : 1 : (N − sh + 1)
  z(k) = cos(C(k + sh) * pi/N);
  f(k) = exp(1/(z(k) + 1.2))/(1 + 25 * (z(k)^2));
end
for k = (N − sh + 2) : 1 : (N + 1)
  z(k) = cos(C(k + sh − N − 1) * pi/N);
  f(k) = exp(1/(z(k) + 1.2))/(1 + 25 * (z(k)^2));
end
```

Our results, together with those obtained by Zhu and Zhu, are displayed in Table 1. We denote by $E(r, f)$ the absolute value of $(r - f)(x)$ and by $E(\hat{r}, f)$ the absolute value of $(\hat{r} - f)(x)$, where $\hat{r}$ denotes the rational interpolant of f in [15].

Moreover, we denote by *shift* the number which corresponds to a shift on the van der Corput sequence which, among all the other shifts, provides the smallest error $E(r, f)$. This number represents a shift of |*shift*| units to the left/right if the sign of *shift* is positive/negative.

Example 4. Finally, in order to compare our method with that of Berrut and Mittelmann, we first apply our method to a function f and then make the "barycentric conversion" with the aid of the relation $b_k = q_k w_k$, $k = 0(1)N$. Subsequently, we insert the barycentric weights b_k into (1.1) and obtain a barycentric rational interpolant br of f.

For this purpose, we take again the function $f(x) = e^{(x+1.2)^{-1}}/(1+25x^2)$ and interpolate it at the Chebyshev points of the second kind on $[-1, 1]$. We calculate the errors at $x = -0.95$ and $x = -0.05$ and compare them with the errors found in [2, Example 3, p. 369].

Our results, together with the ones obtained by Berrut and Mittelmann, are displayed in Table 2. We denote by $E(br, f)$ the absolute value of $(br - f)(x)$ and by $E(\bar{r}, f)$ the absolute value of $(\bar{r} - f)(x)$, where $\bar{r}$ denotes the barycentric rational interpolant of f in [2].

Table 2
Comparison of the errors $E(br, f)$ and $E(\bar{r}, f)$ when interpolating $f(x) = e^{(x+1.2)^{-1}}/(1 + 25x^2)$ at Chebyshev points on $[-1, 1]$

m	n	Shift	$x = -0.95$		$x = -0.05$	
			$E(br, f)$	$E(\bar{r}, f)$	$E(br, f)$	$E(\bar{r}, f)$
2	1	0	2.63463	2.64	1.80029	1.8
4	3	0	1.74338e−1	1.74e−1	4.24836e−1	4.25e−1
8	7	0	1.06580e−7	8.19e−8	2.59348e−13	8.4e−13
16	15	−26	4.44089e−15	5.2e−14	4.44089e−16	0.0
32	31	−26	0.0	0.0	8.88178e−16	3.1e−15

6. Results for the case $r \in \mathscr{R}_{m,1}$

In the sequel we will use the following notation:

$$f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] := f[\{z_0, z_1, \ldots, z_N\} \backslash \{z_k\}].$$

In order to determine the rational interpolant $r = p/q \in \mathscr{R}_{m,1}$ satisfying the interpolation conditions, write $q(z) = v_0 + v_1(z - z_k)$, where z_k is any of the nodes. Then, choose the unknown numbers v_0 and v_1 such that equation (B) holds, i.e., such that

$$f[z_0, \ldots, z_N]v_0 + f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]v_1 = 0. \tag{6.1}$$

Theorem 6.1. *Let $z_0, z_1, \ldots, z_N$ be distinct nodes in $\mathbb{R}$, $f_0, f_1, \ldots, f_N$ the respective values in $\mathbb{R}(\mathbb{C})$ and consider the rational interpolation problem of finding $r = p/q \in \mathscr{R}_{m,1}$, where $m + 1 = N$, $m \geqslant 1$, such that $r(z_k) = f_k$, $k = 0(1)N$. Then, we have the following*:

(1) *If $f[z_0, z_1, \ldots, z_N] = 0$, then the interpolant r is a polynomial which does not depend on the point (z_N, f_N).*
(2) *If $f[z_0, z_1, \ldots, z_N] \neq 0$ and $f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] = 0$ for some $0 \leqslant k \leqslant N$, then the point (z_k, f_k) is unattainable.*
(3) *If $f[z_0, z_1, \ldots, z_N] f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] \neq 0$ for $k = 0(1)N$, then the barycentric weights of r can be given by $b_k = f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]w_k$ and the pole of $r(z)$ is given by $z = y_1$, where*

$$y_1 = z_k + \frac{f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]}{f[z_0, z_1, \ldots, z_N]}. \tag{6.2}$$

Here z_k can be any of the $N + 1$ nodes.

Proof. By Lemma 4.3, we have that $\delta = 1 \Leftrightarrow f[z_0, z_1, \ldots, z_N] \neq 0$. Therefore, Theorem 4.1 implies that if $f[z_0, z_1, \ldots, z_N] = 0$, then the interpolant r is a polynomial which does not depend on (z_N, f_N). This proves part (1).

If $f[z_0, z_1, \ldots, z_N] \neq 0$, then from (6.1) with $v_1 = -f[z_0, z_1, \ldots, z_N]$, we obtain $q(z) = -f[z_0, z_1, \ldots, z_N](z - z_k) + v_0$, where $v_0 = f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]$. Now, if $r = p/q$ has a pole, then from the expression for q it should be equal to

$$z_k + (v_0/f[z_0, z_1, \ldots, z_N]) = z_k + \frac{f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]}{f[z_0, z_1, \ldots, z_N]} = y_1.$$

Hence, the barycentric weights b_k can be given by

$$b_k = q(z_k)w_k = v_0 w_k = f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]w_k, \quad k = 0(1)N.$$

Now, write

$$p(z) = \sum_{k=0}^{N} b_k f_k \prod_{j=0; j\neq k}^{N} (z - z_j), \quad q(z) = \sum_{k=0}^{N} b_k \prod_{j=0; j\neq k}^{N} (z - z_j)$$

and suppose that $f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] = 0$ for some k. Clearly this holds if, and only if, $b_k = 0$, which is equivalent to $y_1 = z_k$. Thus, one gets $p(y_1) = 0 = q(y_1)$, and then (z_k, f_k) is unattainable. This proves part (2).

On the other hand, if $f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] \neq 0$ for $k = 0(1)N$, then $y_1 \neq z_k$ for $k = 0(1)N$ and so by Proposition 4.2 one concludes that $p(y_1) \neq 0$, locating the pole at $z = y_1$. Therefore, $r = p/q$ is the rational interpolant of the data (z_k, f_k), $k = 0(1)N$, and this completes the proof of the theorem. □

Remark 6.1. We point out that the weights b_k given in Theorem 6.1 are in fact simpler than those found by Berrut and Mittelmann in [2, p. 364] for the case $r \in \mathscr{R}_{m,1}$. Indeed, the barycentric weights for the rational interpolant $r \in \mathscr{R}_{m,1}$ can be given (up to a constant factor) by

$$B_k = (-1)^{1-\delta_{k,i}} \frac{f[z_0, \cdots, \widehat{z_k}, \cdots, z_N] \prod_{l \neq i,k} (z_i - z_l)}{f[z_0, \cdots, \widehat{z_i}, \cdots, z_N] \prod_{l \neq i,k} (z_k - z_l)},$$

where $\delta_{k,i}$ is the Kronecker's delta of k and i, and z_i is any of the nodes for which the divided differences in the denominators are all different from zero.

However, since $\lambda := f[z_0, \cdots, \widehat{z_i}, \cdots, z_N] w_i$ does not depend on k, we can simply take the weights b_k as being $\lambda B_k = f[z_0, \cdots, \widehat{z_k}, \cdots, z_N] w_k$.

Remark 6.2. In [6], Larkin shows that if $r \in \mathscr{R}_{m,1}$ has a pole, then this pole can be given by $z = y_1$, where

$$y_1 = z_N + \frac{f[z_0, \ldots, z_{N-1}]}{f[z_0, z_1, \ldots, z_N]}.$$

Since the order of the nodes is irrelevant, this is the same as (6.2). However, Larkin does not give a necessary and sufficient condition for the interpolant to have a pole.

Remark 6.3. The barycentric weights b_k given in Theorem 6.1 can be recursively computed in a very simple way. In fact, the barycentric weights of polynomial interpolation are given by the algorithm (3.2). Furthermore, the divided differences $f[z_0, \ldots, \widehat{z_k}, \ldots, z_N]$ can be easily obtained by using the *divided differences table* and the following basic identity: for $0 \leqslant k \leqslant N$, $f[z_0, \ldots, \widehat{z_k}, \ldots, z_N] = f[z_1, \ldots, z_N] - (z_k - z_0) f[z_0, z_1, \ldots, z_N]$.

Remark 6.4. Let $m \geqslant 1$ be a natural number. The formula

$$z_{j+1} = z_j + \frac{h[z_{j-m}, \ldots, z_{j-1}]}{h[z_{j-m}, \ldots, z_j]} \tag{6.3}$$

can be used to find simple poles of meromorphic functions $h : D \to \mathbb{C}$, where D is some region of the complex plane interior to a closed Jordan curve $\mathscr{C}$, which contains a unique simple pole $\tilde{z}$ of h.

The new approximation z_{j+1} to the required pole of h is proven by Larkin in [6] to be the unique pole α of the interpolant $r(z) \in \mathscr{R}_{m-1,1}$, $r(z_k) = h_k$, $k = j - m(1)j$, given by $r(z) = p(z)/(z - \alpha)$, with $p(\alpha) \neq 0$.

If we define $h = 1/f$, then the method described above is known as Larkin's method for finding the simple zero $\tilde{z}$ of f. In [9] the reader can find a nice algorithm for determining a bracketed zero using Larkin's method. Furthermore, in [8] Neumaier and Schäfer prove that if $f \in \mathbb{P}_n$ has real coefficients and only real roots $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, then if $z_k > \lambda_n$ holds for all initial points z_k, $k = -m(1)0$, then the sequence defined by (6.3) is monotonically decreasing and converges to λ_n. Obviously, the corresponding result also holds for the root λ_1.

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