

Locating invariant tori for a family of two-dimensional Hamiltonian mappings

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ABSTRACT

The location of invariant tori for a two-dimensional Hamiltonian mapping exhibiting mixed phase space is discussed. The phase space of the mapping shows a large chaotic sea surrounding periodic islands and limited by a set of invariant tori. Given the mapping considered is parameterised by an exponent γ in one of the dynamical variables, a connection with the standard mapping near a transition from local to global chaos is used to estimate the position of the invariant tori limiting the size of the chaotic sea for different values of the parameter γ .

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1. Introduction

In the past years the interest in two-dimensional, nonlinear and area-preserving mappings has increased. Particularly, such interest is because of the possible applications that the formalism may describe, special connections can be made and used in the study of channel flows [1,2], waveguide [3], transport properties [4,5], Fermi acceleration [6–8] and also for the study of magnetic field lines in toroidal plasma devices with reversed shear (like tokamaks) and many others [9–12].

In many cases, the phase space of such kind of mappings falls into three different classes including: (i) integrable; (ii) mixed and (iii) ergodic. For integrable cases, only periodic and quasi-periodic orbits are observed. The phase space is regular and chaos, characterised by the exponential divergence of nearby trajectories, is not observed. The scenario changes for case (ii) where one can observe chaotic seas that eventually surround periodic islands [13–17] and the existence of a set of invariant tori. Generally, the size of the chaotic seas are strongly influenced by the control parameters since they control the intensity of the nonlinearity. For case (iii) only a single initial condition is enough to fill up all the phase space and no periodic orbits are observed. Examples of case (iii) resemble to the so-called billiard problems including the Bunimovich stadium [18] and the Sinai billiard [19], also called the Lorentz gas [20]. A billiard denotes a point-like particle reflecting elastically from the boundaries of a compact region where it reflects specularly upon collision. A specular reflection is that in which the angle of incidence is equal to the angle of reflection. Between the collision the particle moves along straight lines.

For a system with a mixed structure in the phase space, the particle can visit different regions of the phase space, in particular it is possible to reach sticky regions [21] leading to the so-called non-uniformity [22]. A combination of these two items produce anomalous diffusion and consequently one can observe anomalous transport. A sticky region *traps* the particle in the phase space and the escape from such a region may happen after a very long time late than the entrance. One of the tools used in the characterisation of such stickiness is the distribution of recurrence time. One of the claims for the mixed structure is that the distribution of such a recurrence time is a power law [23] while for fully developed chaos it is exponential [24].

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In this paper, we concentrate specifically on the location of the first invariant tori (sometimes also called invariant spanning curve) in the phase space using a connection with the standard mapping [13] near a transition from local to global chaos. The numerical and analytical results for an effective control parameter are obtained and the position of the first invariant tori in the phase space is estimated.

The paper is organised as follows. In Section 2, we present the model and discuss the variables and control parameters used. In Section 3, we estimate the location of the first invariant tori using a connection with the standard mapping and obtain our numerical results. Finally, in Section 4, we present our concluding remarks.

2. Definition of the problem and the mapping

In this section we discuss the procedures used to obtain the mapping. We assume that there is a generic two-dimensional integrable system that is slightly perturbed. The Hamiltonian function that describes the system is written as [13]

$$H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \varepsilon H_1(J_1, J_2, \theta_1, \theta_2), \quad (1)$$

where the variables J_i and θ_i with $i = 1, 2$ correspond respectively to the action and angle. To use the characterisation of the dynamics in terms of a mapping, we can now consider a Poincaré section defined by the plane $J_1 \times \theta_1$ and assume θ_2 as constant (mod 2π). A family of two-dimensional mappings which qualitatively describes the behaviour of (1) is (see Refs. [14,16] for recent results and applications)

$$T : \begin{cases} J_{n+1} = J_n + \varepsilon h(\theta_n, J_{n+1}) \\ \theta_{n+1} = [\theta_n + F(J_{n+1}) + \varepsilon p(\theta_n, J_{n+1})] \text{ mod}(2\pi) \end{cases} \quad (2)$$

where h , F and p are assumed to be nonlinear functions of their variables while the index n corresponds to the n th iteration of the mapping. The variables J and θ correspond indeed to J_1 and θ_1 .

Since the mapping (2) should be area preserving, the expressions for $h(\theta_n, J_{n+1})$ and $p(\theta_n, J_{n+1})$ must obey some properties, in particular some special intrinsic relations. The relations are obtained considering that the determinant of the Jacobian matrix equalling to one. After some straightforward algebra, area preservation will be observed only if the condition

$$\frac{\partial p(\theta_n, J_{n+1})}{\partial \theta_n} + \frac{\partial h(\theta_n, J_{n+1})}{\partial J_{n+1}} = 0, \quad (3)$$

is matched. Some results in the literature consider $p(\theta_n, J_{n+1}) = 0$, $h(\theta_n) = \sin(\theta_n)$ and different expressions of F as can be seen in Refs. [25–30].

In this paper, we consider the following expression for the two-dimensional mapping [16]:

$$T : \begin{cases} x_{n+1} = \left[x_n + \frac{a}{y_{n+1}^\gamma} \right] \text{ mod } 1 \\ y_{n+1} = |y_n - b \sin(2\pi x_n)|, \end{cases} \quad (4)$$

where a , b and γ are the control parameters. The determinant of the Jacobian matrix is $\det(J) = \text{sign}(y_n - b \sin(2\pi x_n))$ where $\text{sign}(u) = 1$ if $u > 0$ and $\text{sign}(u) = -1$ if $u < 0$. Note that if $\gamma < 0$, depending on the initial conditions and control parameters, one can observe unlimited growth for the variable \bar{y} which denotes the average value of y in the chaotic sea. It is obtained as

$$\bar{y} = \frac{1}{M} \sum_{i=1}^M \left[\frac{1}{n} \sum_{j=1}^n y_{i,j} \right],$$

where M denotes an ensemble of different initial conditions along the chaotic sea. Note that the average runs over both the orbit and ensemble of initial conditions. Such a growth is observed since large values of y imply in a large number of oscillations for the sine function. Then, in the regime of very large oscillations of the x variable, the sine function behaves more likely a random function yielding in an unlimited growth for \bar{y} . A detailed study of the variable \bar{y} for some control parameters in the mapping (4) and possible connections with other models and universality classes including, critical exponents, was recently discussed in Refs. [14,16].

The phase space generated from iteration of the mapping (4) for different values of the control parameters is shown in Fig. 1. One can see that the phase space is mixed thus containing a set of periodic islands that are surrounded by a large chaotic sea which is limited by a set of invariant spanning curves. The position of the first invariant spanning curve is marked as FISC in Fig. 1. The control parameters used were $a = 1$, $b = 10^{-3}$ and (a) $\gamma = 2/5$ and (b) $\gamma = 1/2$. We noted that the size of the chaotic sea varies as the control parameter γ varies. As γ increases, the position of the lowest invariant spanning tori in Fig. 1 rises. It implies that the control parameter γ plays an important role in the dynamics of the system.

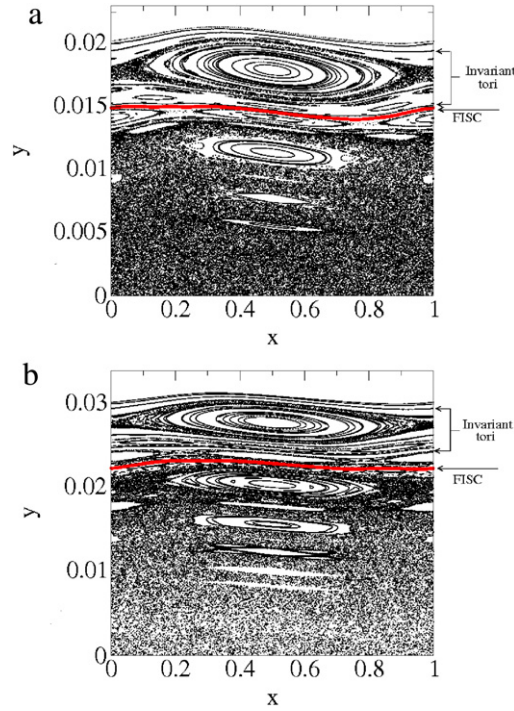


Fig. 1. Phase space generated by the mapping (4) for the control parameters $a = 1$, $b = 10^{-3}$ and different values of γ , namely: (a) $\gamma = 2/5$ and (b) $\gamma = 1/2$.

3. Connections with the standard mapping

In this section we make a short discussion of the standard mapping and show that it is possible to describe the position of the first invariant spanning curve by using a connection of mapping (4) with the standard mapping. The standard mapping is written as [13]

$$S : \begin{cases} I_{n+1} = I_n + K \sin(\Theta) \\ \Theta_{n+1} = \Theta_n + I_{n+1} \text{ mod } 2\pi, \end{cases} \quad (5)$$

where K is a control parameter. For $K \approx 0.9716 \dots$, there is a transition from local to globally chaotic behaviour in the standard mapping [13] in the sense that no invariant spanning curves are observed in the phase space. Thus the chaotic sea can spread over the phase space.

We suppose that near the first invariant spanning curve, which limits the size of the chaotic sea, the variable y can be written as

$$y_{n+1} \cong y^* + \Delta y_{n+1}, \quad (6)$$

where y^* is a typical value along the invariant spanning curve and Δy_{n+1} is a small perturbation of y_{n+1} . Using Eq. (6), the first equation of the mapping (4) is rewritten as

$$x_{n+1} = x_n + \frac{a}{y^{*\gamma}} \left(1 + \frac{\Delta y_{n+1}}{y^*} \right)^{-\gamma}. \quad (7)$$

Expanding the equation above in Taylor series, we obtain that

$$x_{n+1} = x_n + \frac{a}{y^{*\gamma}} \left[1 - \gamma \frac{\Delta y_{n+1}}{y^*} + O\left(\gamma \frac{\Delta y_{n+1}}{y^*} \right)^2 \right], \quad (8)$$

where only terms of first order are taken into account.

The second equation of the mapping (4) can also be written as

$$y^* + \Delta y_{n+1} = y^* + \Delta y_n - b \sin(2\pi x_n). \quad (9)$$

Table 1

Evaluation of K_{eff} at the first invariant spanning curve. The lowest (highest) value of K_{eff} corresponds to the maximum (minimum) value of the variable y^* on the invariant spanning curve.

γ	a	b	K_{eff}
2/5	1	10^{-4}	0.952–1.003
2/5	1	10^{-3}	0.898–0.993
2/5	10	10^{-4}	1.000–1.011
2/5	10	10^{-3}	0.961–0.979
1/2	1	10^{-4}	0.964–0.995
1/2	1	10^{-3}	0.886–0.946
1/2	10	10^{-4}	0.968–0.975
1/2	10	10^{-3}	0.968–0.983
3/4	1	10^{-4}	0.971–0.985
3/4	1	10^{-3}	0.913–0.946
3/4	10	10^{-4}	1.004–1.008
3/4	10	10^{-3}	0.969–0.979

Multiplying both sides of the equation above by $-a \gamma / y^{*(1+\gamma)}$ and adding $a/y^{*\gamma}$ to both sides, and multiplying again by 2π we obtain the following term

$$I_{n+1} = \left(\frac{a}{y^{*\gamma}} - \frac{a \gamma \Delta y_{n+1}}{y^{*(1+\gamma)}} \right) 2\pi. \quad (10)$$

Defining now $\phi_n = 2\pi x_n$, we rewrite mapping (4) as

$$T : \begin{cases} \phi_{n+1} = [\phi_n + I_{n+1}] \bmod(2\pi) \\ I_{n+1} = I_n + 2\pi \left(\frac{ab\gamma}{y^{*(1+\gamma)}} \right) \sin(\phi_n). \end{cases} \quad (11)$$

After a comparison of the equations that describe the standard mapping (see Eq. (5)) and the mapping (11), we see that there is an effective control parameter K_{eff} given by

$$K_{eff} \cong 2\pi \frac{ab\gamma}{y^{*(1+\gamma)}}. \quad (12)$$

To check the accuracy of Eq. (12), we can generate the phase space for mapping (4) using different control parameters and obtain numerically the corresponding coordinates of the first invariant spanning curve. Such coordinates will then furnish y_{min}^* and y_{max}^* . These values may be applied in Eq. (12) to furnish the corresponding range of K_{eff} . It is therefore expected that the values of K_{eff} stay near $K \cong 0.9716 \dots$ since it characterises a transition from local to global chaos in the standard mapping. We have done that. It is shown in Table 1 the range obtained numerically for K_{eff} corresponding the first invariant tori in mapping (4). For each set of control parameters a , b and γ the lowest value of K_{eff} corresponds to the maximum value of the y^* on the invariant spanning curve while the highest K_{eff} corresponds to the minimum y^* .

We observed that as the smaller the control parameter b is, the better the accuracy of the approach. Other simulations were also made and confirmed that the approach used describes well the transition from local to global chaos, as predicted in the standard mapping [13].

4. Conclusion

To summarise our conclusions, we have studied in this work the location of the first invariant spanning curve that characterises the transition from local to globally chaotic dynamics for a family of two-dimensional, area-preserving Hamiltonian mappings. The procedure used consisted in describing locally (near the first invariant spanning curve) the dynamics by using the standard mapping. An effective control parameter was obtained and compared to $K = 0.9716 \dots$ which marks the transition from local to global chaos in the standard mapping [13].

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