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Geodesic Deviation Equation in Locally de Sitter Spacetimes

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Resumo

Como é bem conhecido, a relatividade especial de Einstein, cuja cinemática é governada pelo grupo de Poincaré, deixa de valer na escala de Planck devido à existência de uma escala de comprimento invariante, dada pelo comprimento de Planck. Por essa razão, ela é incapaz de descrever a cinemática naquela escala. Uma solução possível para esse problema, a qual preserva a simetria de Lorentz — e consequentemente a causalidade — é substituir a relatividade especial de Einstein por uma relatividade especial na qual a cinemática é governada pelo grupo de de Sitter. Claro que uma mudança na relatividade especial irá pruduzir mudanças concomitantes na relatividade geral, a qual se torna o que chamamos de relatividade geral modificada por de Sitter. Trabalhando no contexto dessa teoria, o objetivo desse trabalho é deduzir a fórmula geral da aceleração relativa entre duas geodésicas próximas, a qual leva à equação do desvio geodésico modificada por de Sitter. Uma análise simples dos efeitos adicionais induzidos pela cinemática local de de Sitter é apresentada.

Abstract

As is well-known, the Poincaré invariant Einstein special relativity breaks down at the Planck scale due to the presence of an invariant length, given by the Planck length. For this reason, it is unable to describe the spacetime kinematics at that scale. A possible solution to this problem that preserves Lorentz symmetry — and consequently causality — is arguably to replace the Poincaré invariant Einstein special relativity by a de Sitter invariant special relativity. Of course, a change in special relativity produces concomitant changes in general relativity, which becomes what we have called de Sitter modified general relativity. By working in the context of this theory, the purpose of this work is to deduce the general relative acceleration between nearby geodesics, which leads to the de Sitter modified geodesic deviation equation. A simple analysis of the additional effects induced by the local de Sitter kinematics is presented.

Contents

1 Introduction

1.1 The Problem with Einstein Special Relativity

As is well-known, Einstein special relativity is inconsistent with quantum mechanics [1]. The problem is that, for very high energies, or more precisely, for energies of the order of the Planck energy, quantum mechanics predicts the existence of an invariant length scale, dubbed Planck length, which is sometimes interpreted as the minimum attainable length in Nature. Since Lorentz is a subgroup of Poincaré—which is the group that rules the kinematics in ordinary special relativity—and considering that Lorentz is believed not to allow the existence of an invariant length, the kinematics at Planck scale cannot be described by ordinary special relativity. This is the origin of the inconsistency between Einstein special relativity and quantum mechanics. The first idea that comes to mind whenever searching for a quantum kinematics is that, in order to allow the existence of an invariant length, Lorentz symmetry should be broken down [2,3]. However, to accept a possible violation of the Lorentz symmetry is not so easy. Lorentz symmetry is deeply related to causality [4], and any violation of the first implies a violation of the second. The point is that causality is one of the most fundamental principles of Physics, and its violation, even if it is assumed to take place at the Planck scale only, is something we cannot be sure Nature is prepared to afford. One way out of this conundrum is to look for a special relativity that preserves Lorentz symmetry. An interesting possibility in this direction is arguably to assume that, instead of governed by the Poincaré group, the spacetime kinematics is governed by the de Sitter group. As we are going to discuss below, this amounts to replace the Poincaré invariant Einstein special relativity by a de Sitter invariant special relativity.

1.2 de Sitter Invariant Special Relativity

The de Sitter spacetime is usually interpreted as the simplest *dynamical* solution of the sourceless Einstein equation in the presence of a cosmological constant, standing on an equal footing with all other gravitational solutions, like for example Schwarzschild and Kerr. However, as a non-gravitational spacetime, in the sense that its metric does not depend on Newton's gravitational constant, the de Sitter solution should instead be interpreted as a fundamental background for the construction of physical theories, standing on an equal footing with the Minkowski solution. General relativity, for instance, can be constructed on any one of them. Of course, in either case gravitation will have the same dynamics, only their local kinematics will be different. If the underlying spacetime is Minkowski, the local kinematics will be ruled by the Poincaré group of special relativity. If the underlying spacetime is de Sitter, the local kinematics will be ruled by the de Sitter group, *which amounts then to replace ordinary special relativity by a de Sitter-invariant* *special relativity* [5–7]. It is important to mention that the first ideas about a de Sitter special relativity are due to L. Fantappié, who in 1952 introduced what he called *Projective Relativity*, a theory that was further developed by G. Arcidiacono. The relevant literature can be traced back from Ref. [8]. The question then arises. Since the Lorentz group is usually blamed responsible for the non-existence of an invariant length, and considering that it is also a subgroup of de Sitter, how is it possible that in a de Sitter invariant kinematics the existence of an invariant length turns out to be possible? To understand this point, let us first recall that Lorentz transformations can be performed only in *homogeneous spacetimes*. In addition to Minkowski, therefore, they can be performed in de Sitter and anti-de Sitter spaces, which are the unique homogeneous spacetimes in $(1+3)$-dimensions [9]. As a homogeneous spaces, the de Sitter and anti-de Sitter spacetimes have constant sectional curvature. Of course, their Ricci scalar are also constant and have respectively the form

$$R = \pm 12\, l^{-2}, \tag{1.1}$$

where l is a length-parameter, usually called pseudo-radius. Now, by definition, Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed. Since the scalar curvature is given by (1.1), *Lorentz transformations are found to leave the length parameter l invariant* [10]. Although somewhat hidden in Minkowski spacetime, because what is left invariant in this case is an infinite length—corresponding to a vanishing scalar curvature—in de Sitter and anti-de Sitter spacetimes, whose pseudo-radii are finite, this property becomes manifest. Contrary to the usual belief, therefore, *Lorentz transformations do leave invariant a very particular length parameter: that defining the scalar curvature of the homogeneous spacetime in which they are performed*. If the Planck length l_P is to be invariant under Lorentz transformations, it must represent the pseudo-radius of spacetime at the Planck scale, which will be either a de Sitter or an anti-de Sitter space with a Planck cosmological term

$$\Lambda_P = \pm 3/l_P^2 \simeq \pm 1.2 \times 10^{70}\,\mathrm{m}^{-2}. \tag{1.2}$$

In the de Sitter invariant special relativity,* therefore, the existence of an invariant length-parameter at the Planck scale does not clash with Lorentz invariance, which remains a symmetry at all scales. Taking into account the deep relationship between Lorentz symmetry and causality [4], in this theory causality is always preserved, even at the Planck scale. Instead of Lorentz, translation invariance is broken down. In fact, in this theory, physics turns out to be invariant under the so-called *de Sitter translations*, which in stereographic coordinates are given by a combination of translations and proper conformal transformations [11]. We can then say that, in the same way Einstein special relativity may be thought of as a generalization of Galilei relativity for velocities near the speed of light, the de Sitter invariant special relativity may be thought of as a generalization of Einstein special relativity for energies near the Planck energy. It holds, for this reason, at all energy scales.

1.3 General Purposes

When general relativity is constructed on a de Sitter spacetime, the Poincaré invariant Einstein special relativity is naturally replaced by a de Sitter invariant special relativity, changing general relativity to what we have called *de Sitter modified general relativity* [12]. In this theory, the

*From now on, our interest will be restricted to the de Sitter case.

kinematic curvature of the underlying de Sitter spacetime and the dynamical curvature of general relativity are both included in the same Riemann tensor. This means that the cosmological term Λ no longer appears explicitly in Einstein's equation, and consequently the second Bianchi identity does not require it to be constant [13]. Far away from the Planck scale, Λ can consequently assume smaller values, corresponding to larger values of the de Sitter length-parameter l. For low energy systems, like for example the present-day universe, the value of Λ will be very small, and the de Sitter invariant special relativity will approach the Poincaré-invariant Einstein special relativity. Spacetimes that do not reduce locally to Minkowski are known since long and come under the name of Cartan geometry [14]. The particular case in which it reduces locally to de Sitter is known in the literature as de Sitter-Cartan geometry [15]. By considering general relativity in such geometry, the basic purpose of the present work is to derive the de Sitter modified geodesic deviation equation, and then make an attempt to understand the new effects that will show up. It should be mentioned that geodesic deviation is one of the effects present in the Landau-Raychaudhuri equation. This means that the results of the present work could eventually be used to deduce the de Sitter induced modifications in the Landau-Raychaudhuri equation, as well as its implications for Cosmology.

2 Isometries and Killing Vectors

In this chapter we are going to introduce some basic concepts about symmetries of a spacetime. In particular, we are going to see how the invariance of the metric tensor under a local group of transformations gives rise to the so-called Killing vectors. This construction will be illustrated in the specific case of Minkowski spacetime, whose kinematics is governed by the Poincaré group.

2.1 Isometries of spacetime and Killing vector

If the metric of a given spacetime is invariant under a coordinate transformation, then this coordinate transformation is said to be an isometry of the spacetime. Instead of regarding such coordinate transformation as changes from one coordinate system to another, we can adopt an alternative point of view and regard them as changes of position within the same, fixed coordinate system. A metric $g_{\mu\nu}$ is said to be invariant under a given coordinate transformation $x \to x'$ if the new metric tensor is the same function of its argument as the old metric tensor, that is

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) \tag{2.1}$$

for all x. The usual tensor transformation formula is

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x), \tag{2.2}$$

with its inverse given by

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}(x'). \tag{2.3}$$

We consider the special case of a infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu \tag{2.4}$$

where ξ^μ is an arbitrary function of position, with $|\xi^\mu| \ll 1$. For this transformation,

$$\frac{\partial x'^\alpha}{\partial x^\mu} = \delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x^\mu}(x'), \tag{2.5}$$

and the transformation of the metric tensor takes the form

$$g_{\mu\nu}(x) = \left(\delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \xi^\beta}{\partial x^\mu}\right) g'_{\alpha\beta}(x'). \tag{2.6}$$

Neglecting higher order terms, we can write

$$g_{\mu\nu}(x) \approx g'_{\mu\nu}(x') + \frac{\partial \xi^\alpha}{\partial x^\mu} g'_{\alpha\nu}(x') + \frac{\partial \xi^\beta}{\partial x^\nu} g'_{\mu\beta}(x') \,. \tag{2.7}$$

Using the Taylor series expansion as an explicit function of x,

$$g'_{\mu\nu}(x') \approx g'_{\mu\nu}(x) + \xi^\sigma \frac{\partial g'_{\mu\nu}}{\partial x^\sigma} \,, \tag{2.8}$$

the last equation can be rewritten in the form

$$g_{\mu\nu}(x) \approx g'_{\mu\nu} + \epsilon \xi^\sigma \frac{\partial g'_{\mu\nu}(x)}{\partial x^\sigma} + \epsilon \frac{\partial \xi^\alpha}{\partial x^\mu} g'_{\alpha\nu}(x') + \epsilon \frac{\partial \xi^\beta}{\partial x^\nu} g'_{\mu\beta}(x') \,. \tag{2.9}$$

Condition (2.1) is then

$$0 = \xi^\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \frac{\partial \xi^\alpha}{\partial x^\mu} g_{\alpha\nu} + \frac{\partial \xi^\beta}{\partial x^\nu} g_{\mu\beta} \,, \tag{2.10}$$

which is equivalent to

$$0 = \xi^\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \left(\partial_\mu \xi_\nu - \xi^\alpha \partial_\mu g_{\alpha\nu}\right) + \left(\partial_\nu \xi_\mu - \xi^\beta \partial_\nu g_{\mu\beta}\right) \,. \tag{2.11}$$

Combining the three derivatives of the metric tensor into a Christoffel symbols, we get

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\xi_\alpha \Gamma^\alpha{}_{\mu\nu} = 0 \,, \tag{2.12}$$

or more compactly

$$-\nabla_\nu \xi_\mu - \nabla_\mu \xi_\nu = 0 \,. \tag{2.13}$$

This is the Killing equation. The vector fields ξ_α that satisfy (2.13) are called Killing vectors of the metric $g_{\mu\nu}(x)$. The problem of determining all infinitesimal isometries of a given metric is now reduced to the problem of determining all Killing vectors of the metric [9]. Of course, any linear combination of Killing vectors (with constant coefficients) is also a Killing vector.

2.2 Homogeneous Spaces

A metric space is said to be *homogeneous* if there exist infinitesimal isometries (2.4) that carry any given point X into any other point in its immediate neighborhood. This metric must admit Killing vectors that at any given point take all possible values. For any n-dimensional space we can choose a set of n Killing vector $\xi^\mu_\lambda(x, X)$ with

$$\xi^\mu_\lambda(X, X) = \delta^\mu_\lambda \tag{2.14}$$

A metric space is said to be isotropic about a given point X if there exist infinitesimal isometries (2.4) that leave the point X fixed, so that $\xi^\mu(X) = 0$ and for which the first derivatives $\xi_{\lambda;\nu}$ take all possible values, subject only to the antisymmetric condition (2.13). For any n-dimensional space we can choose a set of $n(n-1)/2$ Killing vectors $\xi^{(\mu\nu)}_\lambda(x, X)$:

$$\xi^{(\mu\nu)}_\lambda(x, X) \equiv -\xi^{(\mu\nu)}_\lambda(x; X) \tag{2.15}$$

$$\xi^{(\mu\nu)}_\lambda(X, X) \equiv 0 \tag{2.16}$$

$$\xi^{(\mu\nu)}_{\lambda;\rho}(X, X) \equiv \delta^\mu_\lambda \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\lambda \tag{2.17}$$

Another useful formula relates the second derivative of the Killing vector to the Riemann tensor

$$\nabla_\nu \nabla_\mu \xi_\sigma - \nabla_\mu \nabla_\nu \xi_\sigma = -R^\lambda{}_{\sigma\nu\mu}\xi_\lambda \,. \tag{2.18}$$

If we write the same equation with cyclic permutations of the indices $(\nu\mu\sigma)$, and then add $(\nu\mu\sigma)$ equation to the $(\mu\sigma\nu)$ equation and subtract the $(\sigma\nu\mu)$ equation, we obtain that

$$\nabla_\nu \nabla_\mu \xi_\sigma = -R^\lambda{}_{\sigma\nu\mu}\xi_\lambda \,. \tag{2.19}$$

As an example of a maximally symmetric space, consider an n-dimensional flat space (Minkowski), with vanishing Riemann tensor. We choose a coordinate system with constant metric and vanishing affine connection. From equation (2.19) in this coordinate system, we get

$$\frac{\partial^2 \xi_\sigma}{\partial x^\mu \partial x^\nu} = 0 \,. \tag{2.20}$$

The general solution is

$$\xi_\sigma = a_\sigma + b_{\nu\sigma} x^\nu \tag{2.21}$$

with a_σ and $b_{\nu\sigma}$ constant. This satisfies the Killing equation (2.13) only if

$$b_{\nu\sigma} = -b_{\sigma\nu} \,. \tag{2.22}$$

We can choose a set of $n(n+1)/2$ Killing vectors

$$\xi^{(\nu)}_\sigma \equiv \delta^\nu_\sigma \tag{2.23}$$

and

$$\xi^{(\mu\nu)}_\sigma \equiv \delta^\mu_\sigma x^\nu - \delta^\nu_\sigma x^\mu \,. \tag{2.24}$$

The general Killing vector is

$$\xi_\sigma = a_\mu \xi^{(\nu)}_\sigma + b_{\mu\nu}\xi^{(\mu\nu)}_\sigma \,. \tag{2.25}$$

The n vectors $\xi^{(\nu)}_\sigma$ represent (ordinary) translations and the $n(n-1)/2$ vectors $\xi^{(\mu\nu)}_\sigma$ represent infinitesimal rotation (which for Minkowski are the Lorentz transformations). Thus can see that the Minkowski spacetime admits $n(n+1)/2$ independent Killing vectors.

2.3 Minkowski spacetime as an example

The Killing vectors can be used as a basis for the generators of the kinematic group of special relativity. In the four dimensional Minkowski spacetime, there are ten Killing vectors: six associated to the Lorentz group and four associated to the translation group. The four Killing vectors of translation are

$$\xi^\nu_\mu = \delta^\nu_\mu \,. \tag{2.26}$$

The translation generators can then be written in the form

$$P_\mu = \delta^\nu_\mu \frac{\partial}{\partial x^\nu} . \tag{2.27}$$

It is important to note that the translation Killing vectors appear explicitly in the generators. However, since they are given by Krönecker delta's, it is a usual to rewrite them in the form

$$P_\mu = \frac{\partial}{\partial x^\mu}\,, \tag{2.28}$$

without any mention to the translational Killing vectors. Although mathematically correct, the previous version is more elucidative, as we are going to see when considering the de Sitter spacetime. The translation generators obey the commutation relations

$$\left[P_\mu, P_\nu\right] = 0\,. \tag{2.29}$$

On the other hand, the Killing vector of associated to the Lorentz symmetry are

$$\xi^{(\mu\nu)}_{\sigma} = \delta^\mu_\sigma x^\nu - \delta^\nu_\sigma x^\mu\,. \tag{2.30}$$

The corresponding generators can be split into three rotation generators J_{ij},

$$J_{12} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{2.31}$$

$$J_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \tag{2.32}$$

$$J_{31} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{2.33}$$

and three Lorentz boost generators J_{i0}

$$J_{10} = x\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} \tag{2.34}$$

$$J_{20} = y\frac{\partial}{\partial t} + ct\frac{\partial}{\partial y} \tag{2.35}$$

$$J_{30} = z\frac{\partial}{\partial t} + ct\frac{\partial}{\partial z}\,. \tag{2.36}$$

They can be written in the compact form

$$J_{\mu\nu} = \left(x^\mu\frac{\partial}{\partial x^\nu} - x^\nu\frac{\partial}{\partial x^\mu}\right)\,, \tag{2.37}$$

which are generically called Lorentz generators. In terms of the Killing vectors, they read

$$J_{\mu\nu} = \xi^{\ \alpha}_{(\mu\nu)}\frac{\partial}{\partial x^\alpha}\,. \tag{2.38}$$

The set of ten generators P_μ and $J_{\mu\nu}$ of the Poincaré group obey the commutation relations

$$\left[J_{\mu\nu}, J_{\sigma\rho}\right] = \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} \tag{2.39}$$

$$\left[P_\mu, J_{\nu\rho}\right] = \eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu \tag{2.40}$$

$$\left[P_\mu, P_\nu\right] = 0\,. \tag{2.41}$$

Minkowski space M with zero curvature has as group of motion the Poincaré group

$$\mathcal{P} = \mathcal{L} \oslash \mathcal{T} \,,$$

the semi direct-product bewteen the Lorentz group $\mathcal{L} = SO(3,1)$ and the Abelian group of translation $\mathcal{T}$. In fact, Minkowski is an homogeneous space under $\mathcal{L}$, actually it is the quotient $M \equiv \mathcal{P}/\mathcal{L}$. Lorentz group is a subgroup, which provides a isotropy around a given point of M, and the translation invariance enforces this isotropy around any other point. The translation group $\mathcal{T}$ is then responsible for the equivalence of all points of spacetime. In other words, Minkowski is transitive under translations.

3 Geodesic Deviation in General Relativity

General relativity is constructed on a Riemannian spacetime that reduces locally to Minkowski. This is a consequence of the strong equivalence principle, according to which the laws of physics must reduce locally to that of special relativity. Since the kinematic group of Einstein special relativity is Poincaré, which in turn is the group of motion of Minkowski, any solution of Einstein equation must reduce locally to Minkowski. In this chapter we are going to review the geodesics of general relativity, and then to obtain the corresponding geodesic deviation equation. Its Newtonian limit will also be studied.

3.1 Geodesic Equation

In general relativity, the gravitational interaction is not described by a force, but by a geometric property of the spacetime. More specifically, the presence of gravitation produces a curvature in spacetime, and any particle moving in a gravitational field will follow the curvature of spacetime. The corresponding trajectories are called geodesics. For the sake of comparison with the case of locally de Sitter spacetimes, to be studied in the next chapter, we are going to deduce here the equation of motion of particles in a curved spacetime. If gravity is regarded a manifestation of the curvature of spacetime itself, and not as the action of some force, then the equation of motion of a particle moving only under the influence of gravity must be that of "free" particle in the curved spacetime. Hence, the action for such free particles is given by

$$S = -mc \int_b^a ds = -mc \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \, . \tag{3.1}$$

From the Euler-Lagrange equation, we obtain

$$\frac{d}{ds}\left(g_{\mu\nu} \frac{dx^\nu}{ds} \right) - \frac{1}{2} g_{\alpha\beta,\mu} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \, . \tag{3.2}$$

Using the identity

$$\frac{dg_{\mu\nu}}{ds} = g_{\mu\nu,\alpha} \frac{dx^\alpha}{ds} \, , \tag{3.3}$$

Eq. (3.2) can be rewritten as

$$g_{\mu\nu,\alpha} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - \frac{1}{2} g_{\alpha\beta,\mu} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \, . \tag{3.4}$$

Multiplying both sides of this equation by $g^{\sigma\mu}$, we get

$$\frac{d^2 x^\sigma}{ds^2} + \frac{1}{2} g^{\sigma\mu} \left(2 g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \, . \tag{3.5}$$

In view of the symmetry of $dx^\alpha dx^\beta$ in α and β, the above equation becomes

$$\frac{d^2x^\sigma}{ds} + \Gamma^\sigma{}_{\alpha\beta}\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,, \tag{3.6}$$

or equivalently

$$\frac{du^\sigma}{ds} + \Gamma^\sigma{}_{\alpha\beta}\,u^\alpha u^\beta = 0\,, \tag{3.7}$$

where $u = dx^\sigma/ds$ is the four-velocity of particle. *This is the geodesic equation for spacetimes that reduce locally to Minkowski.*

3.2 Geodesic Deviation Equation

From a general relativity perspective, the relative acceleration of particles in a freely-falling is not caused by variation in the gravitational force (in general relativity there is no the concept of gravitational force). Rather, free particles travel along the geodesics of spacetime, which are trajectories that follow the curvature of spacetime. If two near particles accelerate relative to each other, their geodesics that initially are parallel either converge or diverge, depending on the local curvature. To obtain the geodesic deviation equation, we consider a family of geodesics

$$u^\nu \nabla_\nu u^\mu = 0 \tag{3.8}$$

differing in the value of some parameter λ. This means that, for each λ = constant, $x^\mu = x^\mu(s,\lambda)$ will represent the equation of a geodesic, with s an affine parameter, as for example the proper time. We introduce the four-vector [16]

$$\eta^\mu = \frac{\partial x^\mu}{\partial\lambda}\,\delta\lambda \equiv v^\mu\,\delta\lambda \tag{3.9}$$

joining points on infinitely close geodesics, corresponding to parameters λ and $\lambda + \delta\lambda$, that have the same value of s. From the trivial equality $v^\rho\partial_\rho u^\mu = u^\rho\partial_\rho v^\mu$, it follows that

$$u^\rho\nabla_\rho v^\mu = v^\rho\nabla_\rho u^\mu\,, \tag{3.10}$$

with $u^\mu = \partial x^\mu/\partial s$ the particle four-velocity. We consider now the second derivative

$$\frac{D^2 v^\mu}{Ds^2} = u^\sigma\nabla_\sigma\left(u^\nu\nabla_\nu\eta^\mu\right)\,. \tag{3.11}$$

Using identity (3.10), as well as the relation

$$\left(\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma\right)v^\mu = R^\mu{}_{\nu\sigma\rho}\,v^\nu\,, \tag{3.12}$$

with $R^\mu{}_{\nu\sigma\rho}$ the curvature tensor, after some algebraic manipulation we arrive at

$$\frac{D^2 v^\mu}{Ds^2} = v^\alpha\nabla_\alpha(u^\nu\nabla_\nu u^\mu) + R^\mu{}_{\nu\sigma\rho}u^\nu u^\sigma v^\rho\,. \tag{3.13}$$

The first term on the right-hand side is zero due to the geodesic equation (3.8). Multiplying both sides the factor $\delta\lambda$, we get finally

$$\frac{D^2\eta^\mu}{Ds^2} = R^\mu{}_{\nu\sigma\rho}u^\nu u^\sigma \eta^\rho \,, \tag{3.14}$$

which in general relativity is called the *geodesic deviation equation*. It expresses the relative acceleration between two neighboring geodesics, which is seen to be proportional to the curvature tensor, a concept related to the spacetime geometry. Physically, the acceleration of neighboring geodesics is interpreted as a manifestation of gravitational tidal forces.

3.3 Newtonian Limit of the Geodesic Deviation Equation

An interesting test of the geodesic deviation equation is to obtain its Newtonian limit, where gravity is described in terms of a force. To arrive at this result we have to assumer that the gravitational field is weak, in which case we can consider an expansion around the Minkowski metric $\eta_{\mu\nu}$, in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{3.15}$$

where $h_{\mu\nu}$ is the metric perturbation, which satisfies the condition $|h_{\mu\nu}| \ll 1$. As is well-known, its time component is related to the Newtonian potential Φ through

$$h_{00} = 2\Phi/c^2 \,. \tag{3.16}$$

Also, the metric (3.15) is stationary, which means that all the derivatives $\partial_0 g_{\mu\nu}$ are zero. And finally, as a consequence of the weak field, all particle velocities are assumed to be small compared with the speed of light c, that is, $v \ll c$. Under the above conditions, it is easy to verify that

$$\Gamma^0{}_{00} = 0 \quad \text{and} \quad \Gamma^i{}_{00} = \frac{1}{2}\delta^{ij}\partial_j h_{00} \,. \tag{3.17}$$

The non-vanishing components of the Riemann tensor are then found to be

$$R^i{}_{00j} = -\partial_j \Gamma^i{}_{00} = -\partial_j \left(\frac{1}{2}\delta^{ik}\partial_k h_{00} \right) \,. \tag{3.18}$$

Let us consider now the space components ($\mu = i$) of the geodesic deviation equation (3.14). Since in the Newtonian limit the space components u^j are negligible in relation to the time component $u^0 \simeq 1$, it assumes the form

$$\frac{D^2\eta^i}{Ds^2} = R^i{}_{00j}u^0 u^0\, \eta^j = R^i{}_{00j}\, \eta^j \,, \tag{3.19}$$

Furthermore, considering that the connection components $\Gamma^i{}_{j0}$ vanishes, the covariant derivatives on the left-hand side reduce to ordinary derivatives:

$$\frac{d^2\eta^i}{ds^2} = R^i{}_{00j}\, \eta^j \,. \tag{3.20}$$

Now, the usual Newtonian limit of the time component of the geodesic equation (3.7) yields [16]

$$c dt/ds = \text{constant}.$$

We can then multiply both sides of (3.20) by ds^2/c^2dt^2, which yields

$$\frac{d^2\eta^i}{dt^2} = R^i{}_{00j}\, c^2\, \eta^j \, . \tag{3.21}$$

Substituting (3.18) and using identity (3.16), we get

$$\frac{d^2\eta^i}{dt^2} = -\,\delta^{ik}\left(\partial_j\partial_k\Phi\right)\eta^j \, . \tag{3.22}$$

It should be noted that the force appearing on the right-hand side is proportional to the distance separating the particles, which shows that the Newtonian geodesic deviation in general relativity is cosistent with the Newtonian description of tidal forces [17].

4 de Sitter Space, Group, and All That

In this chapter we are going to review the de Sitter spacetime and group. We present a description about the geometry of the de Sitter spacetime and a discussion about the structure of the de Sitter group. After we will study the contraction limits of de Sitter group by means of *Inönu-Wigner procedure* and, finally we will present the de Sitter Killing vectors.

4.1 The de Sitter Spacetime

The de Sitter spacetime, denoted $dS(4,1)$, is a hyperbolic, maximally symmetric curved spacetime. It can be seen as a hypersurface in a host 5-dimensional pseudo-Euclidean space with metric $\eta_{AB} = (+1,-1,-1,-1,-1)$, whose points in Cartesian coordinates χ^A satisfy the relation

$$\eta_{AB}\chi^A\chi^B = -l^2 \,. \tag{4.1}$$

In terms of four-dimensional coordinates, it assumes the form

$$\eta_{\mu\nu}\chi^\mu\chi^\nu - \left(\chi^4\right)^2 = -l^2 \tag{4.2}$$

where l is the de Sitter length parameter, or pseudo-radius. In addition, the de Sitter length parameter is related with the cosmological constant by

$$\Lambda = \frac{3}{l^2} \tag{4.3}$$

4.1.1 Stereographic coordinates

The stereographic coordinates $\{x^\mu\}$ are obtained by performing a stereographic projection of hyperboloid, represented by equation (4.1), to Minkowski space. This projection is given by

$$\chi^\mu = \Omega x^\mu \qquad \text{and} \qquad \chi^4 = -l\Omega\left(1 + \frac{\sigma^2}{4l^2}\right), \tag{4.4}$$

where

$$\Omega = \frac{1}{1 - \sigma^2/4l^2}\,, \tag{4.5}$$

with σ^2 the Lorentz invariant squared form $\sigma^2 = \eta_{\mu\nu}x^\mu x^\nu$. In these coordinates, the infinitesimal de Sitter quadratic interval assumes the form

$$ds^2 = \Omega^2\eta_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{4.6}$$

from where we see that, in this case, the de Sitter metric is conformally flat:

$$g_{\alpha\beta} = \Omega^2 \eta_{\alpha\beta} \,. \tag{4.7}$$

The Levi-Civita connection of this metric is

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{\Omega}{2l^2}\left(\delta^{\rho}_{\mu}\eta_{\nu\alpha}x^{\alpha} + \delta^{\rho}\eta_{\mu\alpha}x^{\alpha} - \eta_{\mu\nu}x^{\rho}\right) \,. \tag{4.8}$$

The corresponding Riemann tensor is given by

$$R^{\rho}{}_{\mu\nu\sigma} = \frac{\Omega^2}{l^2}\left(\delta^{\rho}_{\nu}\eta_{\mu\sigma} - \delta^{\rho}_{\mu}\eta_{\nu\sigma}\right) \,, \tag{4.9}$$

or equivalently

$$R^{\rho}{}_{\mu\nu\sigma} = \frac{1}{l^2}\left(\delta^{\rho}_{\nu}g_{\mu\sigma} - \delta^{\rho}_{\mu}g_{\nu\sigma}\right) \,, \tag{4.10}$$

with $g_{\mu\sigma}$ the metric (4.7). The Ricci and scalar curvatures are given respectively by

$$R_{\mu\nu} = \frac{3}{l^2}g_{\mu\nu} \quad \text{and} \quad R = \frac{12}{l^2} \,. \tag{4.11}$$

4.2 de Sitter group and algebra

Minkowski spacetime M, with vanishing curvature and Poincaré as its kinematic group, is the simplest case of a homogeneous spacetime. It is actually a homogeneous space under the Lorentz group:

$$M = \mathcal{P}/\mathcal{L} \,. \tag{4.12}$$

Minkoswki is a isotropic and homogeneous spacetime. Its isotropy is ruled by the Lorentz transformations and its homogeneity is ruled by the ordinary translations. de Sitter is also a homogeneous spacetime, with the de Sitter group SO(4,1) as kinematic group. It is homogeneous space under the Lorentz group:

$$dS = SO(4,1)/\mathcal{L} \tag{4.13}$$

Its homogeneity property, as we are going to see, is completely different from Minkowski.

4.2.1 Generators of the de Sitter group

In Cartesian coordinates χ^A , the generators of the infinitesimal de Sitter are written in the form

$$L_{AB} = \eta_{AC}\chi^C \frac{\partial}{\partial\chi^B} - \eta_{BC}\chi^C \frac{\partial}{\partial\chi^A} \,. \tag{4.14}$$

These generators satisfy the commutation relations

$$[L_{AB}, L_{CD}] = \eta_{BC}L_{AD} + \eta_{AD}L_{BC} - \eta_{BD}L_{AC} - \eta_{AC}L_{BD} \tag{4.15}$$

In term of stereographic coordinates $\{x^{\mu}\}$, the de Sitter generators are given by

$$L_{\mu\nu} = \eta_{\mu\rho}x^{\rho}P_{\nu} - \eta_{\nu\rho}x^{\rho}P_{\mu} \quad \text{and} \quad L_{4\mu} = lP_{\mu} - \frac{K_{\mu}}{4l} \,, \tag{4.16}$$

where

$$P_\mu = \partial_\mu \quad \text{and} \quad K_\mu = \left(2\eta_{\mu\nu} x^\nu x^\rho - \sigma^2 \delta_\mu^\rho\right) \partial_\rho \tag{4.17}$$

are the generators of ordinary and proper conformal transformation, respectively. Generators $L_{\mu\nu}$ refer to the Lorentz subgroup, whereas the $L_{4\mu}$ define the transitivity on the homogeneous space. These generators satisfy the commutation relations

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho\mu\sigma} \tag{4.18}$$

$$\left[L_{4\mu}, L_{\nu\rho}\right] = \eta_{\mu\nu} L_{4\rho} - \eta_{\mu\rho} L_{4\nu} \tag{4.19}$$

$$\left[L_{4\mu}, L_{4\nu}\right] = L_{\mu\nu} \tag{4.20}$$

We can see from equation (4.16) that de Sitter spacetime is transitive under combination of ordinary translations and proper conformal transformation — usually called "de Sitter translations". This should be compared to Minkowski, which is transitive under ordinary translations only. We can then say that the de Sitter spacetime naturally introduces the proper conformal transformations as part of spacetime kinematics.

4.3 Inönü-Wigner contractions of de Sitter group

In this section we are going to discuss two possible Inönü-Wigner contractions of the de Sitter group [18]. Starting with a semisimple group, the Inönü-Wigner contraction yields a non-semisimple group with the same dimension of the original group. A trivial example is the contraction limit of the speed of light going to infinity $c \to \infty$, which leads the relativistic Lorentz group to the non-relativistic Galilei group. In the same limit the de Sitter group contracts to the Newton-Hooke group, which is a non-relativistic group in the presence of a non-vanishing cosmological constant. In what follows we are going to consider both the limit of a vanishing and the limit of an infinite cosmological term Λ.

4.3.1 The contraction limit $l \to \infty$

In order to study the limit $l \to \infty$, which corresponds to $\Lambda \to 0$, it is convenient to rewrite the de Sitter generators according to

$$\Pi_\mu \equiv \frac{L_{4\mu}}{l} = P_\mu - \frac{K_\mu}{4l^2}, \tag{4.21}$$

whereas the generators $L_{\mu\nu}$ keep their original form form. In term of theses generators, the commutation relations assume the form

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho\mu\sigma} \tag{4.22}$$

$$\left[\Pi_\mu, L_{\nu\rho}\right] = \eta_{\mu\nu} \Pi_\rho - \eta_{\mu\rho} \Pi_\nu \tag{4.23}$$

$$\left[\Pi_\mu, \Pi_\nu\right] = l^{-2} L_{\mu\nu} \tag{4.24}$$

The last commutation relation shows that the "de Sitter translations" are not really, translations, but rotations. When we take the limit $l \to \infty$, we see from (4.21) that de Sitter generators Π_μ reduces to generators of ordinary translations:

$$\Pi_\mu \to P_\mu \,. \tag{4.25}$$

In this limit, the commutation relations (4.22), (4.23) and (4.23) reduce to

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho\mu\sigma} \tag{4.26}$$

$$\left[P_{\mu}, L_{\nu\rho}\right] = \eta_{\mu\nu} P_{\rho} - \eta_{\mu\rho} P_{\nu} \tag{4.27}$$

$$\left[P_{\mu}, P_{\nu}\right] = 0\,, \tag{4.28}$$

which are the commutation relations of the Poincaré group. This means that, under the limit in consideration, the de Sitter group $SO(4,1)$ contracts to the Poincaré group $\mathcal{P}$. Concomitantly, the de Sitter space $dS = SO(4,1)/\mathcal{L}$ contracts to the Minkowski space $M = \mathcal{P}/\mathcal{L}$, and the curvature tensor vanishes:

$$R^{\rho}_{\mu\nu\sigma} \to 0\,, \quad R^{\sigma\rho} \to 0\,, \quad R \to 0\,. \tag{4.29}$$

4.3.2 The contraction limit $l \to 0$

In order to study the limit $l \to 0$, which corresponds to $\Lambda \to \infty$, we have to rewrite the de Sitter generators according to

$$\bar{\Pi}_{\mu} = 4l L_{4\mu} = 4l^2 P_{\mu} - K_{\mu}\,, \tag{4.30}$$

whereas the generators $L_{\mu\nu}$ keep their original form. In this case the commutation relations assumes the form

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho\mu\sigma} \tag{4.31}$$

$$\left[\bar{\Pi}_{\mu}, L_{\nu\rho}\right] = \eta_{\mu\nu} \bar{\Pi}_{\rho} - \eta_{\mu\rho} \bar{\Pi}_{\nu} \tag{4.32}$$

$$\left[\bar{\Pi}_{\mu}, \bar{\Pi}_{\nu}\right] = 16 l^2 L_{\mu\nu} \tag{4.33}$$

In the contraction limit $l \to 0$, we can see from (4.30) that the de Sitter generators $\bar{\Pi}_{\mu}$ reduce to the generators of proper conformal transformations:

$$\bar{\Pi}_{\mu} = -K_{\mu}\,. \tag{4.34}$$

The commutation relations (4.31), (4.32) and (4.33) assume then the form

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho\mu\sigma} \tag{4.35}$$

$$\left[K_{\mu}, L_{\nu\rho}\right] = \eta_{\mu\nu} K_{\rho} - \eta_{\mu\rho} K_{\nu} \tag{4.36}$$

$$\left[K_{\mu}, K_{\nu}\right] = 0\,. \tag{4.37}$$

We can identify these commutation relations as the Lie algebra of the conformal Poncairé group, $\bar{\mathcal{P}} = \mathcal{L} \oslash \bar{\mathcal{T}}$, the semi-direct product of the Lorentz and the proper conformal. As a result of this algebra and group deformations, the de Sitter spacetime reduces to the homogeneous space

$$dS \to \bar{M} = \bar{\mathcal{P}}/\mathcal{L}\,.$$

Concomitantly, the de Sitter metric reduces to

$$\bar{g}_{\mu\nu} = \sigma^{-4} \eta_{\mu\nu}\,, \tag{4.38}$$

which is invariant under proper conformal transformations. This is similar to the Minkowski metric $\eta_{\mu\nu}$, which is invariant under ordinary translations. Also similar to Minkowski, the corresponding Riemann, Ricci, and scalar curvatures vanish identically:

$$\bar{R}^{\rho}_{\mu\nu\sigma} \to 0\,, \quad \bar{R}^{\sigma\rho} \to 0\,, \quad \bar{R} \to 0\,.$$

From these properties we can infer that $\bar{M}$ is a singular, four-dimensional cone spacetime, transitive under proper conformal transformation [11]

4.4 de Sitter transformation and Killing vectors

The de Sitter spacetime is maximally symmetric in the sense that it can lodge the highest possible number of Killing vectors. In terms of the five-dimensional space coordinates, an infinitesimal de Sitter transformation is written as

$$\delta\chi^C = \frac{1}{2}\epsilon^{AB} L_{AB}\chi^C \tag{4.39}$$

where $\epsilon^{CD} = -\epsilon^{DC}$ are the transformation parameter, and

$$L_{CD} = \eta_{CE}\chi^E \frac{\partial}{\partial\chi^D} - \eta_{DE}\chi^E \frac{\partial}{\partial\chi^C} \tag{4.40}$$

are the de Sitter generators. In terms of stereographic coordinates can express it:

$$\delta x^\mu \equiv \delta_L x^\mu + \delta_\Pi x^\mu = \frac{1}{2}\epsilon^{\nu\rho} L_{\nu\rho} x^\mu + \epsilon^{4\rho} L_{4\rho} x^\mu \tag{4.41}$$

where

$$\delta_L x^\mu = \frac{1}{2}\epsilon^{\nu\rho} L_{\nu\rho} x^\mu$$

are the infinitesimal transformations of Lorentz group, with $L_{\mu\nu}$ the Lorentz generators defined by equation (4.16) and $\epsilon^{\nu\rho}$ are transformation parameters,then

$$\delta_\Pi x^\mu = \epsilon^{4\rho} L_{4\rho} x^\mu \tag{4.42}$$

are the infinitesimal transformations of de Sitter "translation", with $L_{4\rho}$ are the generators that define the transitivity on the de Sitter spacetime. Now, we are going to study separately the cases of large and small values of the de Sitter length parameter l, which correspond to the cases of small and large values of the cosmological term Λ.

4.4.1 Large values of l

For large values of l, it is convenient to redefine the "de Sitter translations" generators according to

$$\Pi_\rho = \frac{L_{4\rho}}{l} = P_\rho - \frac{1}{4l^2} K_\rho\,. \tag{4.43}$$

In this case, the de Sitter transformation assume the form

$$\delta x^\mu \equiv \delta_L x^\mu + \delta_\Pi x^\mu = \frac{1}{2}\epsilon^{\nu\rho} L_{\nu\rho} x^\mu + \epsilon^{4\rho} l \frac{L_{4\rho} x^\mu}{l}$$

$$\delta x^\mu \equiv \delta_L x^\mu + \delta_\Pi x^\mu = \frac{1}{2}\epsilon^{\nu\rho} L_{\nu\rho} x^\mu + \epsilon^\rho \Pi_\rho x^\mu \tag{4.44}$$

where $\epsilon^\rho = \epsilon^{4\rho} l$. Sustituting the generators, the infinitesimal transformations of Lorentz group assumes the form

$$\delta_L x^\mu = \frac{1}{2}\xi^\mu_{\nu\rho}\epsilon^{\nu\rho} \tag{4.45}$$

where $\xi^\mu_{\nu\rho} = \left(\eta_{\nu\lambda}\delta^\mu_\rho - \eta_{\rho\lambda}\delta^\mu_\nu\right) x^\lambda$ are the Killing vectors of the Lorentz group. On the other hand, the "de Sitter translations" assume the form

$$\delta_\Pi x^\mu = \xi^\mu_\rho \epsilon^\rho \tag{4.46}$$

Where

$$\xi^\mu_\rho = \delta^\mu_\rho - \frac{1}{4l^2}\bar{\delta}^\mu_\rho \tag{4.47}$$

are the Killing vectors of the "de Sitter translations", with δ^μ_ρ the Killing vectors of the ordinary translations, and

$$\bar{\delta}^\mu_\nu = 2\eta_{\nu\lambda} x^\lambda x^\mu - \sigma^2 \delta^\mu_\nu \tag{4.48}$$

the Killing vectors of proper conformal transformation. Of course, the ten Killing vectors satisfy the Killing equation:

$$\nabla_\nu \xi^{(\rho\sigma)}_\mu + \nabla_\mu \xi^{(\rho\sigma)}_\nu = 0$$

$$\nabla_\nu \xi^{(\rho)}_\mu + \nabla_\mu \xi^{(\rho)}_\nu = 0\,.$$

In the contraction limit $l \to \infty$, the de Sitter Killing vectors reduce to those of the Poincaré group, and we get back the transformation of the Poncairé group:

$$\delta_L x^\mu = \frac{1}{2}\xi^\mu_{\rho\nu}\epsilon^{\rho\nu} \quad \text{and} \quad \delta_P x^\mu = \delta^\mu_\rho \epsilon^\rho\,. \tag{4.49}$$

4.4.2 Small values of l

For small values of l, it is convenient to rewrite the "de Sitter translations" generators in the form

$$\bar{\Pi}_\rho \equiv 4l L_{4\rho} = 4l P_\rho - K_\rho\,. \tag{4.50}$$

The de Sitter transformations then becomes

$$\delta x^\mu \equiv \delta_L x^\mu + \delta_{\bar{\Pi}} x^\mu = \frac{1}{2}\epsilon^{\nu\rho} L_{\nu\rho} x^\mu + \epsilon^\rho \bar{\Pi}_\rho x^\mu \tag{4.51}$$

with $\epsilon^\rho \equiv \epsilon^{4\rho}/4l$ the transformation parameters. The infinitesimal Lorentz transformation keeps its form

$$\delta_L x^\mu = \frac{1}{2}\xi^\mu_{\nu\rho}\epsilon^{\nu\rho} \tag{4.52}$$

where $\xi^\mu_\rho = \left(\eta_{\mu\lambda}\delta^\mu_\rho - \eta_{\rho\lambda}\delta^\mu_{nu}\right) x^\lambda$ are the Killing vectors of the Lorentz group. The "de Sitter "translation", on the other hand, becomes

$$\delta_{\bar{\Pi}} x^\mu = \bar{\xi}^\mu_\rho \epsilon^\rho \tag{4.53}$$

where

$$\bar{\xi}^{\mu}_{\nu} = 4l^2\delta^{\mu}_{\nu} - \bar{\delta}^{\mu}_{\nu} \tag{4.54}$$

are the Killing vectors of the "de Sitter translations". Of course, the ten Killing vectors satisfy the Killing equations:

$$\nabla_{\nu}\xi^{(\rho\sigma)}_{\mu} + \nabla_{\mu}\xi^{(\rho\sigma)}_{\nu} = 0$$

$$\nabla_{\nu}\xi^{(\rho)}_{\mu} + \nabla_{\mu}\xi^{(\rho)}_{\nu} = 0\,.$$

In the contraction limit $l \to 0$, the de Sitter Killing vectors reduce to the transformation of the conformal Poncairé group:

$$\delta_L x^{\mu} = \frac{1}{2}\xi^{\mu}_{\rho\nu}\epsilon^{\rho\nu} \quad \text{and} \quad \delta_{\bar{\Pi}} x^{\mu} = \bar{\delta}^{\mu}_{\rho}\epsilon^{\rho}\,. \tag{4.55}$$

5 General Relativity in Locally de Sitter Spacetimes

As we have discusses in Chapter 1, the de Sitter spacetime represents a different background for the construction physical theories, standing with an equal footing with Minkowski spacetime. General relativity, for instance, can be constructed on any one of them. Of course, in either case gravitation will have the same dynamics, only their local kinematics will be different. If the underlying spacetime is Minkowski, the local kinematics will be ruled by the Poincaré group of Einstein special relativity. If the underlying spacetime is de Sitter, the local kinematics will be ruled by the de Sitter group, which amounts then to replace ordinary special relativity by a de Sitter-invariant special relativity. When general relativity is constructed on Minkowski, all solutions to Einstein equation will be a spacetime that reduces locally to Minkowski. On the other hand, when general relativity is constructed on de Sitter, all solutions to the corresponding Einstein equation will be a spacetime that reduces locally to de Sitter. Instead of a Riemannian geometry, spacetime will be represented by a more general structure called Cartan geometry. In this chapter, we are going to explore the changes that occur in general relativity when it is constructed on a de Sitter spacetime. Our ultimately purpose is to understand how the concept of local transitivity changes the notion of motion and consequently the notion of geodesics.

5.1 The de Sitter modified Einstein equation

In a locally de Sitter spacetime, the gravitational action is written in the form

$$S_g = -\frac{c^3}{16\pi G}\int R\sqrt{-g}\, d^4x \tag{5.1}$$

where the scalar curvature R represents both the *kinematical curvature* of the underlying de Sitter spacetime and the *dynamical curvature* of general relativity. Let us consider an arbitrary dynamical transformation $\delta g_{\rho\mu}$ of the metric tensor, which is a transformation not related to any spacetime coordinate transformation. Variation of the gravitational action under such metric transformation yields

$$\delta S_g = \frac{c^3}{16\pi G}\int \left(R^{\rho\mu} - \tfrac{1}{2}g^{\rho\mu}R\right)\delta g_{\rho\mu}\sqrt{-g}\, d^4x\,. \tag{5.2}$$

The term between parentheses is the so-called Einstein tensor, which satisfies the contracted form of the second Bianchi identity

$$\nabla_\mu\left(R^{\rho\mu} - \tfrac{1}{2}g^{\rho\mu}R\right) = 0\,, \tag{5.3}$$

with ∇_μ the covariant derivative in the spacetime metric. On the other hand, when considering the variation of the action of a general source field

$$S_m = \frac{1}{c} \int \mathcal{L}_m \sqrt{-g}\, d^4x\,, \tag{5.4}$$

it is necessary to take into account some subtleties. The problem is that the explicit form of the covariantly conserved current depends on the local properties of spacetime. The metric transformation in this case must then take into account those properties. In the locally Minkowski spacetime of ordinary general relativity, such metric transformation has the form

$$\delta_T g_{\nu\mu} = \delta^\rho_\nu\, \delta g_{\rho\mu}\,, \tag{5.5}$$

where δ^ρ_ν are the Killing vectors of translations, which are the transformations that define the transitivity of the Minkowski spacetime, and $\delta g_{\rho\mu}$ is the same arbitrary metric transformation used to compute the variation of the gravitational action. In this case, the variation of the source action (5.4) is

$$\delta S_m = -\frac{1}{2c} \int \delta^\rho_\alpha\, T^{\alpha\mu}\, \delta g_{\rho\mu} \sqrt{-g}\, d^4x\,, \tag{5.6}$$

where

$$\delta^\rho_\alpha\, T^{\alpha\mu} \equiv T^{\rho\mu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g_{\rho\mu}} \tag{5.7}$$

is the symmetric energy-momentum tensor, which is covariantly conserved:

$$\nabla_\mu \left(\delta^\rho_\alpha\, T^{\alpha\mu}\right) = 0\,. \tag{5.8}$$

We have on purpose kept the (trivial) translational Killing vectors δ^ρ_α in the above expressions because they are quite elucidative. For example, remember that Noether's theorem establishes a relation between invariance under ordinary translations and energy-momentum conservation. The presence of the translational Killing vectors in the conserved current leaves it clear that the conservation law (5.8) is a legitimate consequence of Noether's theorem, in the sense that the local properties of spacetime were properly taken into account.* Let us consider now the case of locally de Sitter spacetimes. Analogously to (5.5), the metric transformation in this case is written as

$$\delta_\Pi g_{\nu\mu} = \xi^\rho_\nu\, \delta g_{\rho\mu}\,, \tag{5.9}$$

where ξ^ρ_ν are the Killing vectors of the de Sitter "translations", which are the transformations that define the transitivity of the de Sitter spacetime [18], and $\delta g_{\rho\mu}$ is the same arbitrary metric transformation used to compute the variation of the gravitational action. Using (5.9), the variation of the source action (5.4) is found to be

$$\delta S_m = -\frac{1}{2c} \int T^{\rho\mu}\, \delta_\Pi g_{\rho\mu} \sqrt{-g}\, d^4x\,, \tag{5.10}$$

*In the usual formulation of general relativity, differently from (5.5), the translational Killing vectors are not explicitly shown in the metric transformation. Although this can be done—because the Killing vectors are just Kronecker delta's—it becomes unclear why the variation of the Lagrangian with respect to the metric tensor should give the energy-momentum tensor, a current whose conservation law is related, through Noether's theorem, to the invariance of the source Lagrangian under spacetime translations.

where $T^{\rho\mu}$ is the symmetric energy-momentum tensor (5.7). It can then be written in the form

$$\delta S_m = -\frac{1}{2c} \int \Pi^{(\rho\mu)} \, \delta g_{\rho\mu} \, \sqrt{-g} \, d^4x \,, \tag{5.11}$$

with $\Pi^{(\rho\mu)}$ the symmetric part of the current [19]

$$\Pi^{\rho\mu} = \xi^{\rho}_{\alpha} \, T^{\alpha\mu} \,, \tag{5.12}$$

which is the covariantly conserved current in locally de Sitter spacetimes:

$$\nabla_\mu \Pi^{\rho\mu} = 0 \,. \tag{5.13}$$

Thus, from the variational principle $\delta S_g + \delta S_m = 0$, we get

$$\frac{c^3}{16\pi G} \int \left(R^{\rho\mu} - \frac{1}{2} g^{\rho\mu} R - \frac{8\pi G}{c^4} \, \Pi^{(\rho\mu)} \right) \delta g_{\rho\mu} \sqrt{-g} \; d^4x = 0 \,. \tag{5.14}$$

In view of the arbitrariness of $\delta g_{\rho\mu}$, the de Sitter modified Einstein equation is found to be [19]

$$R^{\rho\mu} - \frac{1}{2} g^{\rho\mu} R = \frac{8\pi G}{c^4} \, \Pi^{(\rho\mu)} \,. \tag{5.15}$$

This is the equation that replaces ordinary Einstein equation when the Poincaré invariant Einstein special relativity is replaced by a de Sitter-invariant special relativity [5–7]. In the contraction limit $l \to \infty$, which corresponds to $\Lambda \to 0$, the underlying de Sitter spacetime contracts to Minkowski, the de Sitter Killing vectors ξ^{μ}_{ρ} reduce to the Killing vectors δ^{μ}_{ρ} of ordinary translations, and we recover the ordinary Einstein equation

$$R^{\rho\mu} - \tfrac{1}{2} g^{\rho\mu} R = \frac{8\pi G}{c^4} \, T^{\rho\mu} \tag{5.16}$$

of locally Minkowski spacetimes.

5.2 Geodesics in Locally de Sitter spacetimes

As is well-known, the de Sitter spacetime is geodesically complete. However, there are points of the space that cannot be joined to each other by any usual geodesics of the de Sitter metric [20]. The reason is that the usual family of de Sitter geodesics describes trajectories whose points are connected to each other by ordinary translation only. Since the de Sitter spacetime is transitive under a combination of translation and proper conformal transformations, there will be points in the de Sitter spacetime that cannot be joined to each other by any geodesics of this family. In what follows we are going to show that, provided the local transitivity properties of spacetime are appropriately taken into account, a new family of geodesics that are able to connect any two points of spacetime is obtained. We are going to obtain these new geodesics from a variational principle, and also from the Mathisson-Papapetrou method.

5.2.1 de Sitter Geodesics from a Variational Principle

A particle of mass m is represented by the action integral

$$S = -mc \int_a^b ds \tag{5.17}$$

with $ds = (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$. In the case of locally de Sitter spacetimes, as we have seen in the previous section, a general coordinate transformation has the form

$$\delta x^\rho \equiv \delta_\Pi x^\rho = \xi^\rho_\mu \delta x^\mu \, , \tag{5.18}$$

where ξ^ρ_μ are the Killing vectors associated to the "de Sitter translations". Under such transformation, the action changes according to

$$\delta S = -mc \int_a^b \left[\frac{1}{2} \partial_\rho g_{\mu\nu} \delta_\Pi x^\rho u^\mu dx^\nu + g_{\mu\nu} u^\mu \delta_\Pi dx^\nu \right] \tag{5.19}$$

where $u^\mu = dx^\mu/ds$ is the usual particle four-velocity. Using the identity

$$\delta_\Pi(dx^\nu) = d(\delta_\Pi x^\nu) \, ,$$

the action variation becomes

$$\delta S = -mc \int_a^b \left[\frac{1}{2} \partial_\rho g_{\mu\nu} \delta_\Pi x^\rho u^\mu dx^\nu + g_{\mu\nu} u^\mu d(\delta_\Pi x^\nu) \right] \, . \tag{5.20}$$

Integrating the last term by parts, and considering that $\delta_\Pi x^\nu$ vanishes at the extremes of integration, the variation reduces to

$$\delta S = -mc \int_a^b \left[\frac{1}{2} \partial_\rho g_{\mu\nu} u^\mu u^\nu - \partial_\sigma g_{\mu\rho} u^\sigma u^\mu - g_{\mu\rho} \frac{du^\mu}{ds} \right] \delta_\Pi x^\rho ds \, . \tag{5.21}$$

Using now the relation

$$\partial_\lambda g_{\mu\nu} = \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \, , \tag{5.22}$$

the variation assumes the form

$$\delta S = mc \int_b^a \left[u^\sigma \nabla_\sigma u_\gamma \xi^\gamma_\rho \right] \delta_\Pi x^\rho ds \tag{5.23}$$

with ∇_σ the covariant derivative in the Christoffel connection of the metric $g_{\mu\nu}$. Considering that the Killing vectors ξ^γ_ρ satisfy locally the Killing equations, we get

$$\delta S = mc \int_b^a \left[u^\sigma \nabla_\sigma \left(u_\gamma \xi^\gamma_\rho \right) \right] \delta_\Pi x^\rho ds \, . \tag{5.24}$$

Defining the anholonomic four-velocity $U_\rho = \xi^\gamma_\rho u_\gamma$, and taking into account the arbitrariness of the variation $\delta_\Pi x^\rho$, we get finally

$$u^\sigma \nabla_\sigma U_\rho \equiv u^\sigma \left[\partial_\sigma U_\rho - \Gamma^\gamma{}_{\rho\sigma} U_\gamma \right] = 0 \, . \tag{5.25}$$

Equivalently, we can write [21]

$$\frac{dU_\rho}{ds} - \Gamma^\gamma{}_{\rho\sigma} U_\gamma u^\sigma = 0\,. \tag{5.26}$$

This is the new family of geodesic, which is consistent with the transitivity properties of locally de Sitter spacetimes. The solutions of this equation are trajectories whose points are connected to each other by the combination of translations and proper conformal transformation. For this reason, they can be interpreted as the true geodesics of locally de Sitter spacetimes. It is important to note that these geodesics introduce a new notion of motion. In fact, in stereographic coordinates, the anholonomic four-velocity assumes the form

$$U^\mu \equiv \xi^\mu_\rho u^\rho = u^\mu - \frac{1}{4l^2}\bar{\delta}^\mu_\rho u^\rho\,, \tag{5.27}$$

from where we see that it takes into account both the translational and proper conformal "directions" of locally Sitter spacetimes. It is also important to note that, similarly to the usual case of locally Minkowski spacetimes, the geodesics (5.26) coincide with the conservation of the for-momentum, which in this case is the de Sitter four-momentum

$$\pi^\mu \equiv mcU^\mu = p^\mu - \frac{1}{4l^2}k^\mu \tag{5.28}$$

where

$$p^\mu \equiv \delta^\mu_\rho p^\rho = mcu^\mu \quad \text{and} \quad k^\mu \equiv \bar{\delta}^\mu_\rho p^\rho = \left(2\eta_{\rho\nu}\, x^\nu x^\mu - \sigma^2 \delta_\rho{}^\mu\right) p^\rho \tag{5.29}$$

are respectively the ordinary four-momentum, and the proper conformal momentum. In fact, it easy to see that (5.26) is equivalent to

$$\frac{d\pi^\mu}{ds} + \Gamma^\mu{}_{\nu\lambda}\pi^\nu u^\lambda = 0\,, \tag{5.30}$$

which is the conservation law of the de Sitter four-momentum. In the contraction limit $l \to 0$, which corresponds to $\Lambda \to 0$, the underlying de Sitter spacetime reduces to Minkowski, and we get the usual conservation of the ordinary four-momentum:

$$\frac{dp^\mu}{ds} - \Gamma^\mu{}_{\nu\lambda} p^\nu u^\lambda = 0\,. \tag{5.31}$$

5.2.2 de Sitter Geodesics from the Mathisson-Papapetrou method

In this section, we are going to derive the geodesic equation using the Papapetrou method. This method considers the simplest kind of a test particle, which is called *single-pole* particle. This means that it has at least some of integrals $\int \Pi^{\mu\nu} dv \neq 0$, while all integral with one or more factors δx^ρ are equal to zero. Thus the particles will describe a narrow tube in four dimensional space. Inside this tube a line C is chosen which will represent the motion of the particle. The coordinates of the points of C denoted as X^σ. To begin with, let us consider the covariant conservation law (5.13), that is,

$$\nabla_\mu \Pi^{\mu\nu} = \partial_\mu \Pi^{\mu\nu} + \Gamma^\mu{}_{\mu\sigma}\Pi^{\sigma\nu} + \Gamma^\nu{}_{\mu\sigma}\Pi^{\sigma\mu} = 0\,, \tag{5.32}$$

where $\Gamma^{\nu}{}_{\mu\sigma}$ is the Christoffel connection of the spacetime metric. Integrating this conservation law in a space section of spacetime, and using the single-pole approximation [22–24], we obtain the momentum conservation law

$$\frac{d\pi^{\mu}}{ds} + \Gamma^{\mu}{}_{\nu\lambda}\,\pi^{\nu}\,u^{\lambda} = 0, \tag{5.33}$$

where

$$\pi^{\mu} \equiv \Pi^{0\mu} = \int dv\sqrt{-g}\,\Pi^{0\mu} \tag{5.34}$$

is the de Sitter four-momentum. This is the conservation law (5.30), here obtained from the energy-momentum conservation (5.32). As already remarked, similarly to what happens in ordinary special relativity, the conservation of the four momentum coincides with the equation of motion, that is, with

$$\frac{dU_{\rho}}{ds} - \Gamma^{\gamma}{}_{\rho\sigma}U_{\gamma}u^{\sigma} = 0\,. \tag{5.35}$$

6 Geodesic Deviation in Locally de Sitter Spacetimes

When general relativity is constructed on Minkowski spacetime, any solution to Einstein equation will be a spacetime that reduces locally to Minkowski. Considering that such spacetime is transitive under ordinary translations, any trajectory in general relativity will be given by world-lines whose points are connected to each other by translations. In this case, the relative acceleration of two nearby geodesics, which is described by the geodesic deviation equation, is proportional to the Riemann tensor. On the other hand, when general relativity is constructed on de Sitter spacetime, any solution to the de Sitter modified Einstein equation will be a spacetime that reduces locally to de Sitter. Considering that such spacetime is transitive under a combination of translations and proper conformal transformations, any trajectory in this theory will be given by world-lines whose points are connected to each other by a combination of translations and proper conformal transformations. Of course, if the very notion of geodesics changes, the relative acceleration of two nearby geodesics, which is described by the geodesic deviation equation, must change accordingly. In what follows we are going to obtain the de Sitter modified geodesic deviation equation.

6.1 The de Sitter Modified Geodesic Deviation Equation

Consider a family of de Sitter modified geodesics (5.26) differing in the value of some parameter λ. This means that, for each λ = constant, $x^\mu = x^\mu(s, \lambda)$ will represent the equation of a geodesic, with s an affine parameter, as for example the proper time. We introduce the four-vector [16]

$$\eta^\mu = \frac{\partial x^\mu}{\partial \lambda}\, \delta\lambda \equiv v^\mu\, \delta\lambda \tag{6.1}$$

joining points on infinitely close geodesics, corresponding to parameters λ and $\lambda + \delta\lambda$, that have the same value of s. From the trivial equality $v^\rho \partial_\rho u^\mu = u^\rho \partial_\rho v^\mu$, it follows that

$$u^\rho \nabla_\rho v^\mu = v^\rho \nabla_\rho u^\mu\,. \tag{6.2}$$

Analogously to the definition of the four-velocity $U^\mu = \xi^\mu_\alpha u^\alpha$, we introduce now the anholonomic four-velocity

$$V^\mu = \xi^\mu_\alpha\, v^\alpha \tag{6.3}$$

which is consistent with the transitivity properties of the de Sitter spacetime. Accordingly,

$$V^\mu \delta\lambda = \xi^\mu_\alpha\, \eta^\alpha \tag{6.4}$$

is a four-vector that is also consistent with the transitivity properties of the de Sitter spacetime. This means that it is always able to join points on infinitely close geodesics, corresponding to parameters λ and $\lambda+\delta\lambda$, with the same value of s. Using relation (6.2), it is easy to verify that U^μ and V^μ satisfy

$$u^\rho \nabla_\rho V^\mu = v^\rho \nabla_\rho U^\mu - \left(u^\alpha\, v^\beta - u^\beta\, v^\alpha\right) \nabla_\beta \xi^\mu_\alpha. \tag{6.5}$$

Consider now the second covariant derivative

$$\frac{D^2 V^\mu}{Ds^2} = u^\sigma \nabla_\sigma \left(u^\rho \nabla_\rho V^\mu\right). \tag{6.6}$$

Using identity (6.5), it can be rewritten in the form

$$\frac{D^2 V^\mu}{Ds^2} = u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho U^\mu\right) - u^\sigma \nabla_\sigma \left[\left(u^\alpha\, v^\beta - u^\beta\, v^\alpha\right) \nabla_\beta \xi^\mu_\alpha\right]. \tag{6.7}$$

Now, the first term on the right-hand side gives

$$u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho U^\mu\right) = u^\sigma\, v^\rho\, \nabla_\rho \nabla_\sigma U^\mu + u^\sigma \nabla_\sigma v^\rho\, \nabla_\rho U^\mu\,. \tag{6.8}$$

Using identity (6.2), it becomes

$$u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho U^\mu\right) = u^\sigma\, v^\rho\, \nabla_\rho \nabla_\sigma U^\mu + v^\sigma \nabla_\sigma u^\rho\, \nabla_\rho U^\mu\,, \tag{6.9}$$

which is equivalent to

$$u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho U^\mu\right) = u^\sigma v^\rho \left(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma\right) U^\mu\,, \tag{6.10}$$

where we have used that $u^\rho \nabla_\rho U^\mu = 0$ on account of the equation of motion (5.25). Substituting into (6.7), we get

$$\frac{D^2 V^\mu}{Ds^2} = u^\sigma v^\rho \left(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma\right) U^\mu - u^\sigma \nabla_\sigma \left[\left(u^\alpha\, v^\beta - u^\beta\, v^\alpha\right) \nabla_\beta \xi^\mu_\alpha\right]. \tag{6.11}$$

Taking into account the relation

$$\left(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma\right) U^\mu = R^\mu{}_{\nu\sigma\rho}\, U^\nu \tag{6.12}$$

it reduces to

$$\frac{D^2 V^\mu}{Ds^2} = R^\mu{}_{\nu\sigma\rho}\, U^\nu u^\sigma v^\rho - u^\sigma \nabla_\sigma \left[\left(u^\alpha\, v^\beta - u^\beta\, v^\alpha\right) \nabla_\beta \xi^\mu_\alpha\right]. \tag{6.13}$$

Multiplying both sides by the constant $\delta\lambda$ and using Eqs. (6.1) and (6.4), we get

$$\frac{D^2}{Ds^2}\left(\xi^\mu_\alpha\, \eta^\alpha\right) = R^\mu{}_{\nu\sigma\rho}\, U^\nu\, u^\sigma \eta^\rho + \frac{DV^\mu}{Ds} \left[\left(\eta^\alpha\, u^\beta - \eta^\beta\, u^\alpha\right) \nabla_\beta \xi^\mu_\alpha\right]. \tag{6.14}$$

Identifying

$$\eta^\alpha\, u^\beta - \eta^\beta\, u^\alpha = \frac{L^{\alpha\beta}}{mc}\,, \tag{6.15}$$

with $L^{\alpha\beta}$ the angular momentum of the particle moving in one geodesics in relation to the particle moving in the other geodesics, we get

$$\frac{D^2}{Ds^2}\left(\xi^\mu_\alpha\, \eta^\alpha\right) = R^\mu{}_{\nu\sigma\rho}\, U^\nu\, u^\sigma \eta^\rho + \frac{1}{mc} \frac{DV^\mu}{Ds} \left(L^{\alpha\beta}\, \nabla_\beta \xi^\mu_\alpha\right). \tag{6.16}$$

Considering furthermore that the angular momentum is covariantly conserved,

$$\nabla_\beta L^{\alpha\beta} = 0\,, \tag{6.17}$$

we get finally

$$\frac{D^2}{Ds^2}\left(\xi^\mu_\alpha\, \eta^\alpha\right) = R^\mu{}_{\nu\sigma\rho}\, U^\nu\, u^\sigma \eta^\rho + \frac{1}{mc}\frac{D}{Ds}\left[\nabla_\beta\left(L^{\alpha\beta}\, \xi^\mu_\alpha\right)\right]. \tag{6.18}$$

This is the de Sitter modified geodesic deviation equation, which is valid in locally de Sitter spacetimes. It should be mentioned that the dependence of the geodesic deviation on the angular momentum $L^{\alpha\beta}$ is expected in the sense that the de Sitter "translations" are not really translations, but rotations. In the contraction limit $l \to \infty$, which corresponds to a $\Lambda \to 0$, the underlying de Sitter spacetime contracts to Minkowski, and we get the usual geodesic deviation equation

$$\frac{D^2\eta^\mu}{Ds^2} = R^\mu{}_{\nu\sigma\rho}\, u^\nu u^\sigma \eta^\rho \tag{6.19}$$

valid in locally Minkowski spacetimes.

7 Conclusions

Both Minkowski and de Sitter are fundamental spacetimes in the sense that, as quotient spaces, they are known *a priori*, independently of Einstein equation. They represent actually different backgrounds for the construction of physical theories. General relativity, for instance, can be constructed on any one of the them. In the first case, a solution to the field equations will be a spacetime that reduces locally to Minkowski. In the second case, a solution to the field equations will be a spacetime that reduces locally to de Sitter. Although these two classes of spacetime are isotropic, their homogeneity properties differ substantially: whereas a locally Minkowski spacetime is transitive under translations, a locally de Sitter spacetime is transitive under a combination of translations and proper conformal transformations — the so-called de Sitter "translations". Now, transitivity is intimately related to the notion of motion. For example, any two points of a locally Minkowski spacetime are connected by a spacetime translation. As a consequence, motion in this spacetime is described by trajectories whose points are connected to each other by ordinary translations. On the other hand, any two points of a locally de Sitter spacetime are connected to each other by a combination of translation and proper conformal transformations. As a consequence, the notion of motion in this spacetime will change in the sense that it will be described by trajectories whose points are connected to each other by a combination of translation and proper conformal transformations. However, in the usual usual form of the geodesic equations, given by

$$\frac{du^\mu}{ds} + \Gamma^\mu{}_{\rho\gamma}\, u^\rho\, u^\gamma = 0\,, \tag{7.1}$$

the appropriate local homogeneity properties of spacetime are not taken into account in the variational principle. As a consequence, there are points in a locally de Sitter spacetime which are not connected by any one of these geodesics [20]. This single fact constitutes a clear evidence that they do not represent the true geodesics of locally de Sitter spacetimes. On the other hand, by taking into account the appropriate local homogeneity properties of locally de Sitter spacetime, we obtain a new family of trajectories, given by [21]

$$\frac{dU^\mu}{ds} + \Gamma^\mu{}_{\rho\gamma}\, U^\rho\, u^\gamma = 0\,, \tag{7.2}$$

where

$$U^\mu = \xi^\mu_\alpha\, u^\alpha \tag{7.3}$$

is an anholonomic four-velocity, with ξ^μ_α the Killing vectors of the de Sitter "translations". In the specific case of stereographic coordinates, they have the form

$$\xi^\mu_\alpha = \delta^\mu_\alpha - \bar{\delta}^\mu_\alpha\,, \tag{7.4}$$

where δ_α^μ are the Killing vectors of ordinary translations, and

$$\bar{\delta}_\alpha^\mu = \frac{\Lambda}{12}\left(2\eta_{\alpha\nu}x^\nu x^\mu - \sigma^2\delta_\alpha^\mu\right) \tag{7.5}$$

are the Killing vectors of proper conformal transformations. We see in this way that the four velocity U^μ takes into account the translational and the proper conformal 'directions' of locally de Sitter spacetime. As a consequence, the corresponding trajectories include both notions of motion: translational and proper conformal. They are, for this reason, able to connect any two points of any locally de Sitter spacetime. Of course, if the very notion of geodesic changes, the geodesic deviation must change accordingly. In this work we have obtained the explicit expression of the *geodesic deviation equation in locally de Sitter spacetimes*, which is given by [cf. Eq. (6.18)]

$$\frac{D^2}{Ds^2}\left(\xi_\alpha^\mu\,\eta^\alpha\right) = R^\mu{}_{\nu\sigma\rho}\,U^\nu\,u^\sigma\eta^\rho + \frac{1}{mc}\frac{D}{Ds}\left[\nabla_\beta\left(L^{\alpha\beta}\,\xi_\alpha^\mu\right)\right]. \tag{7.6}$$

It can be rewritten in the form

$$\frac{D^2}{Ds^2}\left(\xi_\alpha^\mu\,\eta^\alpha\right) = R^\mu{}_{\nu\sigma\rho}\,u^\nu u^\sigma\eta^\rho + \Delta^\mu\,, \tag{7.7}$$

where the first term on the right-hand side represents the usual geodesic deviation in locally Minkowski spacetimes, whereas the second term, given by

$$\Delta^\mu = \left\{-R^\mu{}_{\nu\sigma\rho}\,\bar{\delta}_\alpha^\nu\,u^\alpha u^\sigma\eta^\rho - \frac{1}{mc}\frac{D}{Ds}\left[\nabla_\beta\left(L^{\alpha\beta}\,\bar{\delta}_\alpha^\mu\right)\right]\right\}, \tag{7.8}$$

represents the additional effects produced by the local de Sitter kinematics. That is to say, they are a direct consequence of replacing the Poincaré invariant Einstein special relativity by a de Sitter invariant special relativity. Considering that geodesic deviation is one of the effects appearing in the Landau-Raychaudhuri equation, the additional effects embodied in (7.8) could eventually play a role in the Universe evolution.

Appendices

A Generators in Stereographic Coordinates

In the five-dimensional ambient space Cartesian coordinates χ^A, the generators of infinitesimal de Sitter transformations are written as

$$L_{AB} = \eta_{AC}\chi^C \frac{\partial}{\partial \chi^B} - \eta_{BC}\chi^C \frac{\partial}{\partial \chi^A} \,. \tag{A.1}$$

As we have seen in Chapter 2, the generators in stereographic coordinates stereographic coordinates x^μ are obtained through the projection

$$\chi^\mu = \Omega x^\mu \quad \text{and} \quad \chi^4 = -l\Omega\left(1 + \frac{\sigma^2}{4l^2}\right) , \tag{A.2}$$

where

$$\Omega = \left(1 - \frac{\sigma^2}{4l^2}\right)^{-1} \,. \tag{A.3}$$

The components $L_{\mu\nu}$ are

$$L_{\mu\nu} = \eta_{\mu\sigma}\chi^\sigma \frac{\partial}{\partial \chi^\nu} - \eta_{\nu\sigma}\chi^\sigma \frac{\partial}{\partial \chi^\mu} \,. \tag{A.4}$$

Using the first relation of (A.2), as well as the chain rule

$$\frac{\partial}{\partial \chi^\sigma} = \frac{\partial x^\rho}{\partial \chi^\sigma}\frac{\partial}{\partial x^\rho} = \frac{\partial\left(\Omega^{-1}\chi^\rho\right)}{\partial \chi^\sigma}\frac{\partial}{\partial x^\rho} = \Omega^{-1}\frac{\partial}{\partial x^\sigma} \,, \tag{A.5}$$

equation (A.4) takes the form

$$L_{\mu\nu} = \eta_{\mu\sigma}\Omega x^\sigma \left(\Omega^{-1}\frac{\partial}{\partial x^\nu}\right) - \eta_{\nu\sigma}\Omega x^\sigma \left(\Omega^{-1}\frac{\partial}{\partial x^\mu}\right) , \tag{A.6}$$

or equivalently,

$$L_{\mu\nu} = \eta_{\mu\sigma}x^\sigma \frac{\partial}{\partial x^\nu} - \eta_{\nu\sigma}x^\sigma \frac{\partial}{\partial x^\mu} \,. \tag{A.7}$$

We see from this expression that $L_{\mu\nu}$ are the six generators of the Lorentz group, which rules the isotropy in a de Sitter spacetime. Therefore, the Lorentz symmetry is preserved in presence of a cosmological term Λ. The remaining four generators are

$$L_{4\mu} = \eta_{4C}\chi^C \frac{\partial}{\partial \chi^\mu} - \eta_{\mu C}\chi^C \frac{\partial}{\partial \chi^4} \,. \tag{A.8}$$

Expanding the sum we get

$$L_{4\mu} = \left(\eta_{4\sigma}\chi^{\sigma} + \eta_{44}\chi^{4}\right)\frac{\partial}{\partial\chi^{\mu}} - \left(\eta_{\mu\sigma}\chi^{\sigma} + \eta_{\mu 4}\chi^{4}\right)\frac{\partial}{\partial\chi^{4}}, \tag{A.9}$$

which reduces to

$$L_{4\mu} = -\chi^{4}\frac{\partial}{\partial\chi^{\mu}} - \eta_{\mu\sigma}\chi^{\sigma}\frac{\partial}{\partial\chi^{4}}, \tag{A.10}$$

where we have used that $\eta_{44} = -1$ for the de Sitter group. Now, from the chain rule we have

$$\frac{\partial}{\partial\chi^{4}} = \frac{\partial x^{\rho}}{\partial\chi^{4}}\frac{\partial}{\partial x^{\rho}} = \frac{\partial\left(\Omega^{-1}\chi^{\rho}\right)}{\partial\chi^{4}}\frac{\partial}{\partial x^{\rho}}. \tag{A.11}$$

Using the alternative form of Ω, given by

$$\Omega = \frac{1}{2}\left(1 - \frac{\chi^{4}}{l}\right), \tag{A.12}$$

we obtain

$$\frac{\partial}{\partial\chi^{4}} = \frac{1}{2l}\Omega^{-1}x^{\rho}\frac{\partial}{\partial x^{\rho}}. \tag{A.13}$$

Using relations (A.5) and (A.13), the generators (A.10) can be rewritten in the form

$$L_{4\mu} = l\left(-1 + 2\Omega\right)\Omega^{-1}\frac{\partial}{\partial x^{\mu}} - \eta_{\mu\sigma}\Omega x^{\sigma}\left(\frac{1}{2l}\Omega^{-1}x^{\rho}\frac{\partial}{\partial x^{\rho}}\right), \tag{A.14}$$

which is equivalent to

$$L_{4\mu} = l\left(\frac{\sigma^{2}}{4l^{2}} + 1\right)\frac{\partial}{\partial x^{\mu}} - \frac{1}{2l}\eta_{\mu\sigma}x^{\sigma}x^{\rho}\frac{\partial}{\partial x^{\rho}}. \tag{A.15}$$

This equation can be recast in the form

$$L_{4\mu} = lP_{\mu} - \frac{1}{4l}K_{\mu}, \tag{A.16}$$

where

$$P_{\mu} = \frac{\partial}{\partial x^{\mu}} \qquad \text{and} \qquad K_{\mu} = \left(2\eta_{\mu\sigma}x^{\sigma}x^{\rho} - \sigma^{2}\delta_{\mu}^{\rho}\right)\frac{\partial}{\partial x^{\rho}} \tag{A.17}$$

are, respectively, the translation and the proper conformal generators.

B Commutation Relations

First commutation relation

Let us begin with the Lorentz generators, whose commutator is given by

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \left[\eta_{\mu\lambda} x^{\lambda} \frac{\partial}{\partial x^{\nu}} - \eta_{\nu\lambda} x^{\lambda} \frac{\partial}{\partial x^{\mu}}, \eta_{\sigma\beta} x^{\beta} \frac{\partial}{\partial x^{\rho}} - \eta_{\rho\beta} x^{\beta} \frac{\partial}{\partial x^{\sigma}}\right] . \tag{B.1}$$

Expanding, we get

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \left[\eta_{\mu\lambda} x^{\lambda} \frac{\partial}{\partial x^{\nu}}, \eta_{\sigma\beta} x^{\beta} \frac{\partial}{\partial x^{\rho}}\right] - \left[\eta_{\mu\lambda} x^{\lambda} \frac{\partial}{\partial x^{\nu}}, \eta_{\rho\beta} x^{\beta} \frac{\partial}{\partial x^{\sigma}}\right] - (\mu \leftrightarrow \nu) \tag{B.2}$$

The first term on the right-hand side is

$$\left[\eta_{\mu\lambda} x^{\lambda} \frac{\partial}{\partial x^{\nu}}, \eta_{\sigma\beta} x^{\beta} \frac{\partial}{\partial x^{\rho}}\right] = \eta_{\mu\lambda} \eta_{\sigma\nu} x^{\lambda} \frac{\partial}{\partial x^{\rho}} - \eta_{\sigma\beta} \eta_{\mu\rho} x^{\beta} \frac{\partial}{\partial x^{\nu}} . \tag{B.3}$$

Following a similar procedure for the other terms, we arrive at

$$\left[L_{\mu\nu}, L_{\sigma\rho}\right] = \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\sigma} L_{\nu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho} L_{\mu\sigma} . \tag{B.4}$$

Second commutation relation

Let us consider now the commutation relation

$$\left[L_{4\mu}, L_{\lambda\nu}\right] \equiv \left[l P_{\mu} - \frac{1}{4l} K_{\mu}, L_{\lambda\nu}\right] = l\left[P_{\mu}, L_{\lambda\nu}\right] - \frac{1}{4l}\left[K_{\mu}, L_{\lambda\nu}\right] . \tag{B.5}$$

The first term on the right-hand size yields

$$l\left[P_{\mu}, L_{\lambda\nu}\right] = l\left(\eta_{\mu\lambda} P_{\nu} - \eta_{\mu\nu} P_{\lambda}\right) . \tag{B.6}$$

The second term, on the other hand, is

$$\frac{1}{4l}\left[K_{\mu}, L_{\lambda\nu}\right] = \frac{1}{4l}\left[\left(2\eta_{\mu\sigma} x^{\sigma} x^{\rho} - \sigma^{2} \delta_{\mu}^{\rho}\right) \frac{\partial}{\partial x^{\rho}}, \eta_{\lambda\pi} x^{\pi} \frac{\partial}{\partial x^{\nu}} - \eta_{\nu\pi} x^{\pi} \frac{\partial}{\partial x^{\lambda}}\right] , \tag{B.7}$$

which can be recast in the form

$$\frac{1}{4l}\left[K_{\mu}, L_{\lambda\nu}\right] = \frac{1}{4l}\left[2\eta_{\mu\sigma} x^{\sigma} x^{\rho} \frac{\partial}{\partial x^{\rho}}, \eta_{\lambda\pi} x^{\pi} \frac{\partial}{\partial x^{\nu}}\right] - \frac{1}{4l}\left[\sigma^{2} \delta_{\mu}^{\rho} \frac{\partial}{\partial x^{\rho}}, \eta_{\lambda\pi} x^{\pi} \frac{\partial}{\partial x^{\nu}}\right] - (\lambda \leftrightarrow \nu) . \tag{B.8}$$

After some algebraic manipulation, it reduces to

$$\frac{1}{4l}\left[K_\mu, L_{\lambda\nu}\right] = \frac{1}{4l}\eta_{\mu\lambda}\left(2x_\nu x^\rho - \sigma^2\delta_\nu^\rho\right)\frac{\partial}{\partial x^\rho} - \frac{1}{4l}\eta_{\mu\nu}\left(2x_\lambda x^\rho - \sigma^2\delta_\lambda^\rho\right)\frac{\partial}{\partial x^\rho}\,, \tag{B.9}$$

or equivalently

$$\frac{1}{4l}\left[K_\mu, L_{\lambda\nu}\right] = \frac{1}{4l}\left(\eta_{\mu\lambda}K_\nu - \eta_{\mu\nu}K_\lambda\right)\,. \tag{B.10}$$

Substituting (B.6) and (B.10) into (B.5), we get

$$\left[L_{4\mu}, L_{\lambda\nu}\right] = \eta_{\mu\lambda}\left(lP_\nu - \frac{1}{4l}K_\nu\right) - \eta_{\mu\nu}\left(lP_\lambda - \frac{1}{4l}K_\lambda\right)\,, \tag{B.11}$$

which yields

$$\left[L_{4\mu}, L_{\lambda\nu}\right] = \eta_{\mu\lambda}L_{4\nu} - \eta_{\mu\nu}L_{4\lambda}\,. \tag{B.12}$$

Third commutation relation

Let us consider now the commutation relation

$$\left[L_{4\mu}, L_{4\nu}\right] = \left[lP_\mu - \frac{1}{4l}K_\mu, lP_\nu - \frac{1}{4l}K_\nu\right]\,. \tag{B.13}$$

Expanding the products, we get

$$\left[L_{4\mu}, L_{4\nu}\right] = l^2\left[P_\mu, P_\nu\right] - \frac{1}{4}\left[P_\mu, K_\nu\right] - \frac{1}{4}\left[K_\mu, P_\nu\right] + \frac{1}{16l^2}\left[K_\mu, K_\nu\right]\,. \tag{B.14}$$

The first term on the right-hand side of this equation vanishes identically:

$$\left[P_\mu, P_\nu\right] = 0\,. \tag{B.15}$$

The second term, on the other hand, is

$$\frac{1}{4}\left[P_\mu, K_\nu\right] = \frac{1}{4}\left[\partial_\mu, \left(2\eta_{\nu\lambda}x^\lambda x^\rho - \sigma^2\delta_\nu^\rho\right)\frac{\partial}{\partial x^\rho}\right]\,, \tag{B.16}$$

which can easily seen to be equivalent to

$$\frac{1}{4}\left[P_\mu, K_\nu\right] = \frac{1}{2}\left(\eta_{\nu\mu}x^\rho\frac{\partial}{\partial x^\rho} + \eta_{\nu\sigma}x^\sigma\frac{\partial}{\partial x^\mu} - \eta_{\mu\sigma}x^\sigma\frac{\partial}{\partial x^\nu}\right)\,. \tag{B.17}$$

Similarly, the third term is found to be

$$\frac{1}{4}\left[K_\mu, P_\nu\right] = -\frac{1}{2}\left(\eta_{\mu\nu}x^\rho\frac{\partial}{\partial x^\rho} + \eta_{\mu\sigma}x^\sigma\frac{\partial}{\partial x^\nu} - \eta_{\nu\sigma}x^\sigma\frac{\partial}{\partial x^\mu}\right)\,. \tag{B.18}$$

Finally, as a simple computation shows, the fourth term also vanishes identically:

$$\frac{1}{16l^2}\left[K_\mu, K_\nu\right] = 0\,. \tag{B.19}$$

Substituting theses expressions into equation (B.14), we obtain

$$\left[L_{4\mu}, L_{4\nu}\right] = \eta_{\mu\sigma}x^\sigma\frac{\partial}{\partial x^\nu} - \eta_{\nu\sigma}x^\sigma\frac{\partial}{\partial x^\mu} \equiv L_{\mu\nu}\,. \tag{B.20}$$

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