$SU(1,1)$ thermal group of bosonic strings and D-branes

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All possible Bogoliubov operators that generate the thermal transformations in thermo field dynamics form an $SU(1,1)$ group. We discuss this construction in the bosonic string theory. In particular, the transformation of the Fock space and string operators generated by the most general $SU(1,1)$ unitary Bogoliubov transformation and the entropy of the corresponding thermal string are computed. Also, we construct the thermal D-brane generated by the $SU(1,1)$ transformation in a constant Kalb-Ramond field and compute its entropy.

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I. INTRODUCTION

The physical and geometrical properties of D-branes have been under intense investigation for some time. In particular, the statistics of D-branes has attracted certain attention mainly when formulated in the low energy limit of string theory [1–12]. Progress has been made in understanding the thermodynamics of black holes and string gases [13–31]. Since in this limit the D-branes are solitons of (super)gravity, the statistics of D-branes has been naturally formulated in the Matsubara approach to quantum fields at finite temperature. However, despite the success achieved at low energies, up to now there has been little progress in understanding the microscopic description and the statistical properties of D-branes in the perturbative limit of string theory where the D-branes are described by coherent states in the Fock space of the closed string sector. Because of this interpretation, it is natural to seek a construction of thermal branes in the framework of thermo field dynamics in which the statistical mechanics is implemented by a thermal Bogoliubov operator acting on the Fock space and on the operators. Along this line of thought, a new approach to thermal D-branes has been proposed in [32–35] by using the basic concepts of thermo field dynamics (TFD) [36] (which is known to be equivalent to the Matsubara formalism at thermodynamical equilibrium). The construction from [32–35] has the advantage of maintaining explicitly the interpretation of D-branes as coherent boundary states in the Fock space of the closed bosonic sector. The very interpretation is used to define the D-branes at finite temperature [37–46].

One of the crucial ingredients of TFD is the thermal Bogoliubov transformation that maps the theory from zero to finite temperature. In particular, one can construct a thermal vacuum $|0(\beta_T)\rangle\rangle$ annihilated by thermal annihilation operators and one can express the average value of any observable Q as the expectation value in the thermal vacuum:

$$Z^{-1}(\beta_T)\mathrm{Tr}[\rho \;\; Q] \;=\; \langle\langle 0(\beta_T)|Q|0(\beta_T)\rangle\rangle, \tag{1}$$

where ρ is the distribution function, $\beta_T = (k_B T)^{-1}$, and k_B is the Boltzmann's constant. In order to fulfill the above requirement, the vacuum should belong to the direct product space between the original Fock space by an identical copy of it denoted by a tilde (see below), which justifies the notation used for vectors.

As discussed in [47,48], it is possible to define various Bogoliubov operators for any theory. They form an oscillator representation of the $SU(1,1)$ group for bosons and $SU(2)$ for fermions. If all these operators are taken into account, the thermal transformation is generated by a linear combination of generators of $SU(1,1)$. The coefficients of the generators determine if the transformation is unitary or not and if the generator satisfies the basic requirements of TFD, like the tilde invariance of the most general thermal generator and of vacuum [47,48]. The aim of this paper is to investigate this group in the case of the closed bosonic string and bosonic D-brane. This represents a part of the construction of a TFD-based formulation of thermal bosonic vacuum of string theory and thermal D-branes.

The outline of the paper is as follows. In Sec. II we discuss the $SU(1,1)$ group for the case of the bosonic string and construct the most general $SU(1,1)$ thermal transformation. The choice of the parameters in the general Bogoliubov generator is taken such that the thermal transformation will be unitary. With this choice we compute the entropy of the string. In Sec. III we give the boundary conditions and the D-brane state under the general unitary thermal transformation and compute the entropy of the D-brane. In Sec. IV we discuss the results and the main problems raised by implementing the $SU(1,1)$ thermal group.

II. THERMAL $SU(1,1)$ GROUP FOR CLOSED STRING

Let us consider a closed bosonic string in the Minkowski space-time and in the light-cone gauge $X^0 \pm X^{25}$. The most general solution of the equations of motion with the periodic boundary conditions $X^\mu(\tau,0)=X^\mu(\tau,\pi)$ has a Fourier expansion. Upon the quantization, the Fourier coefficients are interpreted as operators on the Fock space of the closed string. They must obey the canonical commutation relations

$$[\alpha_k^\mu, \alpha_l^\nu] = k\,\delta^{\mu\nu}\delta_{k+l,0}, \tag{2}$$

where α_k denotes the operators for the right-hand moving modes. Similar commutation relations hold for the left-hand moving modes β_k and the right modes and left modes are independent [49]. It is useful to introduce the oscillator operators

$$A_k^\mu = \frac{1}{\sqrt{k}}\alpha_k^\mu, \quad A_k^{\mu\dagger} = \frac{1}{\sqrt{k}}\alpha_{-k}^\mu,$$

$$B_k^\mu = \frac{1}{\sqrt{k}}\beta_k^\mu, \quad B_k^{\mu\dagger} = \frac{1}{\sqrt{k}}\beta_{-k}^\mu \quad \forall \quad k>0. \tag{3}$$

Following the TFD construction, we construct the direct product between the Fock space of the string and an identical copy of it. The quantities referring to the second (unphysical) copy of string are denoted by a tilde. Thus the extended space is given by

$$\hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}}. \tag{4}$$

The vectors of $\hat{\mathcal{H}}$ are constructed by acting with the string operators on the vacuum state

$$|0\rangle\rangle = |0\rangle\rangle_\alpha |0\rangle\rangle_\beta = (|0\rangle_\alpha \widetilde{|0\rangle}_\alpha)(|0\rangle_\beta \widetilde{|0\rangle}_\beta)$$
$$= (|0\rangle_\alpha |0\rangle_\beta)(|0\rangle_\alpha \widetilde{|0\rangle}_\beta), \tag{5}$$

where the vacua of the right-left Fock spaces are given by

$$|0\rangle\rangle_\alpha = |0\rangle_\alpha \otimes \widetilde{|0\rangle}_\alpha = |0,0\rangle_\alpha,$$

$$|0\rangle\rangle_\beta = |0\rangle_\beta \otimes \widetilde{|0\rangle}_\beta = |0,0\rangle_\beta, \tag{6}$$

the original string and its copy with each other. Therefore the operator algebra of the extended system is given by the following relations:

$$[A_k^\mu, A_m^{\nu\dagger}] = [\tilde{A}_k^\mu, \tilde{A}_m^{\nu\dagger}] = \delta_{km}\eta^{\mu\nu},$$

$$[A_k^\mu, \tilde{A}_m^\nu] = [A_k^\mu, \tilde{A}_m^{\nu\dagger}] = [A_k^\mu, \tilde{B}_m^\nu] = \cdots = 0. \tag{7}$$

The Hamiltonian of the extended system is constructed by demanding that the thermal vacuum, which will be constructed later, be invariant under time translation. This can be obtained if we define the following Hamiltonian operator:

$$\hat{H} = H - \tilde{H}$$

$$= \sum_{n>0}^{\infty} n(A_n^\dagger \cdot A_n + B_n^\dagger \cdot B_n - \tilde{A}_n^\dagger \cdot \tilde{A}_n - \tilde{B}_n^\dagger \cdot \tilde{B}_n). \tag{8}$$

The relation (8) shows that, if the new vacuum state is time invariant, then the copy of the string is nonphysical. Indeed, adding the second Fock space corresponds to furnishing new thermal degrees of freedom rather than dynamical degrees of freedom, necessary for discussing the statistics. Consequently, the physical observables are defined by operators without a tilde.

In order to construct a finite temperature model of string, one has to act on zero temperature Fock space and oscillator operators with the Bogoliubov operators which produce a thermal noise [47].

A. Thermal Bogoliubov transformation

The first step in constructing the string theory at finite temperature is to provide a thermal vacuum. This can be obtained by acting on the extended system vacuum (5) with any operator that: (i) mixes the operators A_k^μ, $\tilde{A}_k^{\mu\dagger}$ for the right modes and B_k^μ, $\tilde{B}_k^{\mu\dagger}$ for the left modes; (ii) commutes with the Hamiltonian (8) and (iii) takes the form of a Bogoliubov transformation [47]. The right/left transformations should be independent from the algebra (7). The general form of the Bogoliubov transformation that fixes the form of the generator is given by the following relation [47,48]:

$$\begin{pmatrix} A' \\ \tilde{A}^{\dagger\prime} \end{pmatrix} = e^{-iG}\begin{pmatrix} A \\ \tilde{A}^\dagger \end{pmatrix} e^{iG} = \mathcal{B}\begin{pmatrix} A \\ \tilde{A}^\dagger \end{pmatrix},$$

$$(A^{\dagger\prime} - \tilde{A}') = (A^\dagger - \tilde{A})\mathcal{B}^{-1}, \tag{9}$$

where $\mathcal{B}$ is a 2×2 complex unitary matrix

$$\mathcal{B} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}, \qquad |u|^2 - |v|^2 = 1, \tag{10}$$

and G is the generator of the transformation, called *the Bogoliubov operator*. The operators that satisfy the relations (9) and (10) have the following form [48]:

$$g_{1_k}^\alpha = \theta_{1_k}(A_k \cdot \tilde{A}_k + \tilde{A}_k^\dagger \cdot A_k^\dagger),$$

$$g_{1_k}^\beta = \theta_{1_k}(B_k \cdot \tilde{B}_k + \tilde{B}_k^\dagger \cdot B_k^\dagger),$$

$$g_{2_k}^\alpha = i\theta_{2_k}(A_k \cdot \tilde{A}_k - \tilde{A}_k^\dagger \cdot A_k^\dagger),$$

$$g_{2_k}^\beta = i\theta_{2_k}(B_k \cdot \tilde{B}_k - \tilde{B}_k^\dagger \cdot B_k^\dagger), \tag{11}$$

$$g_{3_k}^\alpha = \theta_{3_k}(A_k^\dagger \cdot A_k + \tilde{A}_k^\dagger \cdot \tilde{A}_k + \delta_{kk} tr\, \eta^{\mu\nu}),$$

$$g_{3_k}^\beta = \theta_{3_k}(B_k^\dagger \cdot B_k + \tilde{B}_k^\dagger \cdot \tilde{B}_k + \delta_{kk} tr\, \eta^{\mu\nu}),$$

where the index α refers to the right-moving modes and β to the left-moving modes, respectively, and the θ's are real parameters depending on the temperature which, for convenience, have been included in the operators. We assume that they are monotonous increasing functions on T. It is easy to verify that the generators (11) satisfy the $SU(1,1)$ algebra

$$[g_{1_k}^{\alpha,\beta}, g_{2_k}^{\alpha,\beta}] = -i\Theta_{123} g_{3_k}^{\alpha,\beta}, \quad [g_{2_k}^{\alpha,\beta}, g_{3_k}^{\alpha,\beta}] = i\Theta_{231} g_{1_k}^{\alpha,\beta},$$
$$[g_{3_k}^{\alpha,\beta}, g_{1_k}^{\alpha,\beta}] = i\Theta_{312} g_{2_k}^{\alpha,\beta}, \tag{12}$$

where we have defined

$$\Theta_{ijk} \equiv 2\frac{\theta_{i_k}\theta_{j_k}}{\theta_{k_k}}. \tag{13}$$

As we can see from Eq. (11), the most general thermal transformation takes the following form

$$G = \sum_k (G_k^\alpha + G_k^\beta), \tag{14}$$

where the right/left generators are given by the relations

$$G_k^\alpha = \lambda_{1_k}\tilde{A}_k^\dagger \cdot A_k^\dagger - \lambda_{2_k} A_k \cdot \tilde{A}_k + \lambda_{3_k}(A_k^\dagger \cdot A_k + \tilde{A}_k^\dagger \cdot \tilde{A}_k + \delta_{kk} tr\, \eta^{\mu\nu}), \tag{15}$$

$$G_k^\beta = \lambda_{1_k}\tilde{B}_k^\dagger \cdot B_k^\dagger - \lambda_{2_k} B_k \cdot \tilde{B}_k + \lambda_{3_k}(B_k^\dagger \cdot B_k + \tilde{B}_k^\dagger \cdot \tilde{B}_k + \delta_{kk} tr\, \eta^{\mu\nu}), \tag{16}$$

and the coefficients represent complex linear combinations of θ's

$$\lambda_{1_k} = \theta_{1_k} - i\theta_{2_k}, \quad \lambda_{2_k} = -\lambda_{1_k}^*, \quad \lambda_{3_k} = \theta_{3_k}. \tag{17}$$

The operator (14) generates the thermal transformation. The dependence on temperature is contained in the complex parameters λ.

There is some arbitrariness in choosing the parameters θ. This freedom can be used to fix the type of transformation. There are two conditions that are imposed on a general thermal transformation: (i) unitarity and (ii) invariance under tilde transformation that acts on any arbitrary operator as follows:

$$(AB)^\sim = \tilde{A}\tilde{B}, \quad (\alpha A)^\sim = \alpha^* \tilde{A}, \tag{18}$$

where α is a complex number and the asterisk is the complex conjugation. The invariance under tilde operation guarantees the invariance of the thermal vacuum under the same operation. However, working with the thermal $SU(1,1)$ implies choosing one type of transformation. The unitarity and the tilde invariance are not always simultaneously compatible [47,48]. In general, the two conditions will select only the generator g_{2_k} from the three generators above and reduce the problem to the TFD with one generator. In what follows, we will stick to the unitarity condition as being the most natural one for the system at hand, i.e., string theory. The other choice will be commented on in the final section.

B. Thermal vacuum and thermal string operators

The thermal vacuum of the system at finite temperature is obtained by acting on the vacuum at zero temperature (5) with the operator (14) [36]

$$|0(\theta)\rangle\rangle = e^{-iG}|0\rangle\rangle. \tag{19}$$

Since the left/right-moving terms commute among themselves, there are two distinct contributions from the α sector and the β sector, respectively. Consider the right-hand thermal vacuum. By applying the disentanglement theorem for $SU(1,1)$ [50,51], one can write the thermal vacuum under the following form:

$$|0(\theta)\rangle\rangle = \prod_k e^{\Gamma_{1_k}(\tilde{A}_k^\dagger \cdot A_k^\dagger)} e^{\log(\Gamma_{3_k})(A_k^\dagger \cdot A_k + \tilde{A}_k^\dagger \cdot \tilde{A}_k + \delta_{kk} tr\, \eta^{\mu\nu})} \times e^{\Gamma_{2_k}(A_k \cdot \tilde{A}_k)} \tag{20}$$

$$\times e^{\Gamma_{1_k}(\tilde{B}_k^\dagger \cdot B_k^\dagger)} e^{\log(\Gamma_{3_k})(B_k^\dagger \cdot B_k + \tilde{B}_k^\dagger \cdot \tilde{B}_k + \delta_{kk} tr\, \eta^{\mu\nu})} \times e^{\Gamma_{2_k}(B_k \cdot \tilde{B}_k)}|0\rangle\rangle, \tag{21}$$

where the coefficients of various generators are given by the relations

$$\Gamma_{1_k} = \frac{-\lambda_{1_k}\sinh(i\Lambda_k)}{\Lambda_k \cosh(i\Lambda_k) + \lambda_{3_k}\sinh(i\Lambda_k)},$$

$$\Gamma_{2_k} = \frac{\lambda_{2_k}\sinh(i\Lambda_k)}{\Lambda_k \cosh(i\Lambda_k) + \lambda_{3_k}\sinh(i\Lambda_k)}, \tag{22}$$

$$\Gamma_{3_k} = \frac{\Lambda_k}{\Lambda_k \cosh(i\Lambda_k) + \lambda_{3_k}\sinh(i\Lambda_k)}, \tag{23}$$

and

$$\Lambda_k^2 \equiv (\lambda_{3_k}^2 + \lambda_{1_k}\lambda_{2_k}). \tag{24}$$

Since the vacuum at zero temeprature is annihilated by A_k^μ and $\tilde{A}_k^\mu$, the only contribution to the thermal vacuum is given by

$$|0(\theta)\rangle\rangle = \prod_k (\Gamma_{3_k})^{2\delta_{kk} tr\, \eta^{\mu\nu}} e^{\Gamma_{1_k}(\tilde{A}_k^\dagger \cdot A_k^\dagger)} e^{\Gamma_{1_k}(\tilde{B}_k^\dagger \cdot B_k^\dagger)}|0\rangle\rangle. \tag{25}$$

The thermal vacuum of the left-moving modes is constructed in the same way. The total vacuum at finite temperature is the direct product between the α and β vacua. The string operators are mapped to finite temperature by the corresponding Bogoliubov generators,

$$A_k^\mu(\theta) = e^{-iG_k^\alpha} A_k^\mu e^{iG_k^\alpha}, \qquad \tilde{A}_k^\mu(\theta) = e^{-iG_k^\alpha} \tilde{A}_k^\mu e^{iG_k^\alpha},$$

$$B_k^\mu(\theta) = e^{-iG_k^\beta} B_k^\mu e^{iG_k^\beta}, \qquad \tilde{B}_k^\mu(\theta) = e^{-iG_k^\beta} \tilde{B}_k^\mu e^{iG_k^\beta}. \tag{26}$$

Similar relations hold for the creation operators. One can easily show that the thermal operators satisfy the same canonical commutation relations as the operators at zero temperature. Alternatively, one can organize the operators in thermal doublets [47,48] and obtain the thermal operators by acting on the doublet with $\mathcal{B}$ matrix

$$\begin{pmatrix} A_k^{\mu}(\theta) \\ \tilde{A}_k^{\mu\dagger}(\theta) \end{pmatrix} = \mathcal{B}_k \begin{pmatrix} A_k^{\mu} \\ \tilde{A}_k^{\mu\dagger} \end{pmatrix}, \tag{27}$$

where the explicit form of $\mathcal{B}_k$ operators is given by the following relation:

$$\mathcal{B}_k = \cosh(i\Lambda_k)\mathbb{I} + \frac{\sinh(i\Lambda_k)}{(i\Lambda_k)} \begin{pmatrix} i\lambda_{3_k} & i\lambda_{1_k} \\ i\lambda_{2_k} & -i\lambda_{3_k} \end{pmatrix}, \tag{28}$$

where $\mathbb{I}$ is the identity matrix. With all these elements at hand, one can construct a solution of the string equations of motion with periodic boundary conditions at finite temperature. This reduces to replacing the string operators by thermal string operators. Since the later satisfy the usual canonical commutations, the equations of motion and its solution satisfy the Virasoro algebra with the following thermal generators

$$L_m^{\alpha}(\theta) = \frac{1}{2} \sum_{k \in \mathcal{Z}} \alpha_{-k}(\theta)\alpha_{k+m}(\theta),$$

$$L_m^{\beta}(\theta) = \frac{1}{2} \sum_{k \in \mathcal{Z}} \beta_{-k}(\theta)\beta_{k+m}(\theta), \tag{29}$$

which guarantees that we are working with thermal strings [32,33].

C. Entropy of thermal string

The entropy operator is defined such that its average value is proportional to the entropy of the bosonic field at thermal equilibrium divided by the Boltzmann's constant [36]. From Eq. (1), the entropy of the bosonic field can be computed as the expectation value of the entropy operator in the thermal vacuum

$$\frac{1}{k_B}\langle\langle 0(\theta)|K|0(\theta)\rangle\rangle = \left\{ \sum_k [(1+n_k)\log(1+n_k) - n_k \log(n_k)] \right\}, \tag{30}$$

where n_k is the density of particles. Consequently, one defines the entropy operator for the bosonic string as

$$K = K^{\alpha} + K^{\beta}, \tag{31}$$

where the entropies of the right- and left-moving modes are given by the following relations:

$$K^{\alpha} = -\sum_k \left[A_k^{\dagger} \cdot A_k \log\left(g \frac{\lambda_{1_k}\lambda_{2_k}}{\Lambda_k^2} \sinh^2(i\Lambda_k) \right) - A_k \cdot A_k^{\dagger} \log\left(1 + g \frac{\lambda_{1_k}\lambda_{2_k}}{\Lambda_k^2} \sinh^2(i\Lambda_k) \right) \right], \tag{32}$$

$$K^{\beta} = -\sum_k \left[B_k^{\dagger} \cdot B_k \log\left(g \frac{\lambda_{1_k}\lambda_{2_k}}{\Lambda_k^2} \sinh^2(i\Lambda_k) \right) - B_k \cdot B_k^{\dagger} \log\left(1 + g \frac{\lambda_{1_k}\lambda_{2_k}}{\Lambda_k^2} \sinh^2(i\Lambda_k) \right) \right]. \tag{33}$$

With this definition of the entropy operator one recovers the entropy of [33] in the case when there is a single transformation generated by g_{2_k}, that is when $\theta_{1_k} = \theta_{3_\kappa} = 0$. Also, this choice gives the operator of [36] when $g = 1$. One obtains the entropy of the bosonic closed string as the average of Eq. (31) in the thermal vacuum,

$$S = k_B \langle\langle 0(\theta)|K|0(\theta)\rangle\rangle = 2k_B \sum_k [(g+n_k)\log(1+n_k) - n_k \log(n_k)], \tag{34}$$

where

$$n_k = g\left[\frac{\lambda_{1_k}\lambda_{2_k}}{\Lambda_k^2} \sinh^2(i\Lambda_k) \right], \tag{35}$$

and

$$g = \langle\langle 0|\tilde{A}_k \cdot \tilde{A}_k^{\dagger}|0\rangle\rangle = \langle\langle 0|\tilde{B}_k \cdot \tilde{B}_k^{\dagger}|0\rangle\rangle. \tag{36}$$

Note that the entropy operator for the bosonic closed string has been constructed as the sum between the right- and the left-moving modes treated as two independent subsystems of the string. Thus the entropy operator obeys the extensivity property of the physical quantity. Also, from Eq. (34), we see that entropy of the bosonic closed string goes to zero when the system is in equilibrium, that is

$$n_k = \frac{e^{-(k_B T)^{-1}\omega}}{1 - e^{-(k_B T)^{-1}\omega}}, \tag{37}$$

and take the limit $T \to 0$. This guarantees that third principle of the Thermodynamics is satisfied [52].

III. BOUNDARY STATES UNDER $SU(1,1)$ TRANSFORMATION

At $T=0$, a rigid Dp-brane is located along the $\{X^a\}$ directions in the target space, $a = 1,2, \ldots, p$ at $\{X^i = y^i\}$, where $i = p+1, \ldots, 24$ in the presence of the constant Kalb-Ramond field F_{ab}. The Dp-brane is described by a superposition of coherent boundary states in the Fock space defined by the boundary conditions in the closed string sector. In order to find the thermal Dp-brane, one has to write down the boundary conditions at finite temperature. This can be achieved by interpreting the string coordinates and their derivatives as operators and then by acting on them with the Bogoliubov transformation. In the case when one considers just a single Bogoliubov generator [32,33], the boundary conditions at finite temperature admit solution. In this section we are going to investigate the case when the thermal transformation is generated by the general $SU(1,1)$ generator, i.e., by the Bogoliubov operator (14).

A. Thermal boundary conditions and boundary states

The boundary conditions in the closed string sector at $T=0$ are given by the following relations:

$$(\partial_\tau X_a(\tau,\sigma)+F_{ba}\partial_\sigma X^b(\tau,\sigma))|_{\tau=0}=0, \tag{38}$$

$$X^i(\tau,\sigma)-y^i|_{\tau=0}=0. \tag{39}$$

One can obtain the corresponding relations for the second copy of the string from Eq. (39) by replacing the string coordinates by the coordinates of the tilde string. In order to find the boundary conditions at finite temperature, one substitutes the general solution of the equations of motion in Eq. (39) and applies the general Bogoliubov (14):

$$X^\mu(\theta)=e^{-iG}X^\mu e^{iG}. \tag{40}$$

The thermal boundary conditions in terms of string operators at finite temperature represent a set of constraints on the Fock space of the extended system of the following form [32,33]:

$$\begin{aligned}
&[(\mathbb{I}+\hat{\mathbb{F}})^a_b A^b_n(\theta)+(\mathbb{I}+\hat{\mathbb{F}})^a_b B^{b\dagger}_n(\theta)]|B(\theta)\rangle\rangle=0,\\
&[(\mathbb{I}+\hat{\mathbb{F}})^a_b A^{b\dagger}_n(\theta)+(\mathbb{I}+\hat{\mathbb{F}})^a_b B^{b}_n(\theta)]|B(\theta)\rangle\rangle=0,\\
&[A^i_n(\theta)-B^{i\dagger}_n(\theta)]|B(\theta)\rangle\rangle=0,\\
&[A^{i\dagger}_n(\theta)-B^{i}_n(\theta)]|B(\theta)\rangle\rangle=0,
\end{aligned} \tag{41}$$

for any $n>0$. The coordinates of the center of mass and their conjugate momenta do not transform under the Bogoliubov transformation, neither does the constant Kalb-Ramond field [32,33]. Then, we have to add to the set (41) the following relations:

$$\hat{p}^a|B(\theta)\rangle\rangle=[\hat{q}^i-y^i]|B(\theta)\rangle\rangle=0. \tag{42}$$

Similar boundary conditions should be imposed for the tilde string and the thermal Dp-brane $|B(\theta)\rangle\rangle$ must satisfy the two of them. It is easy to see that a general solution of the equations (41) and (42) has the following form

$$|B(\theta)\rangle\rangle_1=N_p^2(F,\theta)\,\delta^{(d_\perp)}(\hat{q}-y)\,\delta^{(d_\perp)}(\tilde{\hat{q}}-\tilde{y})\times e^{-\sum_{n=1}^{\infty}A^\dagger_n(\theta)\cdot M\cdot B^\dagger_n(\theta)}e^{-\sum_{n=1}^{\infty}\tilde{A}^\dagger_n(\theta)\cdot M\cdot\tilde{B}^\dagger_n(\theta)}|0(\theta)\rangle\rangle, \tag{43}$$

where

$$M^\mu_\nu=\left[\left(\frac{\mathbb{I}-\hat{\mathbb{F}}}{\mathbb{I}+\hat{\mathbb{F}}}\right)^a_b;-\delta^i_j\right] \tag{44}$$

and $N_p(F,\theta)$ is the thermal normalization constant equal to the one of the tilde system and identical to the normalization constant at $T=0$ [33]

$$N_p(F,\theta)=N_p(F)=\sqrt{-\det(\delta+2\pi\alpha' F)}. \tag{45}$$

Note that Eq. (45) corresponds to the first solution found in [33]. We interpret this solution as describing a thermal Dp-brane and postpone to Sec. IV the discussion of the degeneracy of thermal brane in this case.

B. The entropy of thermal *D*-branes

The thermal D-brane given by Eq. (43) represents a superposition of coherent states in the Fock space of the thermal string. Therefore calculating the entropy of the D-brane is equivalent to computing the average value of the entropy operator (31) in the state (43). One way of doing that is by expressing all the operators and states in terms of operators and states at $T\neq 0$. To this end we need the inverse of the Bogoliubov matrix (28) which has the following form

$$\mathcal{B}_k^{-1}=\cosh(i\Lambda_k)\mathbb{I}-\frac{\sinh(i\Lambda_k)}{(i\Lambda_k)}\begin{pmatrix} i\lambda_{3_k} & i\lambda_{1_k}\\ i\lambda_{2_k} & -i\lambda_{3_k}\end{pmatrix}. \tag{46}$$

By using Eq. (46), we can write the entropy operator K^α in terms of the thermal operators

$$\begin{aligned}
A^\mu_k&=\left[\cosh(i\Lambda_k)-\frac{\sinh(i\Lambda_k)}{\Lambda_k}\lambda_{3k}\right]A^\mu_k(\theta)-\frac{\sinh(i\Lambda_k)}{\Lambda_k}\lambda_{1k}\tilde{A}^{\mu\dagger}_k(\theta),\\
A^{\mu\dagger}_k&=\left[\cosh(i\Lambda_k)-\frac{\sinh(i\Lambda_k)}{\Lambda_k}\lambda_{3k}\right]A^{\mu\dagger}_k(\theta)-\frac{\sinh(i\Lambda_k)}{\Lambda_k}\lambda_{2k}\tilde{A}^{\mu}_k(\theta).
\end{aligned} \tag{47}$$

Then, the entropy in the right-moving sector has the following form:

$$\langle\langle 0(\theta)|K|0(\theta)\rangle\rangle=-\sum_k\left\{[1+2n_k]\mathcal{A}_k\log\left(\frac{n_k}{1+n_k}\right)+n_k\log(n_k)-(g+n_k)\log(1+n_k)\right\}, \tag{48}$$

where

$$n_k=g\frac{\lambda_{1k}\lambda_{2k}}{\Lambda_k}\sinh^2(i\Lambda_k), \tag{49}$$

as in Eq. (35), and

$$\mathcal{A}_k=\left\langle\left\langle B(\theta)\left|\sum_{\mu=1}^{24}N^\mu_k(\theta)\right|B(\theta)\right\rangle\right\rangle. \tag{50}$$

In order to calculate the action of the thermal number operator, we expand the exponentials in the coherent state. Since all the operators are at nonzero temperature, we ignore the θ symbol in the notation. For example, by expanding the nontilde part of the coherent state we obtain the following expression for the $\mathcal{A}_k$:

$$B^2\tilde{B}^2\left\langle\left\langle 0(\theta)\right| e^{\tilde{S}^\dagger}e^{\tilde{S}}\prod_{n=1}^{\infty}\prod_{\mu}\prod_{\nu}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}(A_n^\mu M_{\mu\nu}B_n^\nu)^l \right.$$
$$\left.\times\left|N_m^\alpha\right|\prod_{k}^{\infty}\prod_{\rho}\prod_{\sigma}\sum_{s=0}^{\infty}\frac{(-)^s}{s!}(A_k^{\rho\dagger}M_{\rho\sigma}B_k^{\sigma\dagger})^s\left|0(\theta)\right\rangle\right\rangle, \tag{51}$$

where $\tilde{S}$ represents the exponential operator for the thermal tilde part of the brane state and $B=N_p(F,\theta)\delta^{(d_\perp)}(\hat{q}-y)$. By expanding the products in Eq. (51), one is left with the expression of $\mathcal{A}_m$ in terms of states that describe the number of excitations of string in each direction of space-time and for each oscillation mode. These states are orthogonal and of unit norm and after some short but tedious algebra one can show that Eq. (50) has the following form:

$$B^2\tilde{B}^2\sum_{t_1^{1,1}\cdots t_\nu^{24,24}}\sum_{s_1^{1,1}\cdots s_\nu^{24,24}}\sum_{\rho}\sum_{\sigma}(M_{24,24})^{2t_1^{24,24}}\cdots$$
$$\times(M_{1,1})^{2t_\nu^{1,1}}(M_{24,24})^{2s_1^{24,24}}\cdots(M_{1,1})^{2s_\nu^{1,1}}s_m^{\rho,\sigma}. \tag{52}$$

Here, $s_i^{\rho,\sigma}=0,1,\ldots,\infty$, $i=1,2,\cdots,n$, $n\to\infty$, and $\rho,\sigma=1,\ldots,24$ represent the indices for all excitation s of all frequencies n and in all space-time directions. The expression (52) is not normalized. It contains the full dependence on the Kalb-Ramond field in the entropy (48). The temperature dependence of entropy is contained only in the n_k terms, more exactly in λ's. However, it is not always possible to write down the explicit form of λ's as functions of temperature even in the case of a single generator. (The relation leading to this function is obtained by equating the corresponding coefficient in the Bogoliubov transformation to the statistical distribution in the thermal vacuum [36].) In this case, approximation or numerical methods should be used. The total entropy of the *D*-brane is obtained as twice the Eq. (48) since the left modes contribute with the same amount to it.

IV. FINAL DISCUSSIONS AND CONCLUSIONS

To conclude, we have analyzed the $SU(1,1)$ thermal group formed by all possible unitary thermal Bogoliubov generators in the case of the bosonic string and *D*-brane. The reason for this analysis is that we have constructed the thermal brane and string in the framework of TFD where $SU(1,1)$ represents the most general structure underlying the Bogoliubov transformations. By choosing a certain type of parameters θ the most general Bogoliubov transformation can be fixed to be unitary or nonunitary. We have chosen a unitary trasformation in order to preserve the structure of the Hilbert space at zero temperature and the usual interpretation of quantum mechanics. However, the tilde invariance of the thermal vacuum of the theory is lost with this choice, which is an undesirable feature in TFD. The solution is to construct the thermal vacuum as the direct product between the original thermal vacuum, i.e., the vacuum obtained by acting with the Bogoliubov transformation on the vacuum at zero temperature and its conjugate under the tilde operation. In this case, the thermal vacuum factorizes in a tilde and a nontilde part as does the vacuum at zero temperature. By using a nonunitary transformation, one would have lost states from the Fock space and the isomorphism between the Fock space and its dual conjugate. The normalization constant of the *D*-brane state would have changed and we would have expected that different bra coherent states satisfied the boundary conditions. With two nonisomorphic bra and ket states it would not have been possible to take the average of the entropy operator in one *D*-brane state. Note that the general Bogoliubov transformation will be simultaneously unitary and generated by a tilde invariant Bogoliubov operator, implying that two generators of $SU(1,1)$ do not appear in it. Therefore we might conclude that unless the thermal vacuum is the product of the original thermal vacuum with its tilde conjugate, there is no $SU(1,1)$ thermal group whose general transformation will be compatible with the unitarity of quantum mechanics and the tilde invariance of TFD.

We have obtained the entropy of the closed string in Eq. (34) and the entropy of the bosonic *D*-brane as twice the entropy of the right-modes (48). An analysis of this expression in various temperature limits should be performed in an approximation or a numerical scheme for the θ's. However, if one supposes that the θ's are monotonous functions at least at low temperatures, then, by taking $\lambda_{ik}\sim\epsilon$ for $i=1,2,3$, we see that as $\epsilon\to 0$ the entropy goes as

$$\sum_m(\mathcal{A}_m+\delta_{mm}\mathrm{Tr}\,\delta^{\mu\nu}),$$

which in a normalized theory should be a finite constant. This represents an improved entropy at low temperatures compared with the entropy given in [34] which is divergent in that limit.

Note that the methods used for computing the *D*-brane entropy in [34] and in the present paper are slightly different from the standard TFD. Indeed, since the *D*-branes are states in the Fock space of the bosonic string vacuum different from the vacuum state of the conformal field theory, the entropy of *D*-brane was identified with the entropy of the closed string in this coherent boundary state. A standard TFD would require one to identify the very *D*-brane with some vacuum state in a field theory, but no such approach to *D*-branes is known at present. Also, as noted in [35], our approach to thermal strings is different from the one in literature in which understanding the ideal case of the bosonic strings and the string field theory were the main motivations to implement TFD in ensembles of strings and in string field theory [53–58], much as was done for the standard field theory. In our approach, due to the interpretation of brane, the TFD has been applied to the bosonic vacuum of string theory, i.e., to the two-dimensional conformal field theory describing it, rather than to ensembles of strings or string fields. Therefore we have been working with the thermal bosonic vacuum and its thermal fluctuations some of which are thermal *D*-branes.

Let us end by observing that we have considered just a single solution of *D*-brane type and not all solutions obtained in [32,34]. The reason for that is that the other solutions appear when the vacuum is not invariant under tilde operation. If we impose the tilde invariance as is done in TFD, the degeneracy of *D*-brane solutions should be removed.

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