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Transverse diffeomorphisms and spin-2 particles

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Transverse diffeomorphisms and spin-2 particles

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Orientador: Prof° Dr. Denis Dalmazi

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ESTA DISSERTAÇÃO FOI JULGADA ADEQUADA PARA A OBTENÇÃO DO TÍTULO DE
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“Tu te tornas eternamente responsável por aquilo que cativas.“

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RESUMO

Nesta dissertação, estudamos o modelo TDiff, assim chamado por ser invariante sob difeomorfismos transversais, que descreve a propagação de modos de spin-2 e spin-0. Exploramos a dimensão reducional a la Kaluza-Klein para obter, a partir do modelo TDiff em $D + 1$ dimensões, um modelo massivo D-dimensional. Esta extensão massiva é uma descrição escalar-tensorial cujo limite de massa nula é invariante sob difeomorfismos (Diff). Em seguida, generalizamos a discussão para métricas de fundo curvas e concluímos que espaços de Einstein são sempre exigidos para descrever de modo consistente modelos escalares-tensoriais linearizados, e que é possível obter modelos cujos limites de massa nula são invariantes apenas sob TDiff devido à presença de termos de curvatura não mínimos.

PALAVRAS-CHAVE: TDiff. Modelos escalar-tensoriais. Espaços de Einstein.

ABSTRACT

In this thesis, we study the so-called linearized TDiff model, which is invariant under transverse diffeomorphisms and describes spin-2 and spin-0 propagation modes. We explore Kaluza-Klein dimensional reduction to obtain a D-dimensional massive model from TDiff in $D+1$ dimensions. This massive extension is a scalar-tensor description propagating massive spin-2 and spin-0 modes, and whose massless limit is invariant under diffeomorphisms (Diff). Then, we generalize the discussion to curved backgrounds, and conclude that Einstein spaces are always required to consistently describe linearized scalar-tensor models and it is possible to obtain models whose massless limit is invariant only under TDiff due to the presence of non-minimal curvature terms.

KEYWORDS: TDiff. Scalar-tensor model. Einstein spaces.

SUMÁRIO

1 INTRODUCTION AND THEORETICAL CONSIDERATIONS

The main purpose of this work is the study of the TDiff model (ALVAREZ et al., 2006) - a spin-2 and spin-0 massless model whose action is invariant under transverse diffeomorphisms - and its massive version obtained after a truncated Kaluza-Klein dimensional reduction. The Masters project can be basically divided into two parts. The first one deals with TDiff in the flat Minkowski spacetime, whereas the second one is an extension to curved backgrounds. This thesis is organized as follows.

In this chapter, we introduce the project by establishing the nomenclature, presenting our motivations, and contextualizing our work in the current scenario of alternative formulations of gravity. In chapter 2, we classically analyse the TDiff model by exploring its gauge symmetry properties and verifying the field equations content via Hamiltonian analysis through the Dirac-Bergmann algorithm for constrained systems. We also present the helicity variables decomposition, a useful tool to decouple different sectors of a theory.

Next, in chapter 3, we analyse a massive extension of TDiff obtained through a Kaluza-Klein dimensional reduction. We analyse the Hamiltonian and investigate the massless limit. This is an original work. The first part of the thesis ends in chapter 4 with the (quantum) unitarity analyses.

In the second part of the study, we deal with curved space backgrounds. In chapter 5, we review the work of (BUCHBINDER; KRYKHTIN; PERSHIN, 1999) verifying the consistency of spin-2 massive propagation in arbitrary background manifolds. In chapter 6, we study linearized scalar-tensor theories in those backgrounds, intending to give an answer to the question of whether massive models always support a generalized Diff symmetry in the massless limit. We will show that this is true only for the flat case. This chapter is also original.

Finally, after exposing our conclusions and references, there is an Appendix. Here the reader can find useful information for a proper understanding of the passages throughout the thesis. It includes brief reviews on constrained systems, Kaluza-Klein dimensional reduction, spin-1 and 2 projection operators, differential geometry, as well as some mathematical identities.

This work has been influenced by the references (ALVAREZ et al., 2006; BUCHBINDER; KRYKHTIN; PERSHIN, 1999; HINTERBICHLER, 2012; BONIFACIO; FERREIRA; HINTERBICHLER, 2015). We have chosen to write this Master thesis in English, since this provides an opportunity to practice scientific writing, an important ability for a research career. We apologize to the reader for any language mistakes.

1.1 THE TDIFF MODEL

Before presenting our motivations, let us clarify what is a TDiff model and establish the difference between the names TDiff, Diff, WTDiff and generalized Diff, that will often appear throughout this work. We are closely following the notation of (ALVAREZ et al., 2006).

A general, local, Lorentz-invariant Lagrangian density up to two derivatives with a rank-2 symmetric tensor $h_{(\mu\nu)}$ and without mass is given by

$$\mathcal{L}(1,\beta,a,b) = -\frac{1}{4}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{\beta}{2}\partial^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} - \frac{a}{2}\,\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h\partial^\mu h, \tag{1.1}$$

where we have already normalized the first term and our convention is $\eta_{\mu\nu} = diag(-1,1,...,1)$, as well as $\hbar = 1 = c$. One can show that elimination of spin-1 propagation demands $\beta = 1$, proved in the chapter 4, see (4.8). Furthermore, in this case, for any $a, b \in \mathbb{R}$, $\mathcal{L}(1,1,a,b)$ is invariant under:

$$\delta h_{\mu\nu} = \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu, \tag{1.2}$$

where $\partial^\mu \xi^T_\mu = 0$, i.e. the vector parameter is transverse. Thus, unless there is a special relation between a and b, we say that the family of Lagrangians $\mathcal{L}(1,1,a,b)$ is invariant under **transverse diffeormorphisms**, or simply **TDiff**, a local gauge symmetry.

We can enhance this symmetry by allowing a and b to assume some special values. For example, we can show that $\mathcal{L}(1,1,1,1)$ is invariant under

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{1.3}$$

the linearized **diffeomorphisms** from the general relativity (GR). For this reason, we call such a model as **Diff** invariant or **LEH**, acronym for linearized Einstein-Hilbert, so $\mathcal{L}(1,1,1,1) = \mathcal{L}_{LEH}$. Through the Weyl field redefinition $h_{\mu\nu} \to \hat{h}_{\mu\nu} = h_{\mu\nu} + \lambda\, h\eta_{\mu\nu}$, for $\lambda \neq -1/D$, we obtain a related family $\mathcal{L}(1,1,A,B)$, where A and B are given by

$$A = a + \lambda(D\,a - 2), \tag{1.4}$$
$$B = b + 2\lambda(D\,b - a - 1) + \lambda^2[D^2 b - D(2a+1) + 2]. \tag{1.5}$$

By doing this, from $\mathcal{L}(1,1,a=1,b=1)$, we generate the **Diff family**, i.e, all the Lagrangians $\mathcal{L}(1,1,A,B)$ equivalent to LEH modulo Weyl field redefinitions and which are invariant under **generalized Diff**

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2\lambda \eta_{\mu\nu}\partial_\alpha \xi^\alpha. \tag{1.6}$$

If we set $a = 1 = b$ in (1.4) and (1.5), we can eliminate λ from (1.4) and plug it in (1.5). This leads us to $1 - 2A + A^2(D-1) - B(D-2) = 0$. Thus, let us define an important quantity:

$$d_0 \equiv 1 - 2a + a^2(D-1) - b(D-2). \tag{1.7}$$

It turns out that all models $\mathcal{L}(1,1,a,b)$ which belong to the Diff family must satisfy $d_0 = 0$. However, the opposite is not true since, see (1.4), if $a = \dfrac{2}{D}$, we cannot change the coefficient a via that redefinition (likewise, $A = \dfrac{2}{D}$ cannot be achieved since it would require $\lambda = -1/D$), but we can still have $d_0 = 0$ if $b = \dfrac{D+2}{D^2}$. This is called the **WTDiff** model, invariant under TDiff and Weyl transformations:

$$\delta h_{\mu\nu} = \frac{2}{D}\phi\eta_{\mu\nu}. \tag{1.8}$$

The model $\mathcal{L}\left(1, 1, \frac{2}{D}, \frac{D+2}{D^2}\right)$ corresponds to a linearized version of unimodular gravity (BONIFACIO; FERREIRA; HINTERBICHLER, 2015). The Diff and WTDiff models are the only two possible symmetry extensions of the TDiff models.

Since LEH and WTDiff are also invariant under TDiff, in order to separate those cases from the models only invariant under (1.2), we establish our notation for the rest of this work. By **TDiff** we mean

$$\mathcal{L}_{TDiff} = -\frac{1}{4}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{1}{2}\partial^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} - \frac{a}{2}\,\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h\partial^\mu h, \tag{1.9}$$

such that $d_0 \neq 0$. In summary, once we have a Lagrangian $\mathcal{L}(1, 1, a, b)$ we first check if $a = \frac{2}{D}$ or $a \neq \frac{2}{D}$. In the first case, we can either have a WTDiff model, if $d_0 = 0$, or a TDiff one otherwise. In the second case, if $d_0 = 0$ we have a Diff model, otherwise a TDiff one. We point out that, although we have not taken advantage of, we can always treat TDiff as an **one-parameter** Lagrangian since, without loss of generality, we can always transform $a \neq \frac{2}{D} \to a = 1$, so that TDiff can be worked with either $a = 1$ or $a = \frac{2}{D}$, always bearing in mind that $d_0 \neq 0$. One of the objectives of this work is to show that if $d_0 > 0$, then the TDiff model describes 3 healthy degrees of freedom, associated with spin-2 and spin-0 massless particles.

Both Diff and WTDiff models were already carefully discussed in (LINO DOS SANTOS, 2018), including their Hamiltonian analysis and their massive extensions obtained via Kaluza-Klein (KK) dimensional reductions, basically following (BONIFACIO; FERREIRA; HINTERBICHLER, 2015; BONIFACIO, 2017; SANTOS, Unpublished; KHOUDEIR; MONTEMAYOR; URRUTIA, 2008). These models describe spin-2 massless particles. We can add a specific mass term to LEH, leading to the Fierz-Pauli (FP) model (FIERZ; PAULI, 1939; HINTERBICHLER, 2012), but there is no such extension for WTDiff (ALVAREZ et al., 2006). In fact, the KK dimensional reductions of LEH and WTDiff, from $D + 1$ dimensions, both lead to FP, in D, confirming the known fact that FP is the paradigmatic model for massive spin-2 particles, when using a symmetric rank-2 tensor.

We have been wondering if a consistent massive TDiff model could be obtained after a KK dimensional reduction. The answer is negative since the resulting scalar-tensor massive model is invariant under generalized Diff when $m = 0$. Such demonstration is also one of the goals of this work.

Finally, we analyze the consistency of linearized TDiff models in the presence of an external gravitational field, by using a metric describing fluctuations around a curved space background. We prove that Einstein spaces are required to ensure invariance under TDiff symmetry, as it is the case for linearized Diff (BUCHBINDER; KRYKHTIN; PERSHIN, 1999; DESER; HENNEAUX, 2007).

In the next subsections we present some motivations and contextualize our work. Since TDiff is a modification of Diff, it can also be understood as one of the multiple possible modifications of gravity[1]. Consequently, it makes sense to present the current scenario concerning alternatives to the Einstein

[1] In this thesis, we are not considering TDiff in its non-linear version. See, for example, (ALVAREZ et al., 2006; LOPEZ-VILLAREJO, 2011; CRISTÓBAL, 2014) for extensions.

General Relativity. However, our target here is less appealing: it is to understand, at a classical field theory level, how the aforementioned TDiff invariance affects the spectrum of particles[2]. In addition, TDiff is the minimal symmetry to have a Lorentz invariant S-matrix describing process with spin-2 particles (BIJ; DAM; NG, 1982; HERRERO-VALEA, 2018). From this point of view, the reader may then consider this work as an application of spin-2 classical field theories.

1.2 A PRELUDE TO GRAVITY

General relativity (GR) can be understood as a theory that unifies geometry and gravity. The presence of matter, or energy, affects the structure of the spacetime, curving it. Consequently, we do not have that rigid structure which is the Minkowski space of Special Relativity. Mathematically, the spacetime is defined as a differentiable pseudo-Riemannian manifold. The spacetime information can be stored in a metric field, represented by a rank-2 tensor whose components in some system of coordinates read $g_{\mu\nu}$, and it is possible to express the components of fundamental geometrical quantities like the Riemann or Ricci tensors in terms of $g_{\mu\nu}$ and its derivatives. The metric field describes not only the metric properties of space and time, but also causality properties and contains information about the gravitational field. Having said this, it is clear that differential geometry is an important tool for general relativity, however, in this Master thesis we will restrict ourselves to a minimum presented in the Appendix.

On the other hand, from a field theory perspective, GR can be understood as a field theory whose interaction - gravity - is mediated by a massless spin-2 particle - graviton. In this approach, one often starts with the Einstein-Hilbert (EH) action

$$S_{GR} = \int d^4x \, \sqrt{-g} \left[\frac{c^4}{16\pi G}(R + 2\Lambda) \right] + S_M. \tag{1.10}$$

By applying the variational principle, we obtain the Einstein equations with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.11}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci (curvature) scalar and $T_{\mu\nu}$ is the energy-momentum tensor, which is given by $-\dfrac{2}{\sqrt{-g}} \dfrac{\delta S_M}{\delta g_{\mu\nu}}$, where S_M is the matter action.

These field equations constitute a set of non-linear partial differential equations which make extremely difficult to analyse their solutions, usually found using approximation methods or studying only few special exact cases. Even though improvements have been done with the study of the initial value problem (the Cauchy problem in GR) and numerical methods, important to the description of strong gravity regimes like black holes binaries and neutron stars mergers, recently detected with gravitational waves (ABBOTT et al., 2016; ABBOTT et al., 2017), it makes sense to investigate weak field radiative solutions of GR. By doing so, we can study the following aspects of GR:

[2] In this text, we will be incaustioulsy using the term particle to refer to the fact that when our free models are quantized, they are able to describe free particles propagating the correct number of degrees of freedom.

- learn how gravitational radiation propagates through space (gravitational waves): unless we are interested in strong gravity regimes, or in regions close to the sources, any gravitational radiation is of low intensity.
- understand how the concept of a particle (graviton) emerges: the precise meaning of an elementary particle involves the notion of irreducible representations of the Poincaré group. It requires the study of free weak-field equations.

In this Master thesis, we are exploring the second aspect. We can obtain the linear limit of GR by expanding the metric $g_{\mu\nu}$ around a background metric $\hat{g}_{\mu\nu}$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu}, \tag{1.12}$$

obtaining a quadratic model for the perturbation $h_{\mu\nu}$. See details, for example, in (WEINBERG, 2013). If we assume that the background is a Minkowski space $\eta_{\mu\nu}$, we will obtain the linearized Einstein-Hilbert (LEH) model given by

$$\mathcal{L}_{EH} = -\frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{4}\partial_\mu h \partial^\mu h. \tag{1.13}$$

Then, we can apply our knowledge of classical field theory, and conclude that it describes a massless spin-2 particle: it propagates two degrees of freedom, the ± 2 polarization modes associated with the description of the graviton. By modifying LEH, we mean changing these degrees of freedom (HINTERBICHLER, 2012). Consequently, if we believe that the modified linearized model is the truncation of some non-linear or geometrical model, we are also modifying gravity.

Einstein equations (1.11) are the fundamental equations of GR and among their famous solutions are the Friedmann-Lemaître-Robertson-Walker cosmological solution, black holes, gravitational waves, bending of light and the perihelion shift of Mercury. We can say that it is a successful theory: not only it works very well at the solar system level but also has been strongly supported by recent large scale observational data collected by gravitational waves detectors (ABBOTT et al., 2016). Taking this into account, does modifying gravity make any sense?

1.2.1 Why are we modifying gravity and how can we modify?

The efforts to extend gravity to a quantum regime and also to understand how our Universe fits into the standard cosmological model (today the ΛCDM model) have brought some conceptual and observational questions for GR. There are still theoretical and observational motivations for modifying GR (BULL et al., 2016; CLIFTON et al., 2012).

The search for a quantum theory of gravity has been posing a challenge for physicists. It is known that GR is not perturbative renormalizable (GOROFF; SAGNOTTI, 1986). Consequently, a quantum theory of GR cannot be established in all energy scales and be considered a fundamental theory since we would have to perform an infinite number of experiments to fix the infinite number of parameters introduced in the renormalization procedure. This means that we lose predictive power in the theory. The better we can do is to treat it as an effective field theory (valid up to a certain energy scale)

(WEINBERG, 1995). However, ultimately, it will be necessary to find a better theory if we want to be precise in the description of singularity phenomena where quantum effects come to light, like the surroundings of a black hole or even the Big Bang.

Needless to say that several approaches have been considered so far in order to find an adequate quantum description of gravity (ORITI, 2009). We can, for example, assume that GR must be modified (CLIFTON et al., 2012), or we can give up perturbative quantum field theory and consider fundamental theories such as string theory or loop quantum gravity. Asymptotic safety, on the other hand, proposes a different program where the existence of a non-trivial gravitational fixed point could assure predictivity for the theory even if it is not perturbative renormalizable (EICHHORN, 2019).

Moreover, a second question has come into play after the emergence of the dark universe. The standard model of cosmology (ΛCDM) requires the presence of dark energy and dark matter in huge amounts when compared to baryonic matter. In few words, dark matter is required to properly describe galaxies and clusters, whereas dark energy is needed due to the accelerated expansion of the universe (RIESS et al., 1998; PERLMUTTER et al., 1999). Among the conceptual issues are the "cosmological constant problem" (why is the observed value of the cosmological constant so small in Planck units?), the "coincidence problem" (why is the energy density of the cosmological constant so close to the present matter density?) (WEINBERG, 1989), and there are still unsolved or unresolvable problems on inflation (the theory of very early universe). In the observational side, we refer to production mechanisms (for example, dark-matter candidate particles not observed), observational inconsistencies (simulations and astronomy) (CLIFTON et al., 2012), and the recent and growing tension on the value of the Hubble parameter H_0 (narrowing down the value of H_0 has been problematic): Planck (cosmic microwave background data) $67.4 \pm 0.5 km/s/Mpc$ versus SHoES (astrophysical - supernova - data) $73.5 \pm 1.4 km/s/Mpc$ (MÖRTSELL; DHAWAN, 2018).

The fact that these dark constituents do need to be taken into account, but we do not have a proper comprehension about their origins has also been driving efforts to modify GR (BULL et al., 2016; CLIFTON et al., 2012). According to the Lovelock's theorem[3] (LOVELOCK, 1971; LOVELOCK, 1972), the only possible divergence-free second-order equation of motion $\mathcal{E}_{\mu\nu}$, obtainable in a four-dimensional manifold from a scalar density of the form $\mathcal{L} = \mathcal{L}[g_{\mu\nu}]$, is a linear combination of the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, and the metric tensor $g_{\mu\nu}$:

$$\mathcal{E}_{\mu\nu}[g] = a\, G_{\mu\nu} + b\, g_{\mu\nu}, \qquad (a, b) \in \mathbb{R}. \tag{1.14}$$

Hence, any alternative gravitational theory that can be derived from an action must differ from GR in at least one of the following items (CLIFTON et al., 2012):

- **Introduction of new field content** (tensor, vector and scalar fields besides the metric tensor or, as it is the purpose of massive gravity, mass);
- **Space-time dimension** $(D \neq 4)$;
- **Introduction of non-local terms**;

[3] It is not required to assume symmetry of $\mathcal{E}_{\mu\nu}$, even so the result is symmetric, and consequently does not couple with non-symmetric energy-momentum tensors $\mathcal{E}_{\mu\nu}[g] = T_{\mu\nu}$.

- **Higher-order equations** (higher than second order derivatives of the metric in the field equations)[4].

Furthermore, we could also give up requiring that gravity is derived from an action principle (emergent gravity), or that its field equation is divergence-free (which means the energy-momentum tensor would not satisfy $\nabla_\mu T^{\mu\nu} = 0$). This last possibility, in its linear version, is the case of the TDiff model and massive extensions.

About mass, (HINTERBICHLER, 2012) the massive linear extension of LEH - the standard massive spin-2 Fierz-Pauli (FP) model (FIERZ; PAULI, 1939)- cannot describe gravity because, even in the limit[5] $m \to 0$ (the well-known van Dam-Veltman-Zakharov (vDVZ) discontinuity (VELTMAN; DAM, 1970; ZAKHAROV, 1970)), the model fails at solar system tests (for example, Newtonian potential or bending of light beams). Introduction of non-linear terms could solve the problem (Vainshtein mechanism (VAINSHTEIN, 1972)) but it would add an extra ghost degree of freedom (Boulware-Deser ghost (BOULWARE; DESER, 1972)). The idea of a consistent massive gravity was then stagnated until its revitalization in the 2000s (see e.g. (HINTERBICHLER, 2012; BULL et al., 2016; RHAM, 2014) and their references), after the discovery of cosmic acceleration (SN Ia) (RIESS et al., 1998; PERLMUTTER et al., 1999), and then the interest has been increased over the last years. One of the advantages of a massive gravity would be the existence of self-accelerating solutions, offering a different perspective for the dark energy problem. This could also be achieved by, among other several approaches (BULL et al., 2016), scalar-tensor models, where the scalar dynamics would be responsible for the acceleration. This is basically the motivation for our work from a more empiric and observational viewpoint.

Bimetric theories give rise, in general, to one massive and one massless spin-2 field. The problem of an interacting spin-2 field theory is not new, see for example (ISHAM; SALAM; STRATHDEE, 1971; ARAGONE; DESER, 1971). It was shown that there are not consistent theories of interacting massless spin-2 fields (BOULANGER et al., 2001). Consequently, multiple interacting spin-2 field theories must include a massive graviton. This is one of the advantages of bimetric models. Nonetheless, they are also plagued with the same ghost problems of massive gravity theories. Hence, ghost-free massive gravity is also a promising starting point from which to develop a consistent theory of interacting spin-2 fields.

However, we would like to emphasize our theoretical motivation: to better understand why the graviton mass should be zero, it is natural to ask what implications a non-vanishing mass would have. In other words, even if we are unable to reach a consistent massive gravity, our understanding of GR will be increased.

[4] For field equations that contain higher than second order derivatives of $g_{\mu\nu}$, we have to specify for the Cauchy problem not only the initial values for the metric and its first derivative, but also its higher derivatives on a spacelike surface (STRAUMANN, 2013).

[5] Graviton mass bounds can be given by using different sources, such as galaxy clusters data (DESAI; GUPTA, 2020), $m_g \lesssim 10^{-30} eV$, or gravitational waves data (RHAM et al., 2017), $m_g \lesssim 10^{-23} eV$. Although gravitational waves data bounds are not as strict as previous measurements coming from other sources, the latter ones are largely more dependent model. For this reason, massive gravity articles often present GW150914 LIGO collaboration data as the graviton mass bound: $m_g < 1.2 \times 10^{-22} eV$ (ABBOTT et al., 2016).

In spite of the arguments in favour of a modification, let us remark that any modification of GR potentially harms the healthy attributes of the theory. In fact, there is a myriad of alternative formulations, but none can be claimed as better than GR (BERTI et al., 2015).

1.3 NON-LINEAR TDIFF AND CURVED BACKGROUNDS

In this work, we started the introduction presenting the most general massless rank-2 symmetric tensor up to second order in derivatives. We have discussed that there is a family of Lagrangians invariant under Diff, or generalized Diff, and WTDiff, both describing only spin-2 propagation. However, these are not the minimal symmetries to have such propagation. TDiff is the symmetry that plays such a role (BIJ; DAM; NG, 1982). It eliminates the undesirable spin-1 propagation even though carries an extra scalar degree of freedom, which turns out to be healthy under certain conditions to be explored in this work.

Let us make some comments on what does TDiff mean in a geometrical approach (CRISTÓBAL, 2014). Recall that GR action is invariant under general coordinate transformations and follows the principle of general covariance (WEINBERG, 2013). It is actually a driving principle when formulating a theory of gravity and several alternative formulations follow it. However, as the previous paragraph may suggest, Einstein equations could be obtained from a "smaller symmetry". Indeed, (BUCHMÜLLER; DRAGON, 1988) showed that it was possible to obtain the same content as GR restricting the coordinate transformations to a subgroup that preserves volume.

At least classically, the only difference between GR and this alternative formulation is that whereas in the former, the cosmological constant is a parameter of the Lagrangian, in the latter, it emerges as an integration constant! This formulation is closely related to the approach of unimodular theories of gravity where the determinant of the metric is chosen $|g| = 1$, seen as potential candidates to the cosmological constant problem (WEINBERG, 1989) and also explored as possible formulations for quantum gravity (PERCACCI, 2018). In this context, let us briefly discuss what TDiff and WTDiff are, since it seems that these names are often confounded in the literature.

Associated with general coordinate transformations is the group of general (full) diffeomorphisms $x^\mu \to x'^\mu$. Diffeomorphisms that comprise unimodular transformations, given by the following determinant

$$J \equiv \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| = 1, \tag{1.15}$$

are known as tranverse diffeomorphisms (TDiff). In order to understand this name, consider an infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu$. According to the mathematical identity $\log\det M = Tr(\log M)$, we have $\log J = \delta^\nu{}_\mu \log(\delta^\mu{}_\nu + \partial_\nu \epsilon^\mu) = \delta^\nu{}_\mu(\partial_\nu \epsilon^\mu) = \partial_\mu \epsilon^\mu$, at first order in ϵ. But $J = 1$ implies that $\partial_\mu \epsilon^\mu = 0$. Then, unimodular transformations are equivalent to TDiff

$$x'^\mu = x^\mu + \epsilon^\mu, \qquad \partial_\mu \epsilon^\mu = 0. \tag{1.16}$$

In addition, for general coordinate transformations we have

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \Rightarrow g'(x') = J^{-2} g(x) \tag{1.17}$$

Thus, for unimodular transformations, the determinant of the metric is a scalar $g^{'}(x^{'}) = g(x)$. Since $d^4x^{'}\sqrt{g^{'}} = d^4x\sqrt{g}$ is the invariant volume element, we can say that TDiff is a volume preserving diffeomorphism. Notice that unimodular transformations does not mean unimodular gravity, given by $|g| = 1$. Indeed, due to the fact that $g(x)$ is a scalar field, a non-linear TDiff model is a **scalar-tensor** theory, involving kinetic terms for the determinant of the metric. Let us call it TDiff-gravity.

On the other hand, a WTDiff non-linear theory is related to **unimodular gravity**, through the non-invertible redefinition $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = (-g)^{-\frac{1}{D}}g_{\mu\nu}$, such that $\bar{g} = -1$. It is consequently a volume preserving diffeomorphism, but noticing also that $\bar{g}_{\mu\nu}$ is invariant under Weyl transformations $g^{'}_{\mu\nu} = \phi(x)g_{\mu\nu}$, it means this theory is WTDiff invariant. Another possibility for a unimodular gravity theory is by gauge fixing $g = -1$ in the GR action. This let us with a residual gauge symmetry so that other $D - 1$ conditions can be imposed. It is also a volume preserving theory, but we emphasize it is actually different from TDiff-gravity. In TDiff, the metric is not constrained at all, while in this unimodular version the determinant is fixed. Additionally, in the linearized version, $g = -1$ would correspond to $h = 0$, which means we would have only 9 independent components for $h_{\mu\nu}$, not 10 as we have in (1.1).

In this thesis, we are not going to explore these non-linear possibilities, see for example (LOPEZ-VILLAREJO, 2011; CRISTÓBAL, 2014). However, as a first effort to study the interaction of scalar-tensor models with an external gravitational background, in the chapters 5 and 6 we have generalized our spin-2 linear field equations to a curved background, following (BUCHBINDER; KRYKHTIN; PERSHIN, 1999; BUCHBINDER; GITMAN; PERSHIN, 2000). The motivation behind this approach is the following: our TDiff and its massive extension models are free theories. It is natural to ask about how they react to introduction of interactions. Since every physical theory must interact with gravitation, it is also natural to look for which sorts of couplings TDiff, a free theory, can have to some gravitational background, which is encoded in the background metric $\bar{g}_{\mu\nu}$, where $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.

It can be instructive to review some ideas of (BUCHBINDER; KRYKHTIN; PERSHIN, 1999). There, the authors look for consistent theories of massive spin-2 fields interacting with an external gravity, investigating the fluctuations of $h_{\mu\nu}$ around the background. In such approach, the gravity background is fixed and it is not affected by the fluctuation field . In quantum gravity, this method is known as covariant quantization.

Within the framework of classical field theory, (BUCHBINDER; KRYKHTIN; PERSHIN, 1999) concluded that a ghost-free massive spin-2 field description, consistent with flat limit (3.1), requires the presence of non-minimal couplings in the action (quadratic in $h_{\mu\nu}$), and imposes restriction on the external background since it is only possible to construct a consistent classical action on a specific type of curved spacetime manifolds: Einstein spaces. This is basically the same result we have achieved in this Master thesis for the TDiff model. There exists another way to achieve consistent equations for spin-2 massive fields, but now for arbitrary gravitational backgrounds: by using an infinite series in curvature terms (R/M^2) (BUCHBINDER; KRYKHTIN; PERSHIN, 1999). We will not explore this remarkable possibility.

We finish this introduction writing a few words about recent non-linear results on bimetric and massive gravity (RHAM; GABADADZE; TOLLEY, 2011; HASSAN; ROSEN, 2012b; HASSAN; ROSEN, 2012a; BERNARD; DEFFAYET; STRAUSS, 2015; SCHMIDT-MAY; STRAUSS, 2016). It is necessary to add an additional reference metric $f_{\mu\nu}$ to construct a ghost-free non-linear generalization of Fierz-Pauli. In (RHAM; GABADADZE; TOLLEY, 2011), $f_{\mu\nu}$ was taken to be flat and non-dynamical in a ghost-free (HASSAN; ROSEN, 2012b) massive theory at complete non-linear level. Then, $f_{\mu\nu}$ was promoted to a full dynamical metric by (BERNARD; DEFFAYET; STRAUSS, 2015), leading to a ghost-free background independent massive theory invariant under general coordinate transformations.

2 CLASSICAL ANALYSIS

In this chapter we classically analyse the TDiff model given by

$$S_{TDiff} = \int d^D x \left[-\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{a}{2}\partial^\mu h \partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h \partial^\mu h \right], \tag{2.1}$$

where $d_0 \neq 0$, invariant under $\delta h_{\mu\nu} = \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu$. We will perform the Hamiltonian analysis and conclude that the TDiff model describes three propagating degrees of freedom, verifying for which values of a and b we have physical degrees of freedom in agreement with (ALVAREZ et al., 2006). We start with a few comments on gauge symmetry.

2.1 GAUGE SYMMETRY

In this section we make some comments about gauge symmetry in the Diff and TDiff cases. Assume $D = 3 + 1$ without loss of generality. Let us consider first the **Diff** case, given by the LEH Lagrangian

$$\mathcal{L}_{EH} = -\frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{4}\partial_\mu h \partial^\mu h, \tag{2.2}$$

invariant under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. Since the invariance depends on a 4-parameter vector, it allows us to fix four gauge conditions. For example, by fixing the harmonic (de Donder) gauge

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h \equiv 0, \tag{2.3}$$

we can obtain the D'Alembert wave equation $\Box h_{\mu\nu} = 0$. How about the number of independent degrees of freedom? We know it should be two, those corresponding to the two degrees of freedom of the graviton, a spin-2 massless particle. But how can we count degrees of freedom only by looking at the gauge invariance?

So far, from the initial 10 independent components of the symmetric rank-two tensor, we are left with only 6 independent ones since we have 4 equations (gauge conditions). Let us assume we have chosen the harmonic gauge and then transform $h_{\mu\nu}$ under Diff : $\delta h_{\mu\nu} = \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu$

$$\partial^\mu h^{'}_{\mu\nu} - \frac{1}{2}\partial_\nu h^{'} = \underbrace{\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h}_{\equiv 0} + \Box\Omega_\nu. \tag{2.4}$$

This means we are left with a **residual gauge symmetry**: $\partial^\mu h^{'}_{\mu\nu} - \frac{1}{2}\partial_\nu h^{'} = 0$ if $\Box\Omega_\nu = 0$, i.e. we can still choose four harmonic functions Ω_ν (that is why we call this gauge the harmonic one) and the equations of motion $\Box h_{\mu\nu} = 0$ remain the same. Notice that by choosing Ω_ν we get rid of four more independent components of $h^{'}_{\mu\nu}$. Consequently, we end up with $10 - 4 - 4 = 2$ independent degrees of freedom.

Let us now consider the **TDiff** case (2.1), invariant under $\delta h_{\mu\nu} = \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu$. By proceeding the same way, we could naively conclude that the model propagates $10 - 3 - 3 = 4$ degrees of freedom, however there is a further subtlety. The first difference to the Diff case is that now we are dealing with a 3-parameter transverse vector. As a consequence there is no first order in derivatives covariant expression to fix the gauge. However, we can impose the following transverse condition

$$\partial_\mu \partial^\alpha \partial^\beta h_{\alpha\beta} - \Box \partial^\nu h_{\mu\nu} \equiv 0. \tag{2.5}$$

After that, we can still perform a TDiff transformation $\delta h_{\mu\nu} = \partial_\mu \Psi^T_\nu + \partial_\nu \Psi^T_\mu$:

$$\partial_\mu \partial^\alpha \partial^\beta h'_{\alpha\beta} - \Box \partial^\nu h'_{\mu\nu} = \underbrace{\partial_\mu \partial^\alpha \partial^\beta h_{\alpha\beta} - \Box \partial^\nu h_{\mu\nu}}_{\equiv 0} - \Box\Box \Psi^T_\mu, \tag{2.6}$$

such that the residual symmetry is $\Box\Box\Psi^T_\mu = 0$. However, in the subspace of harmonic functions, Ψ^T_μ possess 4 degrees of freedom, since we can always write $\Psi^T_\mu = \tilde{\Psi}^T_\mu + \partial_\mu \phi$, where $\Box \tilde{\Psi}^T_\mu = 0 = \Box\phi$, and $\tilde{\Psi}^T_\mu$ is a genuinely transverse vector with 3 independent components. It means we end up with $10 - 3 - 4 = 3$ independent degrees of freedom.

2.2 HAMILTONIAN ANALYSIS: TDIFF

In this section we analyse the Hamiltonian of the TDiff model. We will achieve the same conclusions as (ALVAREZ et al., 2006), in which the authors analysed the physical content through a Lagrangian analysis. Our method, on the other hand, consists in applying the orthodox Dirac-Bergmann algorithm and analysing the positivity of the reduced Hamiltonian. For details about the method, consult the Appendix.

We write

$$\mathcal{L}_{TDiff} = -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{a}{2}\partial^\mu h \partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h \partial^\mu h \tag{2.7}$$

as

$$\begin{aligned}\mathcal{L}_{TDiff} = {} & \frac{(2a-b-1)}{4}\dot{h}_{00}\dot{h}_{00} + \frac{1}{4}\dot{h}_{ij}\dot{h}_{ij} + \frac{(b-a)}{2}\dot{h}_{00}\dot{h}_{ii} - \frac{b}{4}\dot{h}_{ii}\dot{h}_{jj} + (1-a)\dot{h}_{00}\partial_i h_{i0} - \dot{h}_{ij}\partial_i h_{0j} + \\ & + a\,\dot{h}_{ii}\partial_j h_{j0} + \frac{(b-1)}{4}\partial_i h_{00}\partial_i h_{00} + \frac{1}{2}\partial_i h_{j0}\partial_i h_{j0} - \frac{1}{2}\partial_i h_{i0}\partial_j h_{j0} - \frac{1}{4}\partial_i h_{kj}\partial_i h_{kj} + \frac{1}{2}\partial_i h_{ij}\partial_k h_{kj} + \\ & - \frac{b}{2}\partial_i h_{00}\partial_i h_{jj} + \frac{b}{4}\partial_i h_{jj}\partial_i h_{kk} + \frac{a}{2}\partial_i h_{00}\partial_j h_{ji} - \frac{a}{2}\partial_i h_{kk}\partial_j h_{ji},\end{aligned} \tag{2.8}$$

from which we obtain the canonical momenta:

$$\pi^{00} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = \frac{(2a-b-1)}{2}\dot{h}_{00} + \frac{(b-a)}{2}\dot{h}_{ii} + (1-a)\partial_i h_{i0}, \tag{2.9}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \frac{1}{2}\dot{h}_{ij} + \frac{(b-a)}{2}\delta_{ij}\dot{h}_{00} - \frac{b}{2}\delta_{ij}\dot{h}_{kk} - \frac{1}{2}(\partial_i h_{0j} + \partial_j h_{0i}) + a\,\delta_{ij}\partial_k h_{k0}, \tag{2.10}$$

$$\pi^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{0i}} = 0. \tag{2.11}$$

We have $D-1$ primary constraints $\varphi^i = \pi^{0i} \approx 0$. Note that we would also have

$$\varphi^0 = \pi^{00} \approx 0 \qquad \text{if } a = b = 1,$$

$$\varphi^0 = \pi^{00} - \pi^{ii} \approx 0 \qquad \text{if } a = \frac{2}{D} \text{ and } b = \frac{(D+2)}{D^2},$$

which would correspond respectively to LEH and WTDiff primary constraints, see (LINO DOS SANTOS, 2018). We will assume that $d_0 \neq 0$, i.e. a and b are not related through (1.7). The reader can consult (LINO DOS SANTOS, 2018) for the cases where $d_0 = 0$. The canonical Hamiltonian density is given by $\mathcal{H} \equiv \mathcal{H}_{TDIFF} = \pi^{\mu\nu}\dot{h}_{\mu\nu} - \mathcal{L}_{TDIFF}$:

$$\begin{aligned}\mathcal{H} =& P\,\pi^{00}\pi^{00} + Q\,\pi^{00}\pi^{ii} + \pi^{ij}\pi^{ij} + R\,\pi^{ii}\pi^{jj} + 2\pi^{00}\partial_i h_{i0} + \pi^{ij}(\partial_i h_{0j} + \partial_j h_{0i}) + \\ &+ \frac{1-b}{4}\partial_i h_{00}\partial_i h_{00} + \frac{1}{4}\partial_i h_{kj}\partial_i h_{kj} - \frac{1}{2}\partial_i h_{ij}\partial_k h_{kj} + \frac{b}{2}\partial_i h_{00}\partial_i h_{jj} \\ &- \frac{b}{4}\partial_i h_{jj}\partial_i h_{kk} - \frac{a}{2}\partial_i h_{00}\partial_j h_{ij} + \frac{a}{2}\partial_i h_{jj}\partial_k h_{ki},\end{aligned} \tag{2.12}$$

where

$$P = \frac{bD-b-1}{d_0}, \qquad Q = \frac{2(b-a)}{d_0}, \qquad R = \frac{b-a^2}{d_0},$$
$$d_0 = 1 - 2a + a^2(D-1) + b(2-D). \tag{2.13}$$

The primary Hamiltonian is given by

$$H_P = \int d^{D-1}x\,\big(\mathcal{H} + \lambda_j(x)\varphi_j(x)\big). \tag{2.14}$$

By applying the consistency condition (A.7) to the constraints (2.11) we obtain the secondary constraints

$$\chi^j = \partial_j \pi^{00} + \partial_i \pi^{ij} \approx 0, \tag{2.15}$$

whose consistency condition allows us to identify another constraint $\partial_j \phi \approx 0$, i.e.

$$\phi = (1-a)\nabla^2 h_{00} + a\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} \approx 0. \tag{2.16}$$

Once its consistency condition gives $\dot{\phi} = -2\partial_j \chi^j \approx 0$, we conclude that there are no new constraints. Since no $D-1$ Lagrange multipliers λ_j could be determined, they remain arbitrary, in total agreement with the TDiff symmetry[1]. All the $2D-1$ constraints of this model are **first class**.

Concerning counting of degrees of freedom, according to (A.10), $n = D(D+1)-2(2D-1)-0 = (D-2)(D-1)$ and, in the Lagrangian formalism,

$$n_L = \frac{(D-2)(D-1)}{2}. \tag{2.17}$$

Hence in $D = 3+1$, $n_L = 3$, in accordance with the description of massless spin-2 and spin-0 particles.

It is needed to verify if this model describes, at least from a classical perspective, physical particles. In this perspective, we recall that by physical particles we mean energy stability manifested by the positivity of the reduced Hamiltonian, which is obtained by using the constraint equations as strong equalities in the primary Hamiltonian.

Through the constraint (2.15) it is easy to show that $2\pi^{00}\partial_i h_{i0} + \pi^{ij}(\partial_i h_{0j} + \partial_j h_{0i})$ strongly vanishes.

Case a $\neq$ 1: Using the constraint (2.16) we have $h_{00} = \omega_{ij}h_{ij} + \frac{a}{a-1}\theta_{ij}h_{ij}$, where we have used projection operators. We can then substitute h_{00} in H_P and, with the help of the spin-2 projection operators, rewrite the coordinate part of the reduced Hamiltonian as $h_{ij}\mathbb{O}^{ij,kl}h_{kl}$, where

$$\begin{aligned}
\mathbb{O} = &-\frac{\nabla^2}{4}\mathbb{1} + \frac{\nabla^2}{2}\left[\frac{P_{ss}^{(1)}}{2} + P_{ww}^{(0)}\right] + \frac{b}{4}\nabla^2\left[(D-2)P_{ss}^{(0)} + P_{ww}^{(0)} + \sqrt{D-2}(P_{sw}^{(0)} + P_{ws}^{(0)})\right] + \\
&-\frac{a}{2}\nabla^2\left[P_{ww}^{(0)} + \frac{\sqrt{D-2}}{2}(P_{sw}^{(0)} + P_{ws}^{(0)})\right] + \frac{a}{2}\nabla^2\left[P_{ww}^{(0)} + \frac{a\sqrt{D-2}}{2(a-1)}(P_{sw}^{(0)} + P_{ws}^{(0)})\right] + \\
&+\left(\frac{b-1}{4}\right)\nabla^2\left[P_{ww}^{(0)} + \frac{a^2(D-2)}{(a-1)^2}P_{ss}^{(0)} + \frac{a\sqrt{D-2}}{(a-1)}(P_{sw}^{(0)} + P_{ws}^{(0)})\right] + \\
&-\frac{b}{2}\nabla^2\left[P_{ww}^{(0)} + \frac{a(D-2)}{(a-1)}P_{ss}^{(0)} + \frac{2a-1}{2(a-1)}\sqrt{D-2}(P_{sw}^{(0)} + P_{ws}^{(0)})\right].
\end{aligned} \tag{2.18}$$

One can show that

$$\mathcal{H}_{h_{ij}} = h_{ij}\left[-\frac{\nabla^2}{4}P_{ss}^{(2)} - \frac{d_0}{4(a-1)^2}\nabla^2 P_{ss}^{(0)}\right]^{ij,kl} h_{kl}. \tag{2.19}$$

Positivity requires $d_0 > 0$, since the eigenvalues of the Laplace operator are negative. This is in agreement with previous subsections where we have just seen that the propagation of degrees of freedom is given by the $P_{ss}^{(2)}$ and $P_{ss}^{(0)}$ operators, spin-2 and spin-0 respectively. Anyway, from $h_{00} = \omega_{ij}h_{ij} + a/(a-1)\,\theta_{ij}h_{ij}$, the trace $h \equiv \eta^{\mu\nu}h_{\mu\nu} = -h_{00} + h_{ii}$ is given by

$$(1-a)h = \theta_{ij}h_{ij} \tag{2.20}$$

so that we can rewrite $hP_{ss}^{(0)}h$ in terms of the $h\nabla^2 h$. By defining Z, following the notation of (ALVAREZ et al., 2006),

$$d_0 = -Z(D-2), \tag{2.21}$$

[1] As worked in the last section, the vector parameter ξ^μ is transverse, thus in D dimensions it possesses $D-1$ independent components.

we rewrite (2.19) as

$$\mathcal{H}_{h_{ij}} = h_{ij}\left[-\frac{\nabla^2}{4}P^{(2)}_{ss}\right]^{ij,kl} h_{kl} + \frac{Z}{4}h\nabla^2 h. \tag{2.22}$$

For the momentum part, using (2.15) we have $\pi^{00} = -\omega_{ij}\pi^{ij} \Rightarrow (\pi^{00})^2 = \pi^{ij}(P^{(0)}_{ww})_{ij,kl}\pi^{kl}$, and

$$\begin{aligned}\nabla^2\pi^{ij}\theta_{ik}\omega_{jl}\pi^{kl} &= \pi^{ij}\theta_{ik}\partial_j\partial_l\pi^{kl}\\ &= -\pi^{ij}\theta_{ik}\nabla^2\partial_j\partial_k\pi^{00} = 0\end{aligned} \tag{2.23}$$

Consequently, note that $\pi^{ij}(P^1_{ss})_{ij,kl}\pi^{kl} = 0$. As it is not possible to relate π^{00} to the transverse part of π^{ij}, the most convenient way of writing the momentum part of the reduced Hamiltonian is

$$\begin{aligned}&\pi^{ij}\left(P^{(2)}_{ss}\right)_{ij,kl}\pi^{kl} + (\pi^{00})^2\left[1+P-Q+R\right] + \pi^{00}(\theta_{ij}\pi^{ij})\left[Q-2R\right] + (\theta_{ij}\pi^{ij})^2\left[\frac{1}{D-2}+R\right] =\\ &= \pi^{ij}\left(P^{(2)}_{ss}\right)_{ij,kl}\pi^{kl} + \frac{1}{d_0(D-2)}\left[(a-1)(\theta_{ij}\pi^{ij}) + a(D-2)(\pi^{00})\right]^2,\end{aligned} \tag{2.24}$$

clearly positive definite for $D > 2$ since $d_0 > 0$. We define the generalized scalar momentum as

$$\Pi = a\pi^{00} + \frac{(a-1)}{(D-2)}(\theta_{ij}\pi^{ij}), \tag{2.25}$$

and write

$$\mathcal{H}_{\pi^{ij}} = \pi^{ij}\left(P^{(2)}_{ss}\right)_{ij,kl}\pi^{kl} - \frac{\Pi^2}{Z}, \tag{2.26}$$

such that the reduced Hamiltonian is

$$H_r = \int d^{D-1}x\left[\pi^{ij}(P^{(2)}_{ss})_{ij,kl}\pi^{kl} + h_{ij}\left(-\frac{\nabla^2}{4}P^{(2)}_{ss}\right)^{ij,kl} h_{kl} - \frac{\Pi^2}{Z} + \frac{Z}{4}h\nabla^2 h\right]. \tag{2.27}$$

Positivity requires $Z < 0$, and H_r contains the correct terms for the propagation of massless spin-2 and spin-0 particles, confirming previous results. Notice that the trace stands for the spin-0 sector.

Case $\mathbf{a = 1}$: For the sake of completeness, we shall show that equation (2.27) is also satisfied if $a = 1$. The parameters d_0 and Z are rewritten as $d_0 = (D-2)(1-b)$ and $Z = b-1$ such that positivity will require $b < 1$.

Basically, the difference between this case and the previous one is in the coordinate part of the reduced Hamiltonian since (2.16) does not allow us to write h_{00} in terms of h_{ij}. We have

$$\begin{aligned}\mathcal{H}_{h_{ij}} =& \frac{(b-1)}{4}h_{00}\nabla^2 h_{00} - \frac{1}{4}h_{ij}\nabla^2 h_{ij} - \frac{b}{2}h_{00}\nabla^2 h_{jj} + \frac{1}{2}h_{ij}\nabla^2\omega_{ik}h_{kj} + \frac{b}{4}h_{jj}\nabla^2 h_{kk} +\\ &+ \frac{1}{2}h_{00}\nabla^2\omega_{ij}h_{ij} - \frac{1}{2}h_{kk}\nabla^2\omega_{ij}h_{ij}.\end{aligned} \tag{2.28}$$

The constraint (2.16) is $\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} = 0 \Rightarrow \nabla^2\theta_{ij}h_{ij} = 0$. Consequently, any one of the contractions $h(P^{(0)}_{ss}, P^{(0)}_{sw}, P^{(0)}_{ws})h$ vanishes. Moreover, $\omega_{ij}h_{ij} = h_{kk}$. Thus (2.28) reads

$$\mathcal{H}_{h_{ij}} = \frac{b}{4}\nabla^2(h_{00} - h_{jj})^2 - \frac{\nabla^2}{4}(h_{00} - \omega_{ij}h_{ij})^2 - \frac{1}{4}h_{kk}\nabla^2\omega_{ij}h_{ij} - \frac{1}{4}h_{ij}\nabla^2 h_{ij} + \frac{1}{2}h_{ij}\nabla^2\omega_{ik}h_{kj}$$
$$= \frac{(b-1)}{4}h\nabla^2 h - \frac{1}{4}h_{ij}\nabla^2(P^{(2)}_{ss})^{ij,kl}h_{kl}, \tag{2.29}$$

positive definite if $b < 1$. The momentum part does not change and we can still use (2.25) and (2.26) with $a = 1$ to write

$$\mathcal{H}_{\pi^{ij}} = -\frac{1}{(b-1)}(\pi^{00})^2 + \pi^{ij}(P^{(2)}_{ss})_{ij,kl}\pi^{kl}, \tag{2.30}$$

again positive if $b < 1$. Therefore, for any value of a we can write

$$H_r = \int d^{D-1}x \left[\pi^{ij}(P^{(2)}_{ss})_{ij,kl}\pi^{kl} + h_{ij}\left(-\frac{\nabla^2}{4}P^{(2)}_{ss}\right)^{ij,kl} h_{kl} - \frac{\Pi^2}{Z} + \frac{Z}{4}h\nabla^2 h\right] \geq 0 \tag{2.31}$$

if $Z < 0$, with Z and Π given respectively by (2.21) and (2.25).

2.3 HELICITY VARIABLES DECOMPOSITION

In this subsection, we introduce a very useful decomposition using helicity variables in order to analyse the physical content of spin-2 models. This method (ALVAREZ et al., 2006; MUKHANOV; FELDMAN; BRANDENBERGER, 1992), also called "the cosmological decomposition", consists in decomposing the symmetric tensor $h_{\mu\nu}$ in its scalar, vector and purely tensor modes, according to their response to spatial rotations $SO(3)$ (BUONINFANTE, 2016). This allows us to decouple these sectors and, consequently, it is little wonder that the calculations become simpler.

Let us concentrate for a moment in $D = 4$ dimensions and then further generalize. We are using the same decomposition as (ALVAREZ et al., 2006). Recall that a symmetric rank-2 tensor has 10 independent components. We conveniently split these components

$$h_{\mu\nu} = \begin{bmatrix} h_{00} & h_{0i} \\ h_{i0} & h_{ij} \end{bmatrix},$$

identifying h_{00} as a scalar, say A, h_{0i} possessing 3 components as a vector v_i, and h_{ij} as a $3D$ spatial 6-component symmetric matrix. Recalling vector calculus (Helmholtz's theorem), it is known that a given vector v_i can be written in terms of a longitudinal part $\partial_i B$ and a transverse part V_i, such that[2] $\partial_i V^i = \vec{\nabla}\cdot\vec{V} = 0$.

Now, we need to understand how to decouple the 6 components of h_{ij}. A piece of helpful information here is to remember that the propagation of spin-2 particles is due to a transverse and traceless mode[3], denoted here by t_{ij}, such that $\partial_i t^{ij} = 0$ and $\eta^{ij}t_{ij} = 0$, a set of 4 equations that reduces

[2] When taking the divergence, only the longitudinal part survives: $\vec{\nabla}\cdot\vec{V} = \nabla^2 B$. On the other hand, taking the rotational, only the transverse part is not identically null, since $\epsilon_{ijk}\partial_j v_k = \epsilon_{ijk}\partial_j V_k$. We can also make this decomposition through the spin-1 projection operators θ_{ij} and ω_{ij}, see Appendix. Whilst the latter gives the longitudinal part of a vector, the former gives the transversal ones.

[3] This mode is nothing all but the application of the spin-2 projection operator $P^{(2)}_{ss}$ to h_{ij}.

the number of independent components of t_{ij} from 6 to 2. There are still 4 remaining components to be dealt with.

There are only two ways of introducing scalars in h_{ij}: by using the 3D metric δ_{ij} and derivatives[4]. Hence, the scalar part of h_{ij} is $\delta_{ij}\psi + \partial_i\partial_j E$, or $\delta_{ij}\psi + \nabla^2\omega_{ij}E$. As we have already introduced all the possible scalars and also a tensor mode, it is natural to finish the decomposition using a vector - more precisely a transverse vector mode - which has only the 2 desired independent components, such that $\partial_i F^i = 0$. Thereby our rank-2 tensor $h_{\mu\nu}$ can be written in terms of

$$h_{00} = A, \qquad h_{0i} = \partial_i B + V_i, \tag{2.32}$$

$$h_{ij} = \psi\delta_{ij} + \nabla^2\omega_{ij}E + 2\partial_{(i}F_{j)} + t_{ij}, \tag{2.33}$$

where V_i and F_i are transverse vectors. The generalization for D dimensions[5] is straightforward. As before, we want to decouple the $\dfrac{D(D+1)}{2}$ independent components of $h_{\mu\nu}$ in its subspaces h_{00}, h_{0i} and h_{ij}, whose dimensions are respectively 1, $(D-1)$ and $\dfrac{D(D-1)}{2}$.

A quick count will convince the reader that the previous decomposition, given by (2.32) and (2.33), also works for $D \geq 4$ dimensions. It is immediate for h_{00} and h_{0i}. For h_{ij}, the transverse vector F_i contributes with $(D-2)$ components, and the two scalars ψ and $\nabla^2 E$, one each. The remaining $\dfrac{D(D-3)}{2}$ components are exactly given by the transverse and traceless tensor $t_{ij} = (P_{ss}^{(2)})^{ij,kl}h_{kl}$. Therefore, the helicity decomposition we have carried out is valid for D dimensions.

2.3.1 TDiff Hamiltonian analysis

In order to appreciate the power of the helicity variables decomposition, let us apply it to the TDiff model. Even though we will obtain the same results of the last subsection and of (ALVAREZ et al., 2006), we will use this method to analyse a massive model in the next chapter. We start rewriting

$$\mathcal{L}_{TDiff} = -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{a}{2}\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h\partial^\mu h \tag{2.34}$$

in terms of the decomposition (2.32) and (2.33). We obtain $\mathcal{L}_{TDiff} = \mathcal{L}^t_{TDiff} + \mathcal{L}^v_{TDiff} + \mathcal{L}^s_{TDiff}$, which is decoupled in its tensor, vector and scalar parts, respectively.

Tensor part: For the tensor part we have

$$\mathcal{L}^t = -\frac{1}{4}\partial_\mu t_{ij}\partial^\mu t_{ij} = \frac{1}{4}\dot{t_{ij}}^2 + \frac{1}{4}t_{ij}\nabla^2 t_{ij}. \tag{2.35}$$

There is no constraint since

$$\pi^{ij} = \frac{\partial\mathcal{L}^t}{\partial\dot{t_{ij}}} = \frac{1}{2}\dot{t_{ij}}. \tag{2.36}$$

[4] Notice that, from the fact that $\delta_{ij} = \omega_{ij} + \theta_{ij}$, we could have written the scalar part in terms of θ and ω, or δ and θ. Choosing two from the three operators is merely a choice of basis. We have chosen δ and ω to compare our results with (ALVAREZ et al., 2006).

[5] We will henceforth refer to the $D \geq 4$ dimensions as D because in this thesis we are not dealing with lower dimensions. Note, however, that there are suitable and simpler decompositions in $D = 3$.

Consequently, the primary and canonical Hamiltonians are equal, given by $\mathcal{H}_c^t = \pi^{ij}\dot{t_{ij}} - \mathcal{L}^t = (\pi^{ij})^2 - \frac{1}{4}t_{ij}\nabla^2 t_{ij}$. As the Laplacian operator has negative eigenvalues (think of the momenta Fourier space), this expression is positively defined and contains no ghosts. We can still write $\mathcal{H}^t$ in terms of the spin-2 projection operator (P_{ss}^2) provided that t_{ij} is a symmetric, transverse and traceless tensor. Thus,

$$H^t = \int d^{D-1}x \, \left\{ \pi^{ij}(P_{ss}^2)_{ij,kl}\pi^{kl} - \frac{1}{4}t_{ij}\nabla^2(P_{ss}^2)_{ij,kl}t_{kl} \right\} \geq 0, \tag{2.37}$$

propagating the $\frac{D(D-3)}{2}$ degrees of freedom of a massless spin-2 particle. Notice that in $D = 3$, it is trivial. Indeed, it is known that there is no propagating degrees of freedom in the EH theory in $D = 3$.

Vector part:

$$\mathcal{L}^v = -\frac{1}{2}(V^i - \dot{F}^i)\nabla^2(V^i - \dot{F}^i) \tag{2.38}$$

from which we calculate the momenta[6]

$$\pi_F^i \equiv \frac{\partial \mathcal{L}^v}{\partial \dot{F}^i} = \nabla^2(V^i - \dot{F}^i) \qquad \Rightarrow \qquad \dot{F}^i = V^i - \frac{\pi_F^i}{\nabla^2}; \tag{2.39}$$

$$\pi_v^i \equiv \frac{\partial \mathcal{L}^v}{\partial \dot{V}^i} = 0 \qquad \Rightarrow \qquad \Phi_{i1}(x) \equiv \pi_v^i(x) \approx 0. \tag{2.40}$$

Thus, the vector canonical Hamiltonian density[7] is $\mathcal{H}_c^v = \pi_F^i\dot{F}^i - \mathcal{L}^v = \pi_F^i V_i - \frac{(\pi_F^i)^2}{2\nabla^2}$, wherewith we write the primary Hamiltonian $\mathcal{H}_p^v = \mathcal{H}_c^v + \lambda_i\pi_v^i$. The consistency condition $\{\Phi_{i1}, \mathcal{H}_p^v\} = 0$ leads us to a new constraint

$$\Phi_{i2}(x) = \pi_F^i(x) \approx 0. \tag{2.41}$$

By noting that $\{\Phi_{i2}, \Phi_{i1}\} = 0$ and $\{\pi_F^i, \mathcal{H}_c^v\} = 0$, the condition $\{\Phi_{i2}, \mathcal{H}_p^v\} = 0$ is trivially satisfied. At this moment, we can treat the constraints as strong equalities and rewrite the primary Hamiltonian - now called the reduced Hamiltonian - which, in this case, vanishes (non-dynamical condition). Indeed, according to the counting rule (A.10) of degrees of freedom, $n = 2 \times 2(D-2) - 2 \times 2(D-2) = 0$, once we have a pair of FCC. In addition, it is not possible to identify the $D - 2$ functions λ_i.

We have demonstrated that there is no vector propagation and the model is consistent with the propagation of a spin-2 massless particle. We shall investigate the scalar part to conclude something about a spin-0 propagation and discuss its physical range (a and b are still arbitrary).

6 Notice that, being F^i and V^i transverse vectors, the momenta π_F^i and π_V^i are likewise. This is important to correctly count the degrees of freedom.

7 Again, in a language abuse, we will be omitting the term density. Just remember that the Hamiltonian is the integral of the Hamiltonian densities over a volume V. This integral plays an important role: when calculating Poisson brackets, we obtain Dirac deltas that should be integrated out to obtain the final result.

Scalar part:

$$\mathcal{L}^s = \frac{2a-b-1}{4}\dot{A}^2 + \frac{d(1-bd)}{4}\dot{\psi}^2 + \frac{1-b}{4}(\nabla^2\dot{E})^2 + \frac{1-bd}{2}\dot{\psi}\nabla^2\dot{E} + \frac{b-a}{2}\dot{A}\nabla^2\dot{E} + \frac{d(b-a)}{2}\dot{A}\dot{\psi} + \\ +\nabla^2 B[(1-a)\dot{A} + (ad-1)\dot{\psi} + (a-1)\nabla^2\dot{E}] + \frac{1-b}{4}A\nabla^2 A + \frac{d-2-bd^2+2ad}{4}\psi\nabla^2\psi + \\ +\frac{2a-b-1}{4}(\nabla^2 E)\nabla^2(\nabla^2 E) + \frac{a+ad-1-bd}{2}\psi\nabla^2(\nabla^2 E) + \frac{b-a}{2}A\nabla^2(\nabla^2 E) + \frac{bd-a}{2}A\nabla^2\psi, \tag{2.42}$$

where $d \equiv D-1$ is the spatial dimension. It is also convenient to define $\xi = \nabla^2 E$. So,

$$\pi_A \equiv \frac{\partial\mathcal{L}^s}{\partial\dot{A}} = \frac{2a-b-1}{2}\dot{A} + \frac{b-a}{2}\dot{\xi} + \frac{d(b-a)}{2}\dot{\psi} + (1-a)\nabla^2 B; \tag{2.43}$$

$$\pi_\psi \equiv \frac{\partial\mathcal{L}^s}{\partial\dot{\psi}} = \frac{d(b-a)}{2}\dot{A} + \frac{1-bd}{2}\dot{\xi} + \frac{d(1-bd)}{2}\dot{\psi} + (ad-1)\nabla^2 B; \tag{2.44}$$

$$\pi_E \equiv \frac{\partial\mathcal{L}^s}{\partial\dot{\xi}} = \frac{b-a}{2}\dot{A} + \frac{1-b}{2}\dot{\xi} + \frac{1-bd}{2}\dot{\psi} + (a-1)\nabla^2 B; \tag{2.45}$$

$$\pi_B \equiv \frac{\partial\mathcal{L}^s}{\partial\dot{B}} = 0 \qquad \Rightarrow \qquad \phi_1 = \pi_B \approx 0. \tag{2.46}$$

The scalar primary Hamiltonian is given by $\mathcal{H}_p^s = \mathcal{H}_c^s + \lambda\phi_1$. Since

$$d_0 = 1 - 2a + a^2(D-1) + b(2-D) = 1 - 2a + a^2 d + b(1-d), \tag{2.47}$$

we write $\mathcal{H}_c^s$ as

$$\mathcal{H}_c^s = \frac{bd-1}{d_0}\pi_A^2 + \frac{d}{d-1}\pi_E^2 + \frac{(a-1)^2}{d_0(d-1)}\pi_\psi^2 + \frac{2(b-a)}{d_0}\pi_A\pi_\psi + \frac{2}{1-d}\pi_E\pi_\psi + 2\nabla^2 B(\pi_A + \pi_E) + \\ +\frac{b-1}{4}A\nabla^2 A - \frac{2(ad-1)+d(1-bd)}{4}\psi\nabla^2\psi - \frac{2a-b-1}{4}\xi\nabla^2\xi - \frac{a-1+d(a-b)}{2}\psi\nabla^2\xi + \\ -\frac{b-a}{2}A\nabla^2\xi - \frac{bd-a}{2}A\nabla^2\psi. \tag{2.48}$$

Observe that the condition $d_0 \neq 0$ is assured, since $d_0 = 0$ corresponds to Diff and WTDiff, cases not considered here. Let us start by analysing the consistency conditions. $\dot{\phi}_1 \approx 0$ gives rise to a secondary constraint[8]

$$\phi_2 = \pi_A + \pi_E \approx 0. \tag{2.49}$$

Analogously, $\{\phi_2, \phi_1\} = 0$ yields the third scalar constraint

$$\phi_3 = (1-a)A + (ad-1)\psi + (a-1)\xi \approx 0. \tag{2.50}$$

Again, $\{\phi_3, \phi_1\} = 0$ and $\{\phi_3, \phi_2\} = 0$. The condition $\dot{\phi}_3 \approx 0$ is trivially satisfied, and the Dirac-

[8] An observation can be made here: actually the constraint which is immediately obtained is $\nabla^2(\pi_A + \pi_E) \approx 0$. However, the solution of $\nabla^2 f = 0$, by requiring that all fields as well as their derivatives vanish at the boundary of the integration volume, is simply $f = 0$.

Bergmann algorithm ends. This also means that λ remains undetermined. Together with the $D-2$ Lagrange multipliers coming from the vector part, they are $D-1$ arbitrary functions, corresponding to the $D-1$ gauge parameters (ξ_μ^T) of the TDiff model.

Therefore, we have 3 scalar FCC which reduce the phase space of 4 scalar fields and their momenta. It follows that we have $n = 8 - 2(3) - 0 = 2 \Rightarrow n_L = 1$ dof, consistent with the propagation of a spin-0 particle. It remains, however, to check positivity.

2.3.2 Positivity of the reduced Hamiltonian

Starting with the momenta sector, by using $\phi_2 = 0$ to eliminate π_E, we have

$$\begin{aligned}\mathcal{H}_\pi^s &= \left(\frac{bd-1}{d_0} + \frac{d}{d-1}\right)\pi_A^2 + \left(\frac{b-a}{d_0} + \frac{1}{d-1}\right)2\pi_A\pi_\psi + \frac{(a-1)^2}{d_0(d-1)}\pi_\psi^2 \\ &= \frac{(ad-1)^2}{d_0(d-1)}\pi_A^2 + \frac{(ad-1)(a-1)}{d_0(d-1)}2\pi_A\pi_\psi + \frac{(a-1)^2}{d_0(d-1)}\pi_\psi^2 \\ &= \frac{1}{d_0(d-1)}\left[(ad-1)\pi_A + (a-1)\pi_\psi\right]^2 . \end{aligned} \tag{2.51}$$

Consequently, $\mathcal{H}_\pi^s \geq 0$ if $d_0 > 0$. By defining $\pi \equiv \dot{h} = \dot{h}_{kk} - \dot{h}_{00}$. and using the definition of momenta, we can show that $(ad-1)\pi_A + (a-1)\pi_\psi = -\frac{d_0}{2}\pi$. Thus,

$$\mathcal{H}_\pi^s = \frac{1}{d_0(d-1)}\left(-\frac{d_0\pi}{2}\right)^2 = -\frac{Z}{4}\pi^2, \tag{2.52}$$

where we have used (2.21). Therefore, positivity requires $Z < 0$, or $d_0 > 0$.

For the coordinate part, we notice that the dependence on B is suppressed by the constraint ϕ_2. The constraint ϕ_3 can be used to eliminate A, if $a \neq 1$. So, it is easy to show that the dependence on ξ disappears, and

$$\mathcal{H}_h^s = -\frac{d_0(d-1)}{4(a-1)^2}\psi\nabla^2\psi. \tag{2.53}$$

We again conclude that $d_0 > 0$ is the necessary and sufficient condition for positivity. Since $h \equiv \eta^{\mu\nu}h_{\mu\nu} = -h_{00} + h_{kk} = -A + \xi + d\,\psi$, and the constraint ϕ_3 is $(d-1)\psi + (a-1)h = 0$, (2.53) is equivalent[9] to

$$\mathcal{H}_h^s = -\frac{d_0}{4(d-1)}h\nabla^2 h = \frac{Z}{4}h\nabla^2 h. \tag{2.54}$$

The complete reduced Hamiltonian is

$$\mathcal{H}_{red}^{s(a\neq 1)} = \frac{\left[(ad-1)\pi_A + (a-1)\pi_\psi\right]^2}{d_0(d-1)} - \frac{d_0}{4(d-1)}h\nabla^2 h = -\frac{Z}{4}\left(\pi^2 - h\nabla^2 h\right). \tag{2.55}$$

For the sake of completeness, if $a = 1$, $\psi = 0$ according to ϕ_3, and $d_0 = (d-1)(1-b)$. We again

[9] We could also rewrite (2.53) in terms of the projection operator $P_{ss}^{(0)}$ since $\theta^{ij}h_{ij} = (d-1)\psi$ and $(d-1)P_{ss}^{(0)} = \theta\theta$.

conclude that $d_0 > 0$ is the desired condition since

$$\mathcal{H}_h^{s(a=1)} = \frac{b-1}{4}(A-\xi)\nabla^2(A-\xi) = -\frac{d_0}{4(d-1)}h\nabla^2 h. \tag{2.56}$$

Hence the final result is

$$H^s_{TDiff} = \int d^{D-1}x \, \frac{d_0}{4(d-1)} \left(\pi^2 - h\nabla^2 h\right) \geq 0, \tag{2.57}$$

valid for any pair (a, b) such that $d_0 > 0$, and whose interpretation is: $\mathcal{L}(1, 1, a, b)$ describes the propagation of spin-2 and spin-0 particles, both massless and physical if $d_0 > 0$.

3 THE MASSIVE MODEL

We have concluded that TDIFF is a healthy model whenever $d_0 > 0$. It describes physical massless spin-2 and spin-0 particles, in agreement with (ALVAREZ et al., 2006). We are now interested in a massive extension of TDiff. The authors in (ALVAREZ et al., 2006) showed that it is not possible to add a mass term to (1.9) and give mass to all particles: it is possible to give mass for the scalar, but not for the tensor mode without introducing ghosts. Likewise, it is not possible to give mass for a WTDiff model. It suggests Diff is the symmetry that must be broken to support massive extensions. In fact, (BONIFACIO; FERREIRA; HINTERBICHLER, 2015; LINO DOS SANTOS, 2018) after performing the Kaluza-Klein dimensional reduction of WTDiff, one obtains Fierz-Pauli

$$\mathcal{L}_{FP} = -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{1}{2}\partial^\mu h \partial^\nu h_{\mu\nu} + \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2), \tag{3.1}$$

which is the massive extension of LEH (FIERZ; PAULI, 1939). It is then natural to ask about the dimensional reduction of the TDiff model, specially because dimensional reductions of restricted symmetries are non-trivial (BONIFACIO; FERREIRA; HINTERBICHLER, 2015).

3.1 THE MASSIVE MODEL: DIMENSIONAL REDUCTION

In this subsection, we present the Kaluza-Klein (KK) dimensional reduction of the TDiff model. For details and references, see Appendix. First, we write the TDiff Lagrangian in $D+1$ dimensions

$$\mathcal{L}_{D+1} = -\frac{1}{4}\partial_A H^{MN}\partial^A H_{MN} + \frac{1}{2}\partial_A H^{AM}\partial_N H^N{}_M - \frac{a}{2}\partial_M H^{MN}\partial_N H + \frac{b}{4}\partial_A H \partial^A H, \tag{3.2}$$

invariant under $\delta H_{MN} = \partial_M \xi^T_N + \partial_N \xi^T_M$ with $\partial^M \xi^T_M = 0$. Notice that the requirement of physical particles in $D+1$ dimensions means

$$d_0[D+1] \equiv D_0 = 1 - 2a - b(D-1) + a^2 D > 0, \tag{3.3}$$

since d_0 in (1.7) is written for D dimensions. We suitably decompose $H_{AB}(x,y)$:

$$H_{\mu\nu}(x,y) = \sqrt{\frac{m}{\pi}} h_{\mu\nu}(x)\cos my, \tag{3.4}$$

$$H_{y\mu}(x,y) = \sqrt{\frac{m}{\pi}} a_\mu(x)\sin my, \tag{3.5}$$

$$H_{yy}(x,y) = \sqrt{\frac{m}{\pi}} \varphi(x)\cos my. \tag{3.6}$$

To obtain the gauge symmetry in D dimensions, we express $\xi_N(x,y)$ as

$$\xi_\mu(x,y) = \sqrt{\frac{m}{\pi}}\psi_\mu(x)\cos my, \tag{3.7}$$

$$\xi_y(x,y) = \sqrt{\frac{m}{\pi}}\psi(x)\sin my. \tag{3.8}$$

In this case, the D-dimensional symmetries are

$$\delta h_{\mu\nu}(x) = \partial_\mu\psi_\nu + \partial_\nu\psi_\mu, \tag{3.9}$$

$$\delta a_\mu(x) = \partial_\mu\psi - m\ \psi_\mu, \tag{3.10}$$

$$\delta\varphi(x) = 2m\ \psi. \tag{3.11}$$

We have, however, an equation between ψ and ψ_μ following from $\partial^M\xi^T_M = 0$:

$$\partial^\mu\psi_\mu + m\psi = 0. \tag{3.12}$$

This constraint forbids us to simultaneously fix $\varphi = 0 = a_\mu$ as gauge conditions. The action in $D+1$ dimensions is

$$S = \int d^{D+1}x\ \mathcal{L}_{(D+1)} = \int_0^L dy \int dx_0\cdots dx_{D-1}\ \mathcal{L}_{(D+1)} = \int_0^{\frac{2\pi}{m}} dy\int d^Dx\ \mathcal{L}_{(D+1)}. \tag{3.13}$$

Integrating in y, we obtain $\mathcal{L}_D$:

$$\begin{aligned}\mathcal{L}_D =&\mathcal{L}_{TDiff} - \frac{1}{4}F^2_{\mu\nu}[a_\mu] - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - b\ h^2) + \frac{a}{2}h_{\mu\nu}\partial^\mu\partial^\nu\varphi - m\ h_{\mu\nu}\partial^\mu a^\nu + m\ a\ h\partial_\mu a^\mu +\\ &- m(a-1)a_\mu\partial^\mu\varphi - \frac{h}{2}[b\,\Box + m^2(a-b)]\varphi - \frac{\varphi}{4}[(b-1)\Box + m^2(2a-b-1)]\varphi,\end{aligned} \tag{3.14}$$

which is invariant under (3.9-3.11). In order to decouple ψ_μ and ψ in (3.10) we define

$$A_\mu = a_\mu - \frac{1}{2m}\partial_\mu\varphi \quad\Rightarrow\quad \delta A_\mu = -m\ \psi_\mu. \tag{3.15}$$

Then, we rewrite (3.14) as

$$\begin{aligned}\mathcal{L}_D =&\mathcal{L}_{TDiff} - \frac{1}{4}F^2_{\mu\nu}[A_\mu] - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - b\ h^2) + \frac{(a-1)}{2}h_{\mu\nu}\partial^\mu\partial^\nu\varphi - m\ h_{\mu\nu}\partial^\mu A^\nu + m\ a\ h\partial_\mu A^\mu +\\ &- m(a-1)A_\mu\partial^\mu\varphi - \frac{(b-a)}{2}h(\Box - m^2)\varphi - \frac{(b-2a+1)}{4}\varphi(\Box - m^2)\varphi.\end{aligned} \tag{3.16}$$

We could already analyse the physical content of (3.16), but we can use the gauge invariance to simplify the calculations. Let us make some comments on this procedure. The first option is the following: we can use the vector parameter ψ_μ from (3.15) to fix $A_\mu = 0$. Because of (3.12), the scalar parameter ψ is determined, and we cannot use it to fix, for example, $\varphi = 0$. Therefore, we have a scalar-tensor model.

A second option would be to get rid of φ by using ψ from (3.11), but (3.12) would not allow us to fix $A_\mu = 0$. Consequently, we would have a vector-tensor description. Even though both options are permissible[1], we chose the first one, since it is simpler to work with scalar-tensor models.

Then, a massive extension of TDiff is given by

$$\mathcal{L}_m \equiv \mathcal{L}_D[A_\mu = 0] = \mathcal{L}_{TDiff} - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - b\,h^2) + \frac{(a-1)}{2}h_{\mu\nu}\partial^\mu\partial^\nu\varphi + \\ - \frac{(b-a)}{2}h(\Box - m^2)\varphi - \frac{(b-2a+1)}{4}\varphi(\Box - m^2)\varphi, \quad (3.17)$$

where $\mathcal{L}_{TDiff}$ is given by (2.7). This result is original. We will verify that it propagates the correct number of degrees of freedom: $n = \frac{D(D-1)}{2}$, which gives 6 in $D = 4$, exactly 5 of the the massive spin-2 particle, and 1 of the massive spin-0. Furthermore, we will show that absence of ghosts requires $D_0 > 0$.

3.2 THE MASSIVE MODEL - HAMILTONIAN ANALYSIS

The symmetries (3.9)-(3.11) allow us to conveniently fix the gauge $A_\mu = 0$. Consequently, due to (3.12) there is not any further symmetry. So, now we are expected to determine every Lagrange multiplier. Our starting point is (3.17):

$$\mathcal{L}_m = \mathcal{L}_{TDiff} - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - b\,h^2) + \frac{(a-1)}{2}h_{\mu\nu}\partial^\mu\partial^\nu\varphi - \frac{(b-a)}{2}h(\Box - m^2)\varphi + \\ - \frac{(b-2a+1)}{4}\varphi(\Box - m^2)\varphi. \quad (3.18)$$

Proceeding as we did in the last chapter, the tensor sector is

$$\mathcal{L}_m^t = \frac{1}{4}t_{ij}(\Box - m^2)t_{ij}, \quad (3.19)$$

which has the same structure as (2.35) (just replace $\nabla^2 \to \nabla^2 - m^2$), and therefore we have the same $D(D-3)/2$ degrees of freedom, corresponding to the ± 2 helicities in $D = 3+1$. The Hamiltonian is

$$H_m^t = \int d^{D-1}x \left\{ \pi^{ij}(P_{ss}^2)_{ij,kl}\pi^{kl} - \frac{1}{4}t_{ij}(\nabla^2 - m^2)(P_{ss}^2)_{ij,kl}t_{kl} \right\} \geq 0. \quad (3.20)$$

However, different results come to light on the vector part:

$$\mathcal{L}_m^v = -\frac{1}{2}(V_i - \dot{F}_i)\nabla^2(V_i - \dot{F}_i) + \frac{m^2}{2}V_i^2 + \frac{m^2}{2}F_j\nabla^2F_j, \quad (3.21)$$

[1] The reader may wonder if it is actually a rigorous procedure to fix these gauges at action level. It is indeed since (3.11) and (3.15) completely fix ψ_μ and ψ, i.e. there is no residual symmetry, see (MOTOHASHI; SUYAMA; TAKAHASHI, 2016).

from which we calculate the momenta

$$\pi_F^i \equiv \frac{\partial \mathcal{L}^v}{\partial \dot{F}^i} = \nabla^2(V^i - \dot{F}^i) \qquad \Rightarrow \qquad \dot{F}^i = V^i - \frac{\pi_F^i}{\nabla^2}, \tag{3.22}$$

$$\pi_V^i \equiv \frac{\partial \mathcal{L}^v}{\partial \dot{V}^i} = 0 \qquad \Rightarrow \qquad \Phi_{i1}(x) = \pi_V^i(x) \approx 0, \tag{3.23}$$

and write the primary Hamiltonian in the vector sector as $\mathcal{H}_p^v = \mathcal{H}_c^v + \lambda_i \Phi_{i1}$, where

$$\mathcal{H}_c^v = -\frac{(\pi_F^i)^2}{2\nabla^2} + \pi_F^i V_i - \frac{m^2}{2} V_i^2 - \frac{m^2}{2} F_j \nabla^2 F_j. \tag{3.24}$$

We apply the Dirac-Bergmann algorithm. We have

$$\dot{\Phi}_{i1} \approx 0 \Rightarrow \Phi_{i2} \equiv \pi_F^i - m^2 V_i \approx 0. \tag{3.25}$$

Differently from the massless case, these two constraints are SCC because $\{\Phi_{i2}, \Phi_{i1}\} = -m^2\delta_{ij}$. This allows us to determine the Lagrange multiplier associated to Φ_{i1}. Once $\{\Phi_{i2}, \mathcal{H}_c^v\} = m^2\nabla^2 F_i$,

$$\dot{\Phi}_{i2} \approx 0 \Rightarrow \lambda_i \approx \nabla^2 F_i. \tag{3.26}$$

Since there are no further constraints, the algorithm is finished. By replacing λ_i in $\mathcal{H}_p^v$, we have

$$\mathcal{H}_p^v = -\frac{(\pi_F^i)^2}{2\nabla^2} + \pi_F^i V_i + \pi_V^i \nabla^2 F_i - \frac{m^2}{2} V_i^2 - \frac{m^2}{2} F_j \nabla^2 F_j, \tag{3.27}$$

and taking the constraints as strong equalities, we obtain the reduced Hamiltonian for the vector sector

$$H_r^v = \frac{1}{2} \int d^{D-1}x \left\{ \frac{(\nabla^2 - m^2)(\pi_F^i)^2}{m^2 \nabla^2} - m^2 F_i \nabla^2 F_i \right\} \geq 0. \tag{3.28}$$

As a comment, if we compare last equation with (3.20), we notice that π_F^i is playing the role of a coordinate (indeed, $\pi_F^i = m^2 V_i$). But this is not a problem since we can make a canonical transformation $\left(\pi_F^i \to \sqrt{-\nabla^2} m \Upsilon_i, F_i \to -\frac{1}{m\sqrt{-\nabla^2}} \pi_\Upsilon^i\right)$, and rewrite H_r^v in a more usual way

$$H_r^v = \frac{1}{2} \int d^{D-1}x \left\{ (\pi_\Upsilon^i)^2 + \Upsilon_i (m^2 - \nabla^2) \Upsilon_i \right\} \geq 0. \tag{3.29}$$

In order to count degrees of freedom, by noting that each one of the constraints is actually a set of $d - 1 = D - 2$ equations (transverse constraints), we have

$$n = 2(2(d-1)) - 2(0) - 1(2(d-1)) = 2(d-1) \Rightarrow n_L = d - 1. \tag{3.30}$$

In $D = 3 + 1$, $n_L = 2$, and this corresponds to the two ± 1 polarizations[2] of the massive spin-2 particle.

[2] Notice that, unlike the usual graviton (which has only the ± 2 helicities), the massive graviton must have 5 dof $(\pm 2, \pm 1, 0)$ since the isotropy group (little-group) is not $ISO(2)$ anymore, but $SO(3) \simeq SU(2)$, from which we derive the $2s + 1$ rule (WEINBERG, 1995) for massive spin-s particles.

In the scalar sector, we have five scalar fields, namely A, B, ξ, ψ and φ:

$$\begin{aligned}\mathcal{L}^s_m =&\mathcal{L}^s + \frac{2a-b-1}{4}\left[\dot{\varphi}^2 - 2\dot{A}\dot{\varphi} + \varphi(\nabla^2 - m^2)\varphi + 2\varphi\nabla^2\xi\right] + \frac{d(a-b)}{2}\dot{\psi}\dot{\varphi} + \frac{a-b}{2}\dot{\xi}\dot{\varphi} + (a-1)\dot{\varphi}\nabla^2 B+\\ &+\frac{b-1}{4}m^2A^2 - \frac{m^2}{2}B\nabla^2 B + \frac{d(bd-1)}{4}m^2\psi^2 + \frac{b-1}{4}m^2\xi^2 + \frac{bd-1}{2}m^2\psi\xi - \frac{bd}{2}m^2A\psi - \frac{b}{2}m^2A\xi+\\ &+\frac{b-a}{2}A(\nabla^2 - m^2)\varphi + \frac{a+1+d(a-b)}{2}\varphi\nabla^2\psi + \frac{b-a}{2}m^2\varphi(\xi + d\,\psi),\end{aligned} \tag{3.31}$$

where $\mathcal{L}^s$ is given by (2.42). Evaluating the momenta - also referring to the massless case results (2.43)-(2.45):

$$\pi^m_A \equiv \pi^{m=0}_A + \frac{b+1-2a}{2}\dot{\varphi}; \quad \pi^m_\psi \equiv \pi^{m=0}_\psi + \frac{d(a-b)}{2}\dot{\varphi}; \quad \pi^m_E \equiv \pi^{m=0}_E + \frac{a-b}{2}\dot{\varphi}; \tag{3.32}$$

$$\pi^m_\varphi \equiv \frac{\partial\mathcal{L}^s_m}{\partial\dot{\varphi}} = \frac{2a-b-1}{2}(\dot{\varphi} - \dot{A}) + \frac{d(a-b)}{2}\dot{\psi} + \frac{a-b}{2}\dot{\xi} + (a-1)\nabla^2 B; \tag{3.33}$$

$$\pi^m_B \equiv \pi_B \qquad \Rightarrow \qquad \phi_1 = \pi^m_B \approx 0. \tag{3.34}$$

Noticing that $\pi^m_A + \pi^m_\varphi = 0$, we define a new PC

$$\phi_2 \equiv \pi^m_A + \pi^m_\varphi \approx 0. \tag{3.35}$$

Then, the primary Hamiltonian is $\mathcal{H}^m_p = \mathcal{H}^m_c + \lambda_1\phi_1 + \lambda_2\phi_2$:

$$\begin{aligned}\mathcal{H}^m_c =& \mathcal{H}^s_c + \frac{b+1-2a}{4}\varphi(\nabla^2 - m^2)\varphi + \frac{a-b}{2}A(\nabla^2 - m^2)\varphi + \frac{d(b-a)+1-a}{2}\psi\nabla^2\varphi+\\ &+\frac{b+1-2a}{2}\xi\nabla^2\varphi - \frac{b-1}{4}m^2A^2 - \frac{d(bd-1)}{4}m^2\psi^2 + \frac{1-b}{4}m^2\xi^2 + \frac{1-bd}{2}m^2\psi\xi+\\ &+\frac{b}{2}m^2A\xi + \frac{bd}{2}m^2A\psi + \frac{a-b}{2}m^2\varphi\xi + d\frac{a-b}{2}m^2\varphi\psi + \frac{m^2}{2}B\nabla^2 B,\end{aligned} \tag{3.36}$$

where $\mathcal{H}^s_c$ is given by expression (2.48), obtained by relabelling its momenta ($\pi_A, \pi_\psi, \pi_E \to \pi^m_A, \pi^m_\psi, \pi^m_E$). Once again we shall apply the Dirac-Bergmann algorithm. Omitting intermediate steps, we have

$$\dot{\phi}_1 \approx 0 : \phi_3 \equiv 2(\pi^m_A + \pi^m_E) + m^2 B \approx 0; \tag{3.37}$$

$$\dot{\phi}_2 \approx 0 : \phi_4 \equiv (1-a)(\nabla^2 - m^2)(\varphi - A) + (1-a)\nabla^2\xi + a\,m^2\xi + (1-ad)\nabla^2\psi + ad\,m^2\psi \approx 0. \tag{3.38}$$

Going to the second round, $\dot{\phi}_3 \approx 0$ determines the first Lagrange multiplier λ_1 since $\{\phi_3, \phi_1\} = m^2$. Back in the scalar primary Hamiltonian, we now have $\mathcal{H}^m_p = \mathcal{H}^m_c + \lambda_1\pi^m_B + \lambda_2\phi_2 \equiv \mathcal{H}' + \lambda_2\phi_2$, written in this way to emphasize that from now on $\{B, \mathcal{H}'\} \neq 0$. In its turn, $\dot{\phi}_4 \approx 0$ leads to

$$\phi_5 \equiv \nabla^2 B - 2\left(1 + \frac{a-b}{d_0}\right)\pi^m_A - 2\left(\frac{a^2-b}{d_0}\right)\pi^m_\psi \approx 0. \tag{3.39}$$

Similarly, $\dot{\phi}_5 \approx 0$ gives

$$\phi_6 \equiv a\, m^2 A + (1-a)m^2\varphi - a\, m^2\xi - ad\, m^2\psi \approx 0. \tag{3.40}$$

The Lagrange multiplier λ_2 can then be determined from $\dot{\phi}_6 \approx 0$ since $\{\phi_6, \phi_2\} = m^2$. This means that the Dirac-Bergmann algorithm ends up with both Lagrange multipliers determined:

$$\lambda_1 = -\frac{\{\phi_3, \mathcal{H}_c^m\}}{m^2}, \lambda_2 = -\frac{\{\phi_6, \mathcal{H}_c^m\}}{m^2}. \tag{3.41}$$

This was expected once the massive Lagrangian (3.17) has no gauge symmetry, as previously mentioned.

So far we can say that ϕ_1, ϕ_2, ϕ_3, and ϕ_6 are all SCC. For the remaining ones, let us evaluate $\{\phi_3, \phi_4\}$:

$$\begin{aligned}\{\phi_3(x), \phi_4(y)\} &= \{2\pi_A^m, (a-1)(\nabla^2 - m^2)A\} + \{2\pi_E^m, (1-a)\nabla^2\xi + a\, m^2\xi\} \\ &= -2m^2(1-a+a)\delta^{(D-1)}(x-y) = -2m^2\delta^{(D-1)}(x-y) \neq 0.\end{aligned} \tag{3.42}$$

Analogously, $\{\phi_1(x), \phi_5(y)\} = \nabla^2\delta^{(D-1)}(x-y) \neq 0$. Consequently, the six scalar constraints are SCC, reducing the 10-dimensional phase space (we have 5 scalar fields A, B, ψ, ξ, φ) down to

$$n = 2(5) - 2(0) - 1(6) = 4 \qquad \Rightarrow \qquad n_L = 2, \tag{3.43}$$

corresponding to the propagation of a massive spin-0 particle, and to the zero helicity mode of a massive spin-2 particle. Together with the tensor and vector results, it demonstrates that the massive model (3.17) is consistent, at least from the perspective of degrees of freedom, with the propagation of a massive spin-2 particle (5 degrees of freedom) and a massive spin-0 particle (1 degree of freedom). Let us check now for which values of (a, b) we have physical particles.

3.2.1 Positivity

As in the section 3.2, we write the reduced Hamiltonian by setting all the constraints to zero (strong equality). Beginning with the scalar momenta sector, we have the constraints $\phi_3 = 0$, and $\phi_5 = 0$ to get rid of two of the four quantities $(\pi_A^m, \pi_E^m, \pi_\psi^m, B)$, by replacing them in (3.36). Let us eliminate π_E^m and π_ψ^m, supposing that $b \neq a^2$. Straightforwardly computing this sector, we have

$$\begin{aligned}\mathcal{H}_\pi^s =& \frac{d_0}{(a^2-b)^2(d-1)}(\pi_A^m)^2 + \frac{a-1}{(a^2-b)(d-1)}m^2\pi_A^m B + \frac{(a-1)d_0}{(a^2-b)^2(d-1)}\pi_A^m\nabla^2 B + \\ &+ \frac{(a-1)^2}{2(a^2-b)(d-1)}m^2 B\nabla^2 B + \frac{d}{4(d-1)}(m^2B)^2 + \frac{(a-1)^2 d_0}{4(a^2-b)^2(d-1)}(\nabla^2 B)^2.\end{aligned}$$

We rewrite the previous equation as

$$\mathcal{H}_\pi^s = \frac{d_0}{d-1}\Pi^2 + \frac{D_0}{4d_0}(m^2B)^2, \tag{3.44}$$

where

$$\Pi = \frac{\pi_A^m}{(a^2-b)} + \frac{(a-1)}{2(a^2-b)}\nabla^2 B + \frac{(a-1)}{2d_0}m^2 B, \tag{3.45}$$

and recall that $D_0 = 1 - 2a + a^2 D - b(D-1)$. Hence, $\mathcal{H}_\pi^s \geq 0$ if $d_0 > 0$ and $D_0 > 0$. Similarly, for $b = a^2$, it is not difficult to show that $\mathcal{H}_\pi^s \geq 0$ for any a:

$$\mathcal{H}_\pi^s = \frac{\Gamma^2}{d-1} + \frac{(m^2 B)^2}{4}, \qquad (\text{if } b = a^2) \tag{3.46}$$

where in this case $d_0 = (a-1)^2 = D_0$, and

$$\Gamma = \pi_\psi^m - \frac{ad-1}{2}\nabla^2 B + \frac{m^2 B}{2}. \tag{3.47}$$

For the coordinates part, we have four scalars φ, A, ξ and ψ. Two of them can be eliminated by replacing the constraints $\phi_4 = 0$ and $\phi_6 = 0$ in (3.36). Let us suppose that $a \neq 1$, and eliminate φ and ξ. Thus, the coordinates scalar sector of the reduced Hamiltonian is

$$\begin{aligned} \mathcal{H}_h^s &= \frac{(m^2-\nabla^2)}{4\nabla^4(1-a)^2}\left[(d-1)d_0\nabla^4\psi^2 - 2(a^2-b)(d-1)m^2 A\nabla^2\psi + (1-2a(a-1)-b)m^4 A^2\right] \\ &= \frac{(d-1)d_0}{4(a-1)^2}\Phi\frac{(m^2-\nabla^2)}{\nabla^4}\Phi + \frac{m^4 D_0}{4d_0}A\frac{(m^2-\nabla^2)}{\nabla^4}A, \end{aligned} \tag{3.48}$$

where $\nabla^4 = \nabla^2\nabla^2$ and

$$\Phi = \nabla^2\psi - \frac{(a^2-b)}{d_0}m^2 A. \tag{3.49}$$

Then, $\mathcal{H}_h^s \geq 0$ if $D_0 > 0$ and $d_0 > 0$. The $a = 1$ case is even simpler:

$$\mathcal{H}_h^s = \frac{d(d-1)}{4}\psi(m^2-\nabla^2)\psi + \frac{(1-b)}{4}\varphi(m^2-\nabla^2)\varphi \geq 0 \qquad (\text{if } b < 1). \tag{3.50}$$

Since whenever $a = 1$, $d_0 = (1-b)(d-1)$ and $D_0 = (1-b)d$, $b < 1$ also means $d_0 > 0$ or $D_0 > 0$.

Therefore, we can finally conclude that the massive scalar reduced Hamiltonian is positive-definite

$$H_m^s = \int d^{D-1}x\ \{\mathcal{H}_\pi^s + \mathcal{H}_h^s\} \geq 0, \tag{3.51}$$

for any real-valued pair (a, b) satisfying $D_0 > 0$ and $d_0 > 0$.

It happens that $D_0 > 0 \Rightarrow d_0 > 0$ for natural dimensions $D = 4, 5, 6, ...$, and this can be easily seen by plotting a graph a versus b. This means that

"$D_0 > 0$ is the necessary and sufficient condition for a consistent massive model (3.17), i.e. it suffices that the higher-dimensional action is a unitary TDiff model."

Therefore, the KK dimensional reduction preserves the degrees of freedom from $D+1$ down to D dimensions, and we end up with massive spin-2 and massive spin-0 particles.

3.3 MASSLESS LIMIT AND FINAL CONSIDERATIONS ABOUT TDIFF

What happens if we set $m = 0$ in (3.18)?

$$\mathcal{L} = \mathcal{L}_{TDiff} + \frac{a-1}{2}\varphi\partial^\mu\partial^\nu h_{\mu\nu} + \frac{a-b}{2}\varphi\Box h + \frac{2a-b-1}{4}\varphi\Box\varphi. \tag{3.52}$$

In order not to draw wrong conclusions, let us analyse the Hamiltonian of (3.52). The tensor and vector parts are exactly the same ones as TDiff (2.7). On the other hand, for the scalar part, we have the PC

$$\phi_1 = \pi_B \approx 0 \quad ; \quad \phi_2 = \pi_A + \pi_\varphi \approx 0 \quad \Rightarrow \mathcal{H}_p^s = \mathcal{H}_c^s + \Lambda_1\phi_1 + \Lambda_2\phi_2. \tag{3.53}$$

Applying the consistency condition, we have

$$\dot{\phi}_1 \approx 0 \Rightarrow \quad \phi_3 = (\pi_A + \pi_E) \approx 0, \tag{3.54}$$
$$\dot{\phi}_2 \approx 0 \Rightarrow \quad \phi_4 = (1-a)(\varphi - A + \xi) + (1-ad)\psi \approx 0, \tag{3.55}$$

while $\dot{\phi}_3 \approx 0$ and $\dot{\phi}_4 \approx 0$ hold identically.

Notice that, now without mass, we have 4 FCC. Concerning the number of dof, $n = 2(5) - 2(4) - 0 = 2 \Rightarrow n_L = 1$, which corresponds to one scalar degree of freedom (spin-0). In addition, the 2 Lagrange multipliers Λ_1 and Λ_2 remain undetermined; this is interesting because, together with the $D-2$ transverse vector Lagrange multipliers λ_j, they become a set of D multipliers. This indicates that (3.52) has a gauge symmetry depending on D parameters (not $D-1$, like TDiff).

Checking positivity, $\mathcal{H}_\pi \geq 0$, since it is the same expression of a previous case. For the coordinate part, we have only one constraint, ϕ_4. If $a \neq 1$, $\varphi = A - \xi + \frac{1-ad}{a-1}\psi$, and

$$\mathcal{H}_s = -\frac{(d-1)d_0}{4(a-1)^2}\psi\nabla^2\psi \geq 0 \tag{3.56}$$

if $d_0 > 0$. On the other hand, if $a = 1$, we have $\psi = 0$ and

$$\mathcal{H}_s^{a=1} = -\frac{d_0}{4(d-1)}(A - \xi - \varphi)\nabla^2(A - \xi - \varphi) \geq 0 \tag{3.57}$$

if $d_0 > 0$, where we have used the fact that when $a = 1$, $d_0 = (d-1)(1-b)$.

Therefore, we conclude that (3.52) describes 3 dof, corresponding to a massless spin-2 and a massless spin-0 particle, physical if $d_0 > 0$.

3.3.1 Investigating the symmetry

We already know that $\mathcal{L}_{TDiff}$ is invariant under $\delta h_{\mu\nu} = \partial_\mu\xi_\nu^T + \partial_\nu\xi_\mu^T$. On the other hand, (3.52) is invariant under some D parameters transformation. Our Ansatz is that it is invariant under

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu; \qquad \delta\varphi = c\nabla\cdot\xi. \tag{3.58}$$

From (1.1) we know that

$$\delta\mathcal{L}_{TDiff} = (a-1)\nabla\cdot\xi\partial^\mu\partial^\nu h_{\mu\nu} + (a-b)h\Box\nabla\cdot\xi \tag{3.59}$$

for an infinitesimal vector ξ_μ, whereas for the part dependent on φ in (3.52)

$$\delta(\mathcal{L}-\mathcal{L}_{TDiff}) = \frac{c(a-1)}{2}\nabla\cdot\xi\partial^\mu\partial^\nu h_{\mu\nu} + \frac{c(a-b)}{2}h\Box\nabla\cdot\xi + \frac{c}{2}(2a-b-1)\varphi\Box\nabla\cdot\xi. \tag{3.60}$$

Note that $\delta\mathcal{L}=0 \Rightarrow c=-2$. So, the model (3.52) is invariant under[3]

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu; \qquad \delta\varphi = -2\nabla\cdot\xi. \tag{3.61}$$

On the other hand, (3.61) suggests the following redefinition

$$\varphi \equiv \phi - h, \tag{3.62}$$

so that[4]

$$\begin{aligned}\mathcal{L} = \ & -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{1}{2}\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{1}{4}\partial_\mu h\partial^\mu h + \qquad \leftarrow \mathcal{L}_{EH}! \\ & + \frac{a-1}{2}\phi(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) + \frac{2a-b-1}{4}\phi\Box\phi\end{aligned} \tag{3.63}$$

is invariant under $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. This a quite peculiar result: $\mathcal{L}_{EH}$ can be deformed to propagate an extra scalar dof - without ghosts if $d_0 > 0$ as above demonstrated. The TDiff model, described by $\mathcal{L}(1,1,a,b)$, is the simplest model describing massless spin-2 and spin-0 particles in the sense that it has no auxiliary fields, however, it is not unique. The model (3.63) has one more field (ϕ), and one more symmetry since ξ^μ is not transverse now.

Anyway, this discovery makes us wonder what would be the effect of (3.62) in the massive model (3.18). Explicitly we have:

$$\begin{aligned}\mathcal{L}_m = \ & -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{1}{2}\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{1}{4}\partial_\mu h\partial^\mu h - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2) + \qquad \leftarrow \mathcal{L}_{FP}! \\ & + \frac{a-1}{2}\phi\,\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1-a}{2}h(\Box - m^2)\phi + \frac{2a-b-1}{4}\phi(\Box - m^2)\phi,\end{aligned} \tag{3.64}$$

a particularly interesting result because it states that the dimensional reduction of TDiff can be derived from an extension of Fierz-Pauli! Indeed, it would suffice to make the replacement $\Box \to (\Box - m^2)$ in (3.63) to obtain (3.64), just like the massive FP theory is obtained from its massless version LEH. Similarly for the WTDiff case, see (BONIFACIO; FERREIRA; HINTERBICHLER, 2015), even though a scalar field φ appears after the dimensional reduction of WTDiff, the massive spin-2 model is a massive deformation of Diff, after a trivial field redefinition $h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{D}\eta_{\mu\nu}h + \eta_{\mu\nu}\varphi$. These

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3 As an observation, (3.52) can be obtained from $\mathcal{L}_{TDiff}$ through a Stückelberg redefinition $h_{\mu\nu} \to h_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{\Box}\varphi$, but this transformation is not local.

4 Notice that ϕ is coupled to the linearized scalar curvature $R_L = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h$ (WEINBERG, 2013).

results indicate that Diff, not TDiff, is the symmetry to be broken when giving mass to the graviton, generalizing the result of (ALVAREZ et al., 2006), which did not consider an extra scalar field. In next chapters we will prove that a massive scalar-tensor model describing 6 degrees of freedom is necessarily invariant under generalized Diff when $m = 0$. As it is not possible to deform TDiff without adding scalar field mass terms (and consequently a massive version other than FP requires at least a scalar-tensor description), this result proves that an exclusive TDiff symmetry does not support any mass.

Finally, notice that we can redefine $h_{\mu\nu} = H_{\mu\nu} - \frac{a-1}{D-2}\varphi$ in (3.63) such that it can be decoupled in the following way:

$$\mathcal{L} = \mathcal{L}_{EH}(H) + \frac{d_0}{4(D-2)}\phi\Box\phi. \tag{3.65}$$

From this expression, the condition for a healthy scalar degree of freedom is easily obtained: $d_0 > 0$. Since (3.63) is obtained from TDiff through a non-local transformation, we could not rigorously assume that TDiff and (3.63) are equivalent. On the other hand, there does not exist a local redefinition for the massive version, given by (3.18), that decouples the tensor and scalar fields if $a \neq 1$. In this case, the massive model obtained after the dimensional reduction will only be a decoupled one if the higher-dimensional massless model is already decoupled.

3.3.2 Equations of motion: massive case

For the sake of completeness, let us finish the classical analysis by looking at the equations of motion of (3.64). Using the notation of chapter 2, we have

$$E_{\mu\nu} = (\Box - m^2)h_{\mu\nu} - \partial^\alpha(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha}) + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \eta_{\mu\nu}(\Box - m^2)h + \\ +(a-1)\partial_\mu\partial_\nu\varphi + (1-a)(\Box - m^2)\eta_{\mu\nu}\varphi = 0, \tag{3.66}$$

and

$$\partial^\mu E_{\mu\nu} = -m^2\partial_\mu h^{\mu\nu} + m^2\partial_\nu h + (a-1)m^2\partial_\nu\varphi = 0. \tag{3.67}$$

The equation of motion with respect to ϕ gives

$$\Phi = (a-1)\partial^\mu\partial^\nu h_{\mu\nu} + (1-a)(\Box - m^2)h + (2a-b-1)(\Box - m^2)\varphi = 0. \tag{3.68}$$

The following combination supplies us with a scalar constraint:

$$\begin{aligned}\Omega &= \frac{a^2-b}{d_0}m^2\eta^{\mu\nu}E_{\mu\nu} + \partial^\mu\partial^\nu E_{\mu\nu} + \frac{a-1}{d_0}m^2\Phi = 0 \\ &= \frac{D_0}{d_0}m^2\left[h + (a-1)\varphi\right] = 0 \Rightarrow \boxed{h + (a-1)\varphi = 0},\end{aligned} \tag{3.69}$$

provided that $D_0 > 0$. With this result, (3.67) implies $\partial^\mu h_{\mu\nu} = 0$. Additionally, (3.66) reads simply as $(\Box - m^2)h_{\mu\nu} = 0$. A traceless and transverse tensor can be easily built:

$$H_{\mu\nu} = h_{\mu\nu} - \frac{\Box\theta_{\mu\nu}h}{m^2(D-1)}. \tag{3.70}$$

Since $(\Box - m^2)h_{\mu\nu} = 0$, we have also $(\Box - m^2)h = 0$, and then

$$\begin{cases} (\Box - m^2)H_{\mu\nu} = 0, \\ \partial^\mu H_{\mu\nu} = 0, \\ \eta^{\mu\nu} H_{\mu\nu} = 0, \end{cases} \tag{3.71}$$

i.e. the Fierz-Pauli conditions for a rank-2 symmetric tensor describing a massive spin-2 particle. The spin-0 particle is represented, for example, by the equation

$$(\Box - m^2)h = 0. \tag{3.72}$$

A similar analysis can be done from the Lagrangian (3.17). Nonetheless, as it does not offer us any new idea and it is not simpler than the last one, we have chosen to omit it. With these observations, we finish the classical analyses for the flat case. Before introducing curved backgrounds, let us make some comments on the quantum analysis of unitarity in the next chapter.

4 QUANTUM ANALYSIS: UNITARITY

In this section we review the models studied in previous chapters by considering a quantum perspective associated with unitarity analysis. In this perspective, by physical models we mean absence of ghosts, particles that violate unitarity. In order to check unitarity of a free quantum field theory we can look at the analytic properties of the propagator (VELTMAN, 1981). For a detailed discussion, see (BUONINFANTE, 2016; HERNASKI, 2011). Here we limit our discussion to the rank-2 symmetric tensor case.

Given an action $S = \int d^D x \mathcal{L}$, we write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta} - \kappa h_{\mu\nu} T^{\mu\nu}. \tag{4.1}$$

In our work, the operator $\mathcal{O}^{\mu\nu,\alpha\beta}$ is quadratic in derivatives, and can be written in terms of the spin-2 projection operators (see Appendix). $T^{\mu\nu}$ is a generic source, and it is coupled with the free theory. The equations of motion are

$$\mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta} - \kappa T^{\mu\nu} = 0. \tag{4.2}$$

We are interested in obtaining the propagator of a model. For our purposes, it suffices to know that it is proportional to the inverse of $\mathcal{O}^{\mu\nu,\alpha\beta}$. The projection operators offer an appealing possibility: it is easy to calculate such an inverse $\mathcal{O}^{-1}$ if we write $\mathcal{O}$ in terms of them. In fact, if

$$\mathcal{O} = A\, P^{(2)}_{ss} + B\, P^{(1)}_{ss} + C\, P^{(0)}_{ss} + \tilde{D}\, P^{(0)}_{ww} + E\, P^{(0)}_{sw} + F\, P^{(0)}_{ws}, \tag{4.3}$$

then

$$\mathcal{O}^{-1} = \frac{1}{A} P^{(2)}_{ss} + \frac{1}{B} P^{(1)}_{ss} + \frac{\tilde{D}}{R} P^{(0)}_{ss} + \frac{C}{R} P^{(0)}_{ww} - \frac{F}{R} P^{(0)}_{sw} - \frac{E}{R} P^{(0)}_{ws}, \tag{4.4}$$

where $R = C\tilde{D} - EF$. There is a subtlety though: if B or R are zero[1], $\mathcal{O}$ is not invertible. However, when $\mathcal{O}$ is singular, it is possible to introduce suitable gauge fixing terms into the Lagrangian and obtain a new operator, which is invertible and whose current-current amplitude does not depend on the gauge fixing parameter. By current-current amplitude we mean the following quantity built with the sources

$$\mathcal{A} = -iT^{*\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\alpha\beta} T^{\alpha\beta}. \tag{4.5}$$

The method then tells us to write the operators in the momenta space ($p_\mu = -i\partial_\mu$), and analyse the sign of the imaginary part of the residues of the amplitude evaluated at the poles of the propagator. If

$$ImRes(\mathcal{A}(p))_{|p^2 \to -M^2} > 0, \tag{4.6}$$

ghosts are absent and we have a physical (unitary) model describing a particle of mass M.

As an application, let us prove that $\beta = 1$ in (1.1) is necessary to eliminate spin-1 propagation.

[1] $A \neq 0$, otherwise we would not have spin-2 propagation.

Consider the following action

$$S_\beta = \int d^D x \left[\frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu} - \frac{\beta}{2} h_\mu{}^\alpha \partial_\alpha \partial_\nu h^{\mu\nu} + \frac{a}{2}\, h \partial_\mu \partial_\nu h^{\mu\nu} - \frac{b}{4} h \Box h \right], \tag{4.7}$$

We can write the operator $\mathcal{O}$ as

$$\mathcal{O}_\beta = \frac{\Box}{2} P_{ss}^{(2)} + \left(\frac{1-\beta}{2} \right) \Box\, P_{ss}^{(1)} + \left[\frac{1 - b(D-1)}{2} \right] \Box P_{ss}^{(0)} + \left(\frac{1 - 2\beta + 2a - b}{2} \right) \Box P_{ww}^{(0)} + \\ + \frac{(a-b)\sqrt{D-1}}{2} \Box (P_{sw}^{(0)} + P_{ws}^{(0)}), \tag{4.8}$$

which means that $\beta = 1$ gets rid of the vector degrees of freedom, justifying $\beta = 1$ in (1.1).

4.1 FIERZ-PAULI

Let us briefly review the Fierz-Pauli case (3.1). For details, see (BUONINFANTE, 2016). It is simpler than LEH because it has no gauge symmetry:

$$\mathcal{L}_{FP} = -\frac{1}{4} \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} + \frac{1}{2} \partial_\mu h^{\mu\nu} \partial^\alpha h_{\alpha\nu} - \frac{1}{2} \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2). \tag{4.9}$$

The operator is

$$\mathcal{O}_{FP} = \frac{(\Box - m^2)}{2} P_{ss}^{(2)} - \frac{m^2}{2} P_{ss}^{(1)} + \frac{(2-D)(\Box - m^2)}{2} P_{ss}^{(0)} + \frac{m^2 \sqrt{D-1}}{2} (P_{sw}^{(0)} + P_{ws}^{(0)}), \tag{4.10}$$

and its inverse is

$$\mathcal{O}_{FP}^{-1} = \frac{2 P_{ss}^{(2)}}{(\Box - m^2)} - \frac{2 P_{ss}^{(1)}}{m^2} + \frac{2(D-2)(\Box - m^2)}{(D-1) m^4} P_{ww}^{(0)} - \frac{2(P_{sw}^{(0)} + P_{ws}^{(0)})}{m^2 \sqrt{D-1}}. \tag{4.11}$$

Therefore,

$$\mathcal{A}_{FP}(p) = 2i T^{*\mu\nu} \left[\frac{P_{ss}^{(2)}}{p^2 + m^2} \right]_{\mu\nu\alpha\beta} T^{\alpha\beta}, \tag{4.12}$$

which has a massive pole at $p^2 = -m^2$. We have

$$\begin{aligned} ImRes(\mathcal{A}_{FP}(p))_{|p^2 \to -m^2} &= Im \lim_{p^2 \to -m^2} \left[(p^2 + m^2)(\mathcal{A}_{FP}(p)) \right] \\ &= 2 T^{*\mu\nu} (P_{ss}^{(2)})_{\mu\nu\alpha\beta} T^{\alpha\beta}. \end{aligned} \tag{4.13}$$

It is possible to show that, in the rest frame $k^\mu = (m, 0, \cdots, 0)$,

$$ImRes(\mathcal{A}_{FP}(p))_{|p^2 \to 0} = 4 \sum_{i>j=1}^{D-1} |T_{ij}|^2 + \sum_{i=2}^{D-1} |T_i|^2 \geq 0. \tag{4.14}$$

In the last expression, the set of T_i are objects obtained from linear combinations $(T_{11} - T_{ii})$, for $i = 2, \cdots, D-1$. As (4.12) is Lorentz invariant, our proof holds for other Lorentz-related frames. We conclude that the model is unitary for $D \geq 3$. For a step-by-step derivation see, for example, (NOGUEIRA, 2018).

4.2 EINSTEIN-HILBERT

Now we consider the LEH case, given by (3.1) when $m = 0$. It is easy to show that the operator is given by $\mathcal{O}_{FP}(m = 0)$. According to (4.10), this operator is singular. Indeed, this was expected since we are dealing with the Diff model, which allows us to choose a gauge condition, for example, the de Donder one. As a result of this, $\mathcal{L}_{EH} \to \mathcal{L}_{EH} + \lambda(\partial_\mu h^{\mu\nu} - 1/2\partial^\nu h)^2$. We evaluate

$$\mathcal{O}_{EH} = \frac{\Box}{2}P_{ss}^{(2)} - \frac{\lambda\Box}{2}P_{ss}^{(1)} - \frac{\lambda\Box}{4}P_{ww}^{(0)} + \frac{(4-2D+\lambda(1-D))\Box}{4}P_{ss}^{(0)} + \frac{\lambda\sqrt{D-1}\Box}{4}(P_{sw}^{(0)} + P_{ws}^{(0)}), \tag{4.15}$$

and its inverse

$$\mathcal{O}_{EH}^{-1} = \frac{2P_{ss}^{(2)}}{\Box} - \frac{2P_{ss}^{(1)}}{\lambda\Box} - 2\frac{4-2D+\lambda(1-D)}{\lambda(2-D)\Box}P_{ww}^{(0)} + \frac{2P_{ss}^{(0)}}{(2-D)\Box} + \frac{2\sqrt{D-1}}{(2-D)\Box}(P_{sw}^{(0)} + P_{ws}^{(0)}). \tag{4.16}$$

Although the inverse depends on the parameter λ, the amplitude does not. In fact, from the gauge symmetry $\delta h_{\mu\nu} = \partial_\mu\phi_\nu + \partial_\nu\phi_\mu$, and from (4.1), we have conserved sources $\partial_\mu T^{\mu\nu} = 0$, or $k_\mu T^{\mu\nu} = 0$. Consequently, the source satisfies $(P_{ww}^{(0)})_{\mu\nu\alpha\beta}T^{\alpha\beta} = 0$ and $T^{*\mu\nu}(P_{sw}^{(0)} + P_{ws}^{(0)})_{\mu\nu\alpha\beta}T^{\alpha\beta} = 0$. Furthermore, by multiplying (4.2) by $P_{ss}^{(1)}$, we have $(P_{ss}^{(1)})_{\mu\nu\alpha\beta}T^{\alpha\beta} = 0$. Therefore,

$$\mathcal{A}_{EH}(p) = 2iT^{*\mu\nu}\left[\frac{P_{ss}^{(2)}}{p^2} + \frac{P_{ss}^{(0)}}{(2-D)p^2}\right]_{\mu\nu\alpha\beta}T^{\alpha\beta}, \tag{4.17}$$

which has a massless pole at $p^2 = 0$. We have

$$\begin{aligned} ImRes(\mathcal{A}_{EH}(p))_{|p^2\to 0} &= Im \lim_{p^2\to 0}\left[p^2(\mathcal{A}_{EH}(p))\right] \\ &= 2T^{*\mu\nu}\left(P_{ss}^{(2)} - \frac{P_{ss}^{(0)}}{D-2}\right)_{\mu\nu\alpha\beta}T^{\alpha\beta}. \end{aligned} \tag{4.18}$$

It is possible to show that, in the frame $k^\mu = (\kappa, 0, 0, \cdots, 0, \kappa)$,

$$ImRes(\mathcal{A}_{EH}(p))_{|p^2\to 0} = 4\sum_{i>j=1}^{D-2}|T_{ij}|^2 + \sum_{i=2}^{D-2}|T_i|^2 \geq 0. \tag{4.19}$$

In the last expression, the set of T_i are objects obtained from linear combinations $(T_{11} - T_{ii})$, for $i = 2, \cdots, D-2$. As (4.17) is Lorentz invariant, our proof holds for other Lorentz-related frames. Notice that we do not have terms for $D = 3$, i.e. there are no (massless) gravitons. For $D \geq 4$ the model is unitary.

It is interesting to compare the LEH amplitude with the massless limit of Fierz-Pauli one. We only have to use explicit expressions for the projection operators. From (4.12) we have

$$\mathcal{A}_{FP}(p) \backsim \frac{i}{p^2 - m^2} T^{*\mu\nu} \left[\frac{1}{2} \left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha} \right) - \frac{1}{D-1} \theta_{\mu\nu}\theta_{\alpha\beta} \right] T^{\alpha\beta}, \tag{4.20}$$

whereas from (4.17)

$$\mathcal{A}_{EH}(p) \backsim \frac{i}{p^2} T^{*\mu\nu} \left[\frac{1}{2} \left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha} \right) - \frac{1}{D-2} \theta_{\mu\nu}\theta_{\alpha\beta} \right] T^{\alpha\beta}. \tag{4.21}$$

It is worthy to point out two properties. The first one is relevant when it comes to dimensional reduction: the structure of the amplitudes are related by $EH(D+1) \longrightarrow FP(D)$. This means that we have unitary massive gravity for $D \geq 3$. The second property is about the massless limit of Fierz-Pauli: it is different from Einstein-Hilbert! This difference is a manifestation of the well-known vDVZ discontinuity (VELTMAN; DAM, 1970).

4.3 TDIFF

Now we analyse the TDiff model given by (2.1). The operator is again singular,

$$\mathcal{O}_T = \frac{\Box}{2} P_{ss}^{(2)} + \frac{(1 - b(D-1))\Box}{2} P_{ss}^{(0)} + \frac{(2a - b - 1)\Box}{2} P_{ww}^{(0)} + \frac{\sqrt{D-1}(a-b)\Box}{2} (P_{sw}^{(0)} + P_{ws}^{(0)}), \tag{4.22}$$

as a consequence of the TDiff symmetry. Since $P_{ss}^{(1)} \mathcal{O}_T h = 0$, from (4.2) we have $P_{ss}^{(1)} T = 0$. However, after conveniently gauge fixing

$$\mathcal{L}_{TDiff} \to \mathcal{L}_{TDiff} + \frac{\lambda}{2} (\partial_\alpha \partial_\mu \partial_\nu h^{\mu\nu} - \Box \partial^\mu h_{\alpha\mu})^2 = \mathcal{L}_{TDiff} + \frac{\lambda}{2} h_{\mu\nu} \left(-\frac{\Box^3}{2} P_{ss}^{(1)} \right)^{\mu\nu\alpha\beta} h_{\alpha\beta}, \tag{4.23}$$

we calculate the inverse operator

$$\mathcal{O}_T^{-1} = \frac{2P_{ss}^{(2)}}{\Box} - \frac{2P_{ss}^{(1)}}{\lambda \Box^3} - 2\frac{2a - b - 1}{d_0 \Box} P_{ss}^{(0)} - 2\frac{1 - b(D-1)}{d_0 \Box} P_{ww}^{(0)} + \frac{2(a-b)\sqrt{D-1}}{d_0 \Box} (P_{sw}^{(0)} + P_{ws}^{(0)}), \tag{4.24}$$

where d_0 is given by (1.7). Consequently, as $P_{ss}^{(1)} T = 0$, the amplitude does not depend on the λ parameter and, by omitting indices, it is given by

$$\mathcal{A}_T(p) = 2iT^* \left[\frac{P_{ss}^{(2)}}{p^2} - \frac{2a - b - 1}{d_0} \frac{P_{ss}^{(0)}}{p^2} - \frac{1 - b(D-1)}{d_0} \frac{P_{ww}^{(0)}}{p^2} + \frac{(a-b)\sqrt{D-1}}{d_0} \frac{(P_{sw}^{(0)} + P_{ws}^{(0)})}{p^2} \right] T. \tag{4.25}$$

The most general source satisfying TDiff symmetry is not conserved, i.e. $\partial^\mu T_{\mu\nu} \neq 0$, unlike the LEH source. Basically, TDiff enforces us to work with a source whose longitudinal part is a gradient. From (4.1), under TDiff we have

$$\delta S = 0 \Rightarrow 0 = \int d^D x \delta h_{\mu\nu} T^{\mu\nu} = 2 \int d^D x \partial_\mu \xi_\nu^T T^{\mu\nu} = -2 \int d^D x \xi_\nu^T \partial_\mu T^{\mu\nu} \Rightarrow \partial_\mu T^{\mu\nu} = \partial^\nu J. \tag{4.26}$$

Hence, written in the momenta space, it satisfies

$$p^\mu T_{\mu\nu} = p_\nu J, \tag{4.27}$$

for some arbitrary scalar function $J(x)$. Here we point out that (ALVAREZ et al., 2006) considered a less general source in their analysis, using the trace of the source instead of an arbitrary scalar function. Let us then generalize their result. From (4.27) we have $p^\mu p^\nu T_{\mu\nu} = p^2 J \Rightarrow \omega^{\mu\nu} T_{\mu\nu} = J \Rightarrow T^* P^{(0)}_{ww} T = J^2$. On the other hand, if $T \equiv \eta^{\mu\nu} T_{\mu\nu}$, $\theta^{\mu\nu} T_{\mu\nu} = T - J \Rightarrow T^* P^{(0)}_{ss} T = \dfrac{|T-J|^2}{D-1}$. In addition, $T^* P^{(0)}_{sw} T = \dfrac{J^*(T-J)}{2\sqrt{D-1}} + h.c. = T^* P^{(0)}_{ws} T$. By writing $T^* P^{(2)}_{ss} T = T^{*\mu\nu} T_{\mu\nu} - T^* P^{(0)}_{ss} T - T^* P^{(0)}_{ww} T$, we rewrite the amplitude as

$$\mathcal{A}_T(p) = \frac{2i}{p^2}\left[T^{*\mu\nu}T_{\mu\nu} - \frac{a^2-b}{d_0}|T-J|^2 - \frac{1-b(D-1)+d_0}{d_0}|J|^2 + \frac{(a-b)}{d_0}(J^*(T-J)+h.c)\right]. \tag{4.28}$$

For simplicity[2], we decompose the source in a particular basis, see (BITTENCOURT, 2019), given by $k^\mu = (1,0,\ldots,0,1)$, $\tilde{k}^\mu = (1,0,\ldots,0,-1)$, $\epsilon^1_\mu = (0,1,0,\ldots,0,0)$, $\epsilon^2_\mu = (0,0,1,\ldots,0,0)$, until, $\epsilon^{(D-2)}_\mu = (0,0,0,\ldots,1,0)$. In such a basis, we can write a rank-2 symmetrical tensor, in particular the source, as

$$T_{\mu\nu} = c_{ij}\varepsilon^i_\mu \varepsilon^j_\nu + b_i(\varepsilon^i_\mu k_\nu + \varepsilon^i_\nu k_\mu) + \tilde{b}_i(\varepsilon^i_\mu \tilde{k}_\nu + \varepsilon^i_\nu \tilde{k}_\mu) + \tilde{c}(k_\mu \tilde{k}_\nu + k_\nu \tilde{k}_\mu) + \tilde{d} k_\mu k_\nu + f \tilde{k}_\mu \tilde{k}_\nu, \tag{4.29}$$

where $i = 1,2,\ldots,D-2$, $c_{ij} = c_{ji}$, and we have light-like momenta vectors (as it is should be for a massless case) $k_\mu k^\mu = 0 = \tilde{k}_\mu \tilde{k}^\mu$ such that $k_\mu \tilde{k}^\mu = -2$ and $k^\mu \varepsilon_\mu = 0 = \tilde{k}^\mu \varepsilon_\mu$. The reader can verify that (4.29) describes the $D(D+1)/2$ independent components of $T_{\mu\nu}$. The coefficients are not independent due to (4.27) since, by choosing $p^\mu = k^\mu$, $k^\mu T_{\mu\nu} = k_\nu J$ implies

$$-2\tilde{b}_i \varepsilon^i_\nu - 2\tilde{c} k_\nu - 2f\tilde{k}_\nu = k_\nu J \Rightarrow f = 0 = \tilde{b}_i,\ \tilde{c} = -\frac{J}{2}. \tag{4.30}$$

We have $T^{*\mu\nu}T_{\mu\nu} = c^*_{ij}c_{ij} + 2|J|^2$ and $T = \delta_{ij}c_{ij} + 2J = c + 2J$. But, we can also decompose $c_{ij} = \bar{c}_{ij} + \dfrac{\delta_{ij}}{D-2}c$, where $\delta_{ij}\bar{c}_{ij} = 0$. Then, it is easy to show from (4.28) that

$$ImRes(\mathcal{A}_T(p))_{|p^2\to 0} = 2[\underbrace{\bar{c}^*_{ij}\bar{c}_{ij}}_{\geq 0} + \underbrace{\frac{(a-1)^2}{d_0(D-2)}|c|^2 + \frac{a^2(D-2)}{d_0}|J|^2 + \frac{a(1-a)}{d_0}(c^*J + h.c)}_{\dfrac{|(a-1)c - a(D-2)J|^2}{d_0(D-2)}}]. \tag{4.31}$$

Hence, if $d_0 > 0$,

$$ImRes(\mathcal{A}_T(p))_{|p^2\to 0} \geq 0, \tag{4.32}$$

i.e. TDiff is unitary $\forall (a,b) \in \mathbb{R}^2$ such that $d_0 > 0$. On the other hand, if $d_0 < 0$, we have, for generic sources, a scalar ghost.

[2] We thank Raphael Schimidt Bittencourt for suggesting the basis.

4.4 MASSIVE SCALAR-TENSOR MODEL

Last, let us analyse (3.64). The first piece of new information here is that now we have a scalar field that hinders us from writing $\mathcal{O}$ as a quadratic in the derivatives operator. In fact, the best we can do is to integrate on ϕ, and substitute it in the Lagrangian. Then, (3.64) becomes[3]

$$\mathcal{L} = \frac{1}{4} h_{\mu\nu}(\Box - m^2)h^{\mu\nu} - \frac{1}{2} h_{\mu}{}^{\alpha}\partial_\alpha\partial_\nu h^{\mu\nu} + x\, h\partial_\mu\partial_\nu h^{\mu\nu} + y\, h(\Box - m^2)h + f\, h^{\mu\nu}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta h^{\alpha\beta}, \quad (4.33)$$

where $x = \frac{1}{2}\left(1 + \frac{(a-1)^2}{2a-b-1}\right)$, $y = -\frac{1}{4}\left(1 + \frac{(a-1)^2}{2a-b-1}\right)$, and $f = -\frac{(a-1)^2}{4(\Box - m^2)(2a-b-1)}$. Consequently, the operator $\mathcal{O}$ is given by

$$\mathcal{O} = \frac{1}{2}(\Box - m^2)P^{(2)}_{ss} - \frac{m^2}{2}P^{(1)}_{ss} + \frac{a_{ss}}{2}P^{(0)}_{ss} + \frac{a_{ww}}{2}P^{(0)}_{ww} + \frac{a_{sw}\sqrt{D-1}}{2}(P^{(0)}_{ws} + P^{(0)}_{sw}), \quad (4.34)$$

where $a_{ss} = (\Box - m^2)\left[(2-D) - \frac{(D-1)(a-1)^2}{2a-b-1}\right]$, $a_{ww} = -\frac{m^4(a-1)^2}{(2a-b-1)(\Box - m^2)}$, and also $a_{sw} = \frac{m^2(a^2-b)}{2a-b-1}$. The inverse is given by (4.4), where $R = -\frac{D_0}{2a-b-1}$, with $D_0 = 1 - 2a + a^2D - b(D-1)$. The amplitude reads as

$$\mathcal{A}(p) = 2iT^*\left[\frac{P^{(2)}_{ss}}{p^2+m^2} + \frac{(a-1)^2}{D_0}\frac{P^{(0)}_{ss}}{(p^2+m^2)}\right]T, \quad (4.35)$$

which means

$$ImRes(\mathcal{A})_{|p^2\to -m^2} = \underbrace{2T^*P^{(2)}_{ss}T}_{ImRes(\mathcal{O}_{FP})} + \underbrace{2\frac{(a-1)^2}{D_0}T^*P^{(0)}_{ss}T}_{\mathcal{A}_0}. \quad (4.36)$$

We have already shown that $ImRes(\mathcal{O}_{FP}) \geq 0$. In the frame $p^\mu = (m, \vec{0})$, it is easy to show that $T^{*ij}(P^{(0)}_{ss})_{ijkl}T^{kl} = \frac{|T|^2}{D-1}$, where $T = \delta_{ij}T^{ij}$. Then, $\mathcal{A}_0 = \frac{(a-1)^2|T|^2}{(D-1)D_0} \geq 0$ if $D_0 > 0$. The conclusion is that, for $D_0 > 0$,

$$ImRes(\mathcal{A})_{|p^2\to -m^2} > 0. \quad (4.37)$$

In this case, we have healthy massive spin-2 and spin-0 particles. For $D_0 < 0$ we have ghosts. Finally, if $a = 1$ we have a decoupled Lagrangian $\mathcal{L} = \mathcal{L}_{FP} + \mathcal{L}_\phi$, where it is clear that we have a massive graviton and a massive scalar particle, which is healthy if $b < 1$, or $D_0 > 0$ since $D_0 = (D-1)(1-b)$.

[3] The case when $b = 2a - 1$ should be analysed separately since in this case the Lagrangian is linear in ϕ, which becomes a mere auxiliary field. However, as mentioned in the last chapter, we can redefine $h_{\mu\nu}$ and make the term $\phi\Box\phi$ reappears. For this reason, we are not computing this case here.

5 SPIN-2 FIELD THEORY IN CURVED BACKGROUNDS

In this chapter, we start the second part of the project: spin-2 field theories in curved backgrounds. Mainly driven by (BUCHBINDER; KRYKHTIN; PERSHIN, 1999), we will analyse the consistency of massless and massive Diff and TDiff equations of motion in the presence of external gravitational fields. It is the purpose of this chapter and the next one to investigate under which conditions we have the correct number of degrees of freedom in a curved background. By external field, we mean a field that is not dynamical so that the background is fixed. The full metric is given by

$$g_{\mu\nu}(h) = g_{\mu\nu}(h=0) + \lambda\, h_{\mu\nu}, \tag{5.1}$$

which should be understood as some perturbation $h_{\mu\nu}$ around a curved background $g_{\mu\nu}(h=0) \equiv g_{\mu\nu}$, for some constant λ. In this parametrization, recall that $h^{\mu\alpha} = g^{\mu\nu} g^{\alpha\beta} h_{\nu\beta}$. If $g_{\mu\nu} = \eta_{\mu\nu}$, the background is Minkowski - the flat limit - and by expanding the Einstein-Hilbert action S_{GR}, given by (1.10), around $\eta_{\mu\nu}$ up to second order in $h_{\mu\nu}$, we obtain the linearized Einstein-Hilbert action S_{LEH} (1.13). On the other hand, if $g_{\mu\nu}$ has a non-trivial curvature, another action will arise after expanding S_{GR}. Concerning degrees of freedom[1], see next section, it is necessary that the spin-2 massless action is invariant under Diff:

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{5.2}$$

where the covariant derivatives are defined in terms of the background metric $g_{\mu\nu}$. Kindly remember that, in the presence of curvature, the derivatives are replaced by covariant derivatives, which are defined in terms of the Christoffel symbol. In addition, we have the Riemann and Ricci tensors, and the Ricci scalar. When acting on tensor objects, the covariant derivatives do not commute and the calculations are trickier. In the flat limit, all of these quantities vanish and we are back to S_{LEH}. We are excluding torsion and assuming metric compability. It is a well-known fact that a massless spin-2 particle can only consistently propagate in an Einstein space (ARAGONE; DESER, 1980; DESER, 1987; DESER; HENNEAUX, 2007):

$$R_{\mu\nu} = \frac{R}{D} g_{\mu\nu}, \tag{5.3}$$

which is a solution of the Einstein equations in the vacuum with a cosmological constant $\Lambda = \dfrac{D-2}{2D} R$. It includes interesting spaces, like the maximally symmetric spaces such as de Sitter (dS) or anti-de Sitter (AdS). In fact, the resulting action can be obtained from the expansion up to second order of S_{GR} around a background metric satisfying (5.3), see for example (HIGUCHI, 1987). Alternatively, we will impose invariance of a general action under (5.2), and end up with the restriction (5.3) on the background metric. By demanding TDiff invariance, we also need Einstein spaces.

We will show that, from a general quadratic, and consistent with the flat limit spin-2 massive action on a curved background, the only possibility for massive gravitons is to propagate in Einstein spaces,

[1] Here we are assuming that the property of particles as irreducible representations of the Poincaré group is also valid on curved backgrounds. So, consistency with the flat limit requires that spin-2 (massive) massless particles propagate (5) 2 degrees of freedom in $D = 3 + 1$.

following (BUCHBINDER; KRYKHTIN; PERSHIN, 1999; FORTES, 2018). Once in the flat limit we must recover the flat action, in these arbitrary backgrounds, we must promote flat derivatives to covariant derivatives but, in general, it is necessary to add non-minimal terms containing curvature tensors.

5.1 PRELUDE: CONSTRAINTS IN THE FLAT LIMIT

Consider the Fierz-Pauli (FP) flat action $S_{FP} = \int d^D x \, \mathcal{L}_{FP}$,

$$\mathcal{L}_{FP} = -\frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu{}_\rho - \frac{1}{2}\partial^\mu h \partial^\nu h_{\mu\nu} + \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2). \quad (5.4)$$

From its (Euler-Lagrange) equations of motion $E_{\mu\nu} = 0$, we derive the Klein-Gordon equation $(\Box - m^2)h_{\mu\nu} = 0$, as well as the $D+1$ constraints

$$\phi_\nu = \partial^\mu E_{\mu\nu} = m^2\partial_\nu h - m^2\partial^\mu h_{\mu\nu} = 0, \quad (5.5)$$

$$\phi = \frac{m^2}{D-2}\eta^{\mu\nu}E_{\mu\nu} + \partial^\mu\partial^\nu E_{\mu\nu} = \frac{D-1}{D-2}m^4 h = 0, \quad (5.6)$$

which reduce the configuration space to the desired number of degrees of freedom, i.e 5 in $D = 3+1$. Now, if we wish to establish a massive spin-2 action on a curved space background, we should be able to build exactly $D+1$ constraints from the equations of motion $E_{\mu\nu} = 0$, in order to have 5 degrees of freedom in $D = 3+1$. Recall that in the flat limit we must recover FP. Taking this into account, let us generalize S_{FP} to a curved background, following (BUCHBINDER; KRYKHTIN; PERSHIN, 1999).

5.2 GENERALIZING FIERZ-PAULI TO A CURVED BACKGROUND

Since we must recover FP in the flat limit, we replace the derivatives in (3.1) by covariant derivatives (with respect to the background metric $g_{\mu\nu}$) and add non-minimal terms. Recalling that $d^D x\sqrt{-g}$ is the invariant volume element, we establish the notation $S \equiv \int d^D x \sqrt{-g}\mathcal{L}$. The most general quadratic Lagrangian up to second order in derivatives is written as

$$\mathcal{L} = -\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h \nabla^\mu h - \frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \\ + \frac{a_1}{2}R h_{\mu\nu}h^{\mu\nu} + \frac{a_2}{2}R h^2 + \frac{a_3}{2}R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \frac{a_4}{2}R^{\mu\beta}h_{\mu\nu}h^\nu{}_\beta + \frac{a_5}{2}R^{\mu\nu}h_{\mu\nu}h. \quad (5.7)$$

Recall that there is an arbitrariness in the definition of minimal coupling. It is ambiguous due to the non-commutation of covariant derivatives: the terms $\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha}$ and $\nabla^\mu h_{\mu\nu}\nabla_\alpha h^{\nu\alpha}$ are only equivalent in the flat case; here, they are actually different. The difference in choosing the first or the second form is given by curvature non-minimal terms, which are already being taken into account with arbitrary coefficients, so that there is no problem in such an ambiguity.

The equations of motion are

$$E_{\mu\nu} \equiv \Box h_{\mu\nu} - \nabla^\alpha(\nabla_\nu h_{\mu\alpha} + \nabla_\mu h_{\nu\alpha}) + \nabla_\nu \nabla_\mu h + g_{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta} - g_{\mu\nu}\Box h - m^2(h_{\mu\nu} - g_{\mu\nu}h) +$$
$$+ 2a_1 R h_{\mu\nu} + 2a_2 R g_{\mu\nu} h + 2a_3 R_{\mu\alpha\nu\beta} h^{\alpha\beta} + a_4(R_{\mu\beta}h_\nu{}^\beta + R_{\nu\beta}h_\mu{}^\beta) + a_5(R_{\mu\nu}h + g_{\mu\nu}R^{\alpha\beta}h_{\alpha\beta}) = 0. \tag{5.8}$$

Keeping in mind the commutation relation (D.4), we have

$$\nabla^\mu E_{\mu\nu} = (2a_1 R - m^2)\nabla^\mu h_{\mu\nu} + (2a_2 R + m^2)\nabla_\nu h + 2a_3 R_{\mu\alpha\nu\beta}\nabla^\mu h^{\alpha\beta} + a_4 R^{\mu\alpha}\nabla_\mu h_{\alpha\nu}$$
$$+ (a_4 - 2)R_{\alpha\nu}\nabla_\mu h^{\alpha\mu} + a_5 R^{\mu\alpha}\nabla_\nu h_{\mu\alpha} + (a_5 + 1)R^{\mu\nu}\nabla_\mu h + \cdots = 0, \tag{5.9}$$

where $\cdots$ denote terms not involving the derivatives of $h_{\mu\nu}$ since we have derivatives of curvature terms. So, as there are no second derivative of $h_{\mu\nu}$, we have D vector constraints $\phi_\nu = \nabla^\mu E_{\mu\nu} = 0$, for any curved background and arbitrary values of $a_1, \ldots, a_5$.

Just like (5.7) contains non-minimal terms, the curved generalization of (5.6) should contain additional terms too, addressed by two constants c_1 and c_2,

$$\phi = \frac{m^2}{D-2} g^{\mu\nu} E_{\mu\nu} + \nabla^\mu \nabla^\nu E_{\mu\nu} + c_1 R g^{\mu\nu} E_{\mu\nu} + c_2 R^{\mu\nu} E_{\mu\nu}. \tag{5.10}$$

With the traceless Ricci tensor $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{R}{D} g_{\mu\nu}$, and by writing the Riemann tensor in terms of the Ricci and (traceless) Weyl tensors (see Appendix), we have[2]

$$\phi = C_1 R \nabla^\mu \nabla^\nu h_{\mu\nu} + C_2 R \Box h + 2a_3 C^{\mu\alpha\nu\beta}\nabla_\mu\nabla_\nu h_{\alpha\beta} + \left[2(a_4 - 1) - 2c_2 - \frac{4a_3}{D-2}\right]\tilde{R}^{\mu\nu}\nabla_\mu\nabla^\alpha h_{\alpha\nu} +$$
$$+ (a_5 + c_2 + \frac{2a_3}{D-2})\tilde{R}^{\mu\nu}\Box h_{\mu\nu} + (a_5 + c_2 + \frac{2a_3}{D-2} + 1)\tilde{R}^{\mu\nu}\nabla_\mu\nabla_\nu h + o(\nabla h, h) = 0, \tag{5.11}$$

where

$$C_1 = 2a_1 + c_1(D-2) + c_2\frac{(D-2)}{D} - \frac{2a_3}{D(D-1)} + \frac{2a_4}{D} - \frac{2}{D},$$
$$C_2 = 2a_2 - c_1(D-2) - c_2\frac{(D-2)}{D} + \frac{2a_3}{D(D-1)} + \frac{2a_5}{D} + \frac{1}{D}.$$

Now, note that if we want to get rid of all the second derivatives (not only the accelerations), each bracket above should vanish, but this is not possible (see the second line of (5.11)!). In order that ϕ be a scalar constraint, the gravitational background **must** fulfil the condition $\tilde{R}_{\mu\nu} = 0$, i.e.

$$R_{\mu\nu} = \frac{1}{D} g_{\mu\nu} R. \tag{5.12}$$

Curved manifolds satisfying this condition are known as **Einstein spacetimes**. They possess constant curvature since $\nabla_\mu R = 0$, which can be easily seen from the Bianchi identity $\nabla^\mu G_{\mu\nu} = \nabla^\mu(R_{\mu\nu} - 1/2 R g_{\mu\nu}) = 0$. Notice that (5.12) fixes the Ricci-part of the Riemann tensor, but the Weyl tensor is

[2] The coefficients of $g^{\mu\nu}E_{\mu\nu}$ and $\nabla^\mu\nabla^\nu E_{\mu\nu}$ in (5.10) assure that the terms $m^2\nabla^\mu\nabla^\nu h_{\mu\nu}$ and $m^2\Box h$ vanish.

kept arbitrary. From an operational viewpoint, it simplifies a lot the calculations.

Furthermore, also from (5.11), we have to impose $a_3 = 0$, otherwise the Weyl tensor would have to vanish, which is a more restrictive condition (maximally symmetric spacetimes).

Since R contains the information about $R_{\mu\nu}$, due to (5.12), the coefficients a_4 and a_5 can be absorbed into a_1 and a_2, see (5.7). The same for c_2 (into c_1). So, it is reasonable to set $a_4 = 0 = a_5$ (and $c_2 = 0$). Thus,

$$\begin{cases} C_1 = 2a_1 + c_1(D-2) - \dfrac{2}{D} = 0, \\ C_2 = 2a_2 - c_1(D-2) + \dfrac{1}{D} = 0, \end{cases} \tag{5.13}$$

which means that $2a_1 + 2a_2 - 1/D = 0$ and $c_1(D-2) = 2/D - 2a_1$. Again following (BUCHBINDER; KRYKHTIN; PERSHIN, 1999), $a_1 \equiv \dfrac{\gamma}{D} \Rightarrow a_2 = \dfrac{1-2\gamma}{2D}$, and the action, which describes the massive graviton in Einstein spaces, is

$$S_{\text{FP}} = \int d^D x\sqrt{-g}\, [-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h + \\ -\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \frac{\gamma}{2D}R\, h_{\mu\nu}h^{\mu\nu} + \frac{1-2\gamma}{4D}R\, h^2]. \tag{5.14}$$

Notice that γ is arbitrary and **minimal coupling** is not consistent. The last two terms cannot be simultaneously zero for a non-vanishing scalar curvature. From the equations of motion, we have

$$\nabla^\mu E_{\mu\nu} = \left[\frac{2(1-\gamma)}{D}R + m^2\right](\nabla_\nu h - \nabla^\mu h_{\mu\nu}) = 0, \tag{5.15}$$

and from the scalar constraint $\phi = \left(\dfrac{m^2}{D-2} + \dfrac{2(1-\gamma)R}{D(D-2)}\right) g^{\mu\nu}E_{\mu\nu} + \nabla^\mu\nabla^\nu E_{\mu\nu} = 0$, we get

$$\left[\frac{2(1-\gamma)}{D}R + m^2\right]\left[R - \frac{D(D-1)m^2}{2(D-1)\gamma - D}\right] h = 0. \tag{5.16}$$

If $R \neq \dfrac{Dm^2}{2(\gamma-1)}$ and $R \neq \dfrac{D(D-1)m^2}{2(D-1)\gamma - D}$, then $h = 0$, as well as $\nabla^\mu h_{\mu\nu} = 0$. So, we have derived the usual Fierz-Pauli constraints (they are recovered in the flat limit).

The equations of motion read

$$E_{\mu\nu} = (\Box - m^2)h_{\mu\nu} + 2R_{\alpha\mu\beta\nu}h^{\alpha\beta} + 2\left(\frac{\gamma-1}{D}\right)Rh_{\mu\nu} = 0, \tag{5.17}$$

which is the analog of the Klein-Gordon (KG) equation. As noticed in (FORTES, 2018), KG (first term in (5.17)) could not be obtained in curved backgrounds because of $[\Box - m^2, \nabla^\mu]h_{\mu\nu} \neq 0$.

We have a **partially massless** model if $R = \dfrac{D(D-1)m^2}{2(D-1)\gamma - D}$. In this case, $h \neq 0$, and the action (5.14) has a gauge symmetry $\delta h_{\mu\nu} = \nabla_\mu\nabla_\nu\lambda + \dfrac{m^2}{D-2}g_{\mu\nu}\lambda$, which takes 2 degrees of freedom away from the massive theory, propagating one fewer degree of freedom than the usual[3] (4 in $D = 3+1$).

[3] For this value of R, the scalar constraint is trivially zero. If it was not the gauge symmetry, we would have 6 degrees

For this reason, the corresponding model is said to be partially massless (DESER; NEPOMECHIE, 1984; DESER; WALDRON, 2001; HINTERBICHLER, 2012; FORTES; DALMAZI, 2019).

Note that, for $\gamma \neq 1$, $R = \dfrac{Dm^2}{2(\gamma-1)}$ implies that the vector constraint is identically zero, which means we have found a Diff symmetry for this special value of R. Indeed, by substituting m^2 in (5.14), we obtain (5.24), an action that is invariant under Diff and will be derived in the next section. Hence, this case can be also called a massless one. It is not possible to simultaneously have $R = \dfrac{Dm^2}{2(\gamma-1)}$ and $R = \dfrac{D(D-1)m^2}{2(D-1)\gamma - D}$, since it would demand $D = 2$.

It is interesting to consider the case when $\gamma = 1$, namely

$$\left[\left(\frac{2-D}{D}\right) R + m^2(D-1)\right] h = 0 \tag{5.18}$$

and

$$\nabla_\nu h - \nabla^\mu h_{\mu\nu} = 0. \tag{5.19}$$

Unless $R = m^2 \dfrac{D(D-1)}{D-2}$, we have $h = 0$ and $\nabla^\mu h_{\mu\nu} = 0$. Finally, replacing them in $E_{\mu\nu} = 0$, we derive the Klein-Gordon (KG)-like equation for a curved background:

$$(\Box - m^2) h_{\mu\nu} + 2R_{\alpha\mu\beta\nu} h^{\alpha\beta} = 0, \tag{5.20}$$

which depends now on the Riemann tensor and also tends to KG in the flat limit. In the next section, we will see that $\gamma = 1$ is the choice to be made when requiring Diff symmetry in the massless limit. For $\gamma \neq 1$, Diff symmetry is not recovered when $m \to 0$ because of the curvature terms, only when both $m \to 0$ and $R \to 0$. Therefore, we have reviewed the work of (BUCHBINDER; KRYKHTIN; PERSHIN, 1999).

5.3 THE MASSLESS CASE IN AN EINSTEIN SPACE

We have seen that (5.14) is the most general consistent linearized action in Einstein spaces for a massive graviton. If we set $m \to 0$, we hope to establish an action for the massless graviton. To keep consistency with the LEH flat limit, and in accordance with the gauge symmetry section in this chapter, we expect that this massless action be invariant under diffeomorphisms **Diff**:

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{5.21}$$

It turns out that it is not true for every value of γ. From the equations of motion previously obtained, we see that $\gamma = 1$ is the wisest choice. Indeed, from (5.15), we immediately have a trivial equation $\nabla^\mu E_{\mu\nu} = 0$ for $m = 0$ only if $\gamma = 1$. Since

$$\delta S = \int d^D x \sqrt{-g} \frac{\delta S}{\delta h_{\mu\nu}} \delta h_{\mu\nu} = 2 \int d^D x \sqrt{-g} \frac{\delta S}{\delta h_{(\mu\nu)}} \nabla_\mu \xi_\nu = - \int d^D x \sqrt{-g} \xi_\nu \nabla_\mu E^{\mu\nu} \tag{5.22}$$

of freedom, instead of 5. The symmetry allows us to fix one scalar condition, and the residual symmetry eliminates another degree of freedom: $10 - 4 - 2 = 4$ in $D = 3 + 1$.

where we have introduced the notation

$$E_{\mu\nu} \equiv 2\frac{\delta S[\mathcal{L}]}{\delta h_{\mu\nu}}. \tag{5.23}$$

If $\nabla^\mu E_{\mu\nu} = 0$ we have symmetry $\delta S = 0$. Thus, $\gamma = 1$ is a necessary condition. Therefore, the linearized action for massless spin-2 particles in a curved background is given by[4]

$$S_{\text{Diff}} = \int d^D x\sqrt{-g}\left[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h + \right.$$
$$\left. +\frac{R}{2D}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right)\right] \quad \leftarrow \text{non-minimal terms are necessary to ensure the invariance!} \tag{5.24}$$

From another viewpoint, the action (5.24) can be obtained from S_{GR} in vacuum but with a cosmological constant[5] $\Lambda = (D-2)R/2D$ by expanding, up to second order, the metric around a background Einstein-space. This is the approach[6] of (HIGUCHI, 1987). Is there any other consistent model in an arbitrary background? Apparently, the answer is negative, Einstein spaces are required and (5.24) is the only consistent massless model invariant under Diff, see (DESER; HENNEAUX, 2007).

[4] Notice the presence of non-minimal terms, which are required to assure invariance under Diff. The physical implications of such terms in equations of motion, or in computation of observable quantities, are not discussed in this work.

[5] Recall that Einstein equations in vacuum read $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0$. Then, from $R_{\mu\nu} = R/Dg_{\mu\nu}$, we obtain $\Lambda = \frac{D-2}{2D}R$.

[6] For completeness, in (HIGUCHI, 1987) A. Higuchi was interested in studying unitary spin-2 representation in dS spaces (de Sitter), which is an Einstein space with positive curvature. He found that unitarity holds only if $m^2 > 2\Lambda/(D-1)$, where m is the graviton mass. Notice that the zero mass limit does not make any sense in this case. The same result was obtained by (BENGTSSON, 1995), whose approach is a Hamiltonian formulation. On the other hand, in AdS spaces the massless limit not only exists, but also does not contain vDVZ discontinuity (HINTERBICHLER, 2012).

6 SCALAR-TENSOR IN CURVED BACKGROUNDS

As in the last chapter, we shall proceed by looking for curved space actions, this time for the TDiff model and its massive version. We will conclude that Einstein spaces are also required to describe the correct number of degrees of freedom. Since we have concluded that, even starting from a massless TDiff model, Diff is the symmetry to be broken when giving mass to gravitons in the flat spacetime, we are interested in investigating whether a different conclusion can arise in curved backgrounds. For this reason, we start writing the most general scalar-tensor theory in the presence of an external gravitational background, looking for generalized Diff invariance, which will make it easier to recover Diff case (5.24), the desired TDiff case, and other generalizations. Then, we finish the chapter by studying not only the curved massive extension of TDiff, but also general massive scalar-tensor models in curved backgrounds.

6.1 GENERALIZED DIFF INVARIANCE

The most general second-order linearized scalar-tensor theory in the presence of an **arbitrary** external gravitational background can be written as

$$\mathcal{L} = \mathcal{L}_\nabla + \frac{a_1}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{a_2}{2} R h^2 + \frac{a_3}{2} R^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} + \frac{a_4}{2} R^{\mu\beta} h_{\mu\nu} h^{\nu}{}_{\beta} + \frac{a_5}{2} R^{\mu\nu} h_{\mu\nu} h + \\ + \frac{b_1}{2} R \varphi^2 + \frac{b_2}{2} R h \varphi + \frac{b_3}{2} R^{\mu\nu} h_{\mu\nu} \varphi, \qquad (6.1)$$

where $\mathcal{L}_\nabla$ is written with the minimal coupling as

$$\mathcal{L}_\nabla = -\frac{1}{4} \nabla_\mu h^{\nu\rho} \nabla^\mu h_{\nu\rho} + \frac{1}{2} \nabla_\mu h^{\mu\rho} \nabla_\nu h^{\nu}{}_{\rho} - \frac{a}{2} \nabla^\mu h \nabla^\nu h_{\mu\nu} + \frac{b}{4} \nabla_\mu h \nabla^\mu h + \frac{x}{2} \varphi \nabla_\mu \nabla_\nu h^{\mu\nu} + \frac{y}{2} h \Box \varphi + \frac{z}{4} \varphi \Box \varphi. \qquad (6.2)$$

Now we require invariance of (6.1) under (infinitesimal) generalized Diff

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + c\, g_{\mu\nu} \nabla \cdot \xi, \qquad \delta \varphi = k\, \nabla \cdot \xi, \qquad (c, k) \in \mathbb{R}. \qquad (6.3)$$

Below, we collect the transformation of each term in (6.2) under (6.3).

$$\delta(\nabla_\alpha h_{\mu\nu} \nabla^\alpha h^{\mu\nu}) = 4 \nabla_\alpha h_{\mu\nu} \nabla^\alpha \nabla^\mu \xi^\nu + 2c \nabla_\alpha h \nabla^\alpha \nabla \cdot \xi, \qquad (6.4)$$

$$\delta(\nabla^\mu h_{\mu\nu} \nabla_\alpha h^{\alpha\nu}) = 2 \nabla^\mu h_{\mu\nu} (\Box \xi^\nu + R^{\lambda\nu} \xi_\lambda) + 2(c+1) \nabla_\nu \nabla \cdot \xi \nabla_\alpha h^{\alpha\nu}, \qquad (6.5)$$

$$\delta(\nabla^\mu h \nabla^\nu h_{\mu\nu}) = (c+1) \nabla_\mu h \nabla^\mu \nabla \cdot \xi + \nabla_\mu h R^{\mu\lambda} \xi_\lambda + \nabla_\mu h \Box \xi^\mu + (2 + cD) \nabla_\mu \nabla \cdot \xi \nabla_\nu h^{\mu\nu}, \qquad (6.6)$$

$$\delta(\nabla_\mu h \nabla^\mu h) = (4 + 2cD) \nabla_\mu \nabla \cdot \xi \nabla^\mu h, \qquad (6.7)$$

$$\delta(\nabla_\mu \varphi \nabla_\nu h^{\mu\nu}) = k \nabla_\mu \nabla \cdot \xi \nabla_\nu h^{\mu\nu} + \nabla_\mu \varphi \left[(1+c) \nabla^\mu \nabla \cdot \xi + \Box \xi^\mu + R^{\mu\lambda} \xi_\lambda\right], \qquad (6.8)$$

$$\delta(h \Box \varphi) = (cD + 2) \nabla \cdot \xi \Box \varphi + k h \Box \nabla \cdot \xi, \qquad (6.9)$$

$$\delta(\varphi \Box \varphi) = 2k \varphi \Box \nabla \cdot \xi. \qquad (6.10)$$

This means that (6.2) changes as

$$\delta\mathcal{L}_\nabla = h\Box\nabla\cdot\xi\left[\frac{y}{2}k - \frac{b}{2}(2+cD) + \frac{a}{2}(c+2) + \frac{c}{2}\right] + \varphi\Box\nabla\cdot\xi\left[\frac{zk}{2} + \frac{y}{2}(cD+2) + \frac{x}{2}(2+c)\right] +$$
$$+ \nabla\cdot\xi\nabla_\mu\nabla_\nu h^{\mu\nu}\left[\frac{x}{2}k + \frac{a}{2}(cD+2) - (1+c)\right] + h\nabla_\mu(R^{\mu\lambda}\xi_\lambda)\,[a] + \varphi\nabla_\mu(R^{\mu\lambda}\xi_\lambda)\,[x] +$$
$$+ (\nabla^\mu h_{\mu\nu})R^{\lambda\nu}\xi_\lambda\,[1] + h_{\mu\nu}R^{\lambda\mu}\nabla_\lambda\xi^\nu\,[1] + h_{\mu\nu}R^{\nu\lambda\mu\alpha}(\nabla_\alpha\xi_\lambda)\,[-2] +$$
$$+ h_{\mu\nu}(\nabla_\alpha R^{\nu\lambda\mu\alpha})\xi_\lambda\,[-1] + h_{\mu\nu}\Box\nabla^\mu\xi^\nu\,[0]\,. \tag{6.11}$$

For the non-minimal terms we have

$$\delta(Rh_{\mu\nu}h^{\mu\nu}) = 4Rh_{\mu\nu}\nabla^\mu\xi^\nu + 2cRh\nabla\cdot\xi, \tag{6.12}$$
$$\delta(Rh^2) = 2(cD+2)Rh\nabla\cdot\xi, \tag{6.13}$$
$$\delta(R^{\mu\nu\alpha\lambda}h_{\mu\alpha}h_{\nu\lambda}) = 2R^{\mu\nu\alpha\lambda}h_{\mu\alpha}(\nabla_\nu\xi_\lambda + \nabla_\lambda\xi_\nu) + 2cR^{\mu\nu}h_{\mu\nu}\nabla\cdot\xi, \tag{6.14}$$
$$\delta(R^{\mu\alpha}h_{\mu\nu}h^\nu{}_\alpha) = 2R^{\mu\alpha}h_{\mu\nu}(\nabla_\alpha\xi^\nu + \nabla^\nu\xi_\alpha) + 2cR^{\mu\nu}h_{\mu\nu}\nabla\cdot\xi, \tag{6.15}$$
$$\delta(R^{\mu\nu}h_{\mu\nu}h) = (cD+2)R^{\mu\nu}h_{\mu\nu}\nabla\cdot\xi + 2R^{\mu\nu}h\nabla_\mu\xi_\nu + cRh\nabla\cdot\xi, \tag{6.16}$$
$$\delta(R\varphi^2) = 2kR\varphi\nabla\cdot\xi, \tag{6.17}$$
$$\delta(Rh\varphi) = (cD+2)R\varphi\nabla\cdot\xi + kRh\nabla\cdot\xi, \tag{6.18}$$
$$\delta(R^{\mu\nu}h_{\mu\nu}\varphi) = kR^{\mu\nu}h_{\mu\nu}\nabla\cdot\xi + 2R^{\mu\nu}\varphi\nabla_\mu\xi_\nu + cR\varphi\nabla\cdot\xi. \tag{6.19}$$

Remark 1. Not all terms are independent since we can use the commutation relation (D.4). For example, we have used

$$\nabla^\mu\Box\xi^\nu = \Box\nabla^\mu\xi^\nu - R^{\lambda\mu}\nabla_\lambda\xi^\nu + 2R^{\mu\alpha\nu\lambda}\nabla_\alpha\xi^\lambda + (\nabla_\alpha R^{\nu\lambda\mu\alpha})\xi_\lambda, \tag{6.20}$$
$$\nabla_\mu\Box\xi^\mu = \Box\nabla\cdot\xi + \nabla_\alpha(R^{\alpha\lambda}\xi_\lambda). \tag{6.21}$$

Besides, we can integrate by parts once the Lagrangian is meant to be under integration (action).

Remark 2. Notice in (6.2) that we could have written $(\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu})$ instead of $(\nabla_\mu h^{\mu\rho}\nabla_\nu h^\nu{}_\rho)$, since both are obtainable through minimal coupling of the flat term (modulo boundary term). It is easy to show that

$$\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu} = (\nabla^\mu h_{\mu\nu})^2 + R^{\lambda\nu\mu\alpha}h_{\lambda\mu}h_{\nu\alpha} - R^{\lambda\alpha}h_{\lambda\nu}h^\nu{}_\alpha. \tag{6.22}$$

Therefore, in order to compare our results with those obtained in previous calculation (where it was used $\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu}$, following (BUCHBINDER; KRYKHTIN; PERSHIN, 1999)) we define $a_3 = A_3 + 1$ and $a_4 = A_4 - 1$. The equivalent Lagrangian is then

$$\mathcal{L} = \mathcal{L}'_\nabla + \frac{a_1}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{a_2}{2}Rh^2 + \frac{A_3}{2}R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \frac{A_4}{2}R^{\mu\beta}h_{\mu\nu}h^\nu{}_\beta + \frac{a_5}{2}R^{\mu\nu}h_{\mu\nu}h +$$
$$+\frac{b_1}{2}R\varphi^2 + \frac{b_2}{2}Rh\varphi + \frac{b_3}{2}R^{\mu\nu}h_{\mu\nu}\varphi, \tag{6.23}$$

where $\mathcal{L}'_\nabla$ reads

$$\mathcal{L}'_\nabla = -\frac{1}{4}\nabla_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho}+\frac{1}{2}\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu}-\frac{a}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu}+\frac{b}{4}\nabla_\mu h\nabla^\mu h+\frac{x}{2}\varphi\nabla_\mu\nabla_\nu h^{\mu\nu}+\frac{y}{2}h\Box\varphi+\frac{z}{4}\varphi\Box\varphi. \tag{6.24}$$

Taking into account the transformation of each term of (6.1), we have several coefficients that should vanish. Let us split this analysis in a convenient way. By collecting terms with Riemann tensors, that come from (6.5), (6.14) and (6.20), we have

$$\delta\mathcal{L} = (A_3+1)R^{\mu\nu\alpha\lambda}h_{\mu\alpha}(\nabla_\nu\xi_\lambda+\nabla_\lambda\xi_\nu) - 2h_{\mu\nu}R^{\mu\alpha\nu\lambda}\nabla_\alpha\xi_\lambda - h_{\mu\nu}\xi_\lambda\nabla_\alpha R^{\nu\lambda\mu\alpha} + \ldots . \tag{6.25}$$

Integration by parts gives

$$A_3R^{\mu\nu\alpha\lambda}h_{\mu\alpha}(\nabla_\nu\xi_\lambda+\nabla_\lambda\xi_\nu) - (\nabla_\lambda R^{\mu\lambda\nu\alpha})h_{\mu\nu}\xi_\alpha + \ldots . \tag{6.26}$$

Now we use the fact that we can decompose the Riemann tensor in irreducible parts, being the Weyl tensor one of them (see Appendix). By doing so, and integrating by parts, the last equation contributes with

$$(2A_3+1)h_{\mu\lambda}\xi_\alpha\nabla_\nu C^{\nu\mu\lambda\alpha} - A_3C^{\mu\nu\alpha\lambda}\left(\xi_\lambda\nabla_\nu+\xi_\nu\nabla_\lambda\right)h_{\mu\alpha}, \tag{6.27}$$

which must vanish for any arbitrary functions $h_{\mu\lambda}\xi_\alpha$. We have two sufficient conditions: i) $A_3 = 0$ and $\nabla_\nu C^{\nu\mu\lambda\alpha} = 0$, or ii) $C^{\mu\nu\alpha\lambda} = 0$. The latter one is more drastic since it states that the metric tensor is conformally flat. Let us then assume that i) holds. We can also decompose the Ricci and Riemann tensors in terms of the traceless part of the Ricci tensor and Ricci scalar. Conveniently, we collect the following terms from $\delta\mathcal{L}$

$$\tilde{R}^{\lambda\nu}h_{\mu\nu}\nabla^\mu\xi_\lambda[-1+(A_4-1)-\frac{1}{D-2}-\frac{2A_3}{D-2}]+\tilde{R}^{\lambda\nu}h_{\mu\nu}\nabla_\lambda\xi^\mu[1+(A_4-1)-\frac{1}{D-2}-\frac{2A_3}{D-2}]. \tag{6.28}$$

The coefficients[1] cannot be simultaneously put to zero! Since the invariance must be true for any $h_{\mu\nu}\nabla^\mu\xi_\lambda$, we conclude that

$$\tilde{R}_{\mu\nu} = 0, \tag{6.29}$$

and we have **Einstein spaces** (5.12).

Once we are in Einstein spaces, we can replace the Ricci tensors by Ricci scalars. Then, according to (6.1), we can make the following replacements $a_1 + \frac{A_4}{D} \equiv A_1$, $a_2 + \frac{a_5}{D} \equiv A_2$, and $b_2 + \frac{b_3}{D} \equiv B_2$.

[1] For both brackets, the last two terms come from (6.26), the second one comes from the correspondent non-minimal coupling, whereas the first one comes from the very same term: $(\nabla^\mu h_{\mu\nu})^2$, see (6.5) and (6.20), which basically explains why we have Einstein spaces regardless of the values of a, b, x, y, z. Note that the value of A_3 does not affect the result.

In addition, derivatives of the curvature quantities vanish. We end up collecting the remaining terms:

$$\delta\mathcal{L} = \frac{R}{D}h_{\mu\nu}\nabla^\mu\xi^\nu[2(A_1D-1)] + \frac{R}{D}h\nabla\cdot\xi\left[a + A_1cD + A_2(cD+2)D + \frac{B_2}{2}kD\right] + \\ + \frac{R}{D}\varphi\nabla\cdot\xi\left[x + b_1kD + \frac{B_2}{2}(cD+2)D\right] + h\Box\nabla\cdot\xi\left[\frac{a}{2}(c+2) - \frac{b}{2}(2+cD) + \frac{c}{2} + \frac{y}{2}k\right] + \\ + \nabla_\mu\nabla\cdot\xi\nabla_\nu h^{\mu\nu}\left[1 + c - \frac{a}{2}(cD+2) - \frac{x}{2}k\right] + \varphi\Box\nabla\cdot\xi\left[\frac{x}{2}(2+c) + \frac{y}{2}(2+cD) + \frac{z}{2}k\right] = 0. \tag{6.30}$$

Notice from the first term that $A_1 = \frac{1}{D}$, regardless of the values of the real parameters. The following two brackets determine up to two non-minimal coupling coefficients in terms of others. The remaining brackets constrain the symmetry factors c and k in terms of the Lagrangian coefficients.

We can recover the result of the last chapter for a Diff invariant theory. In this case, $\varphi = 0 = c$, $a = 1 = b$, and then

$$\delta\mathcal{L}_{Diff} = -\frac{2R}{D}h_{\mu\nu}\nabla^\mu\xi^\nu + \frac{R}{D}\nabla\cdot\xi h + 2A_1Rh_{\mu\nu}\nabla^\mu\xi^\nu + 2A_2R\nabla\cdot\xi h \tag{6.31}$$

from which we conclude that $A_1 = \frac{1}{D}$ and $A_2 = -\frac{1}{2D}$. This gives exactly (5.24):

$$\mathcal{L}_{Diff} = -\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h + \frac{R}{2D}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right). \tag{6.32}$$

6.2 MASSLESS SCALAR-TENSOR

The simplest massless scalar-tensor model propagating 3 degrees of freedom, is given by the promotion of TDiff (1.9) to curved spaces. According to the previous section, with $\varphi = 0 = c$ and $\nabla\cdot\xi = 0$, we have

$$\delta\mathcal{L}_{TDiff} = -\frac{2R}{D}h_{\mu\nu}\nabla^\mu\xi^\nu + 2A_1Rh_{\mu\nu}\nabla^\mu\xi^\nu \Rightarrow A_1 = \frac{1}{D}, \tag{6.33}$$

A_2 is free, and the equivalent Lagrangian is

$$\mathcal{L}_{TDiff} = -\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{a}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{b}{4}\nabla_\mu h\nabla^\mu h + \frac{R}{2D}h_{\mu\nu}h^{\mu\nu} + \frac{A_2}{2}Rh^2. \tag{6.34}$$

so that the action reads

$$S_{TDiff} = \int d^Dx\sqrt{-g}\left[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{a}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{b}{4}\nabla_\mu h\nabla^\mu h + \\ + \frac{R}{2D}h_{\mu\nu}h^{\mu\nu} + \gamma Rh^2\right], \tag{6.35}$$

invariant under $\delta h_{\mu\nu} = \nabla_\mu \xi^T_\nu + \nabla_\nu \xi^T_\mu$ on Einstein spaces. We notice that h^2 is kept invariant under TDiff ($A_2 \equiv 2\gamma$ is arbitrary). Comparing (6.35) with (5.24), we notice that if $a = 1 = b$, $S_{TDiff}[\gamma = -R/4D] = S_{Diff}$, showing that Diff is also an enhancement of the TDiff symmetry in curved space backgrounds.

Alternatively, we can formulate an Einstein space extension of Diff propagating 3 degrees of freedom. In this case, from (3.63), $a = 1 = b$, $x = A - 1$, $y = 1 - A$ and $z = 2A - B - 1$. Then, by solving (6.30), we have $A_1 = \frac{1}{D}$, $A_2 = -\frac{1}{2D}$, $B_2 = \frac{1-A}{D}$, with $c = 0 = k$:

$$\mathcal{L}^{\nabla}_{(3.63)} = -\frac{1}{4}\nabla_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho} + \frac{1}{2}\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu} - \frac{1}{2}\nabla^\mu h \nabla^\nu h_{\mu\nu} + \frac{1}{4}\nabla_\mu h \nabla^\mu h + \frac{A-1}{2}\varphi\nabla_\mu\nabla_\nu h^{\mu\nu} + \\ + \frac{1-A}{2}h\Box\varphi + \frac{2A-B-1}{4}\varphi\Box\varphi + \frac{R}{2D}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) + \frac{b_1}{2}R\varphi^2 + \frac{1-A}{2D}Rh\varphi. \quad (6.36)$$

Although it describes 3 degrees of freedom, its symmetry is still Diff, and the action reads:

$$S = \int d^D x\sqrt{-g}\Bigg[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h + \\ + \frac{R}{2D}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) + \frac{a-1}{2}\phi\nabla_\mu\nabla_\mu h^{\mu\nu} + \frac{1-a}{2}h\Box\phi + \frac{2a-b-1}{4}\phi\Box\phi + \frac{1-a}{2D}Rh\varphi + \gamma R\varphi^2\Bigg]. \quad (6.37)$$

One can ask whether there exists a non-linear and geometrical theory (like $\sqrt{-g}R$) which becomes (6.35) when linearized around an Einstein space. Similarly, we can look for a WTDiff formulation in curved backgrounds, and verify if it is an enhancement of the TDiff symmetry. For these extensions, see for example (LOPEZ-VILLAREJO, 2011; CRISTÓBAL, 2014).

6.3 MASSIVE SCALAR-TENSOR

For the massive case, we proceed in the same way as we did in section 5.3: in order to preserve the flat limit number of degrees of freedom, we reduce the dimension of configuration space by imposing constraints. Differently from the FP case, we now have a scalar field coupled to $h_{\mu\nu}$. So, in $D = 4$, we have a set of 11 independent variables, $h_{(\mu\nu)}$ and ϕ, that must be reduced to 6 (5 for the massive graviton and 1 for the massive scalar); thus, like in the FP case, we have to find out 5 constraints from the equations of motion.

Notice that the massless theory $\mathcal{L}(1,1,1,1)$ is essential for $\nabla^\mu E_{\mu\nu} = 0$ to be a vector constraint in the massive theory. It is not the case of $\mathcal{L}(1,1,a,b)$. However, now we have the equation of motion of φ, which means we can still construct a constraint. By using (6.2), which is the minimal coupling version of TDiff, we start generalizing (3.17) by writing

$$\mathcal{L}_m = \mathcal{L}_{TDiff} - \frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - b\,h^2) + \frac{a-1}{2}\varphi\nabla_\mu\nabla_\nu h^{\mu\nu} + \frac{a-b}{2}h(\Box - m^2)\varphi + \frac{2a-b-1}{4}\varphi(\Box - m^2)\varphi + \\ + \frac{a_1}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{a_2}{2}Rh^2 + \frac{a_3}{2}C^{\mu\nu\alpha\beta}h_{\mu\alpha}h_{\nu\beta} + \frac{a_4}{2}\tilde{R}_{\mu\beta}h^\mu{}_\nu h^{\nu\beta} + \frac{a_5}{2}\tilde{R}_{\mu\nu}h^{\mu\nu}h + \\ + \frac{b_1}{2}R\varphi^2 + \frac{b_2}{2}Rh\varphi + \frac{b_3}{2}\tilde{R}_{\mu\nu}\varphi h^{\mu\nu}, \quad (6.38)$$

where we have already decomposed the Riemann tensor in its irreducible parts: the Weyl tensor $C^{\mu\nu\alpha\beta}$, the traceless Ricci tensor $\tilde{R}_{\mu\nu}$, and the Ricci scalar.

By evaluating the equations of motion of (6.38) with respect to $h_{\mu\nu}$, we have

$$E_{\mu\nu} = \Box h_{\mu\nu} - \nabla^\alpha(\nabla_\nu h_{\mu\alpha} + \nabla_\mu h_{\nu\alpha}) + a\nabla_\mu\nabla_\nu h + ag_{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta} - bg_{\mu\nu}\Box h + \\ +(a-1)\nabla^\mu\nabla^\nu\varphi + (a-b)g_{\mu\nu}\Box\varphi + \mathbf{o}(h,\varphi) = 0, \tag{6.39}$$

where $\mathbf{o}(h,\varphi)$ denotes terms that do not contain two covariant derivatives. Explicitly, we have

$$\mathbf{o}(h,\varphi) = 2a_1 R h_{\mu\nu} + 2a_2 R g_{\mu\nu} h + 2a_3 C_{\mu\alpha\nu\beta} h^{\alpha\beta} + a_4(\tilde{R}_{\mu\beta} h_\nu{}^\beta + \tilde{R}_{\nu\beta} h_\mu{}^\beta) + a_5(\tilde{R}_{\mu\nu} h + g_{\mu\nu} R^{\alpha\beta} h_{\alpha\beta}) + \\ + (b-a)m^2 g_{\mu\nu}\varphi + b_2 g_{\mu\nu} R\varphi + b_3 \tilde{R}_{\mu\nu}\varphi. \tag{6.40}$$

Note that these terms will only be important when deriving the scalar constraint. On the other hand, we notice that $\nabla^\mu E_{\mu\nu}$ contains the following terms

$$\nabla^\mu E_{\mu\nu} = (a-1)\nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + (a-b)\nabla_\nu\Box h + (2a-b-1)\nabla_\nu\Box\varphi + \cdots = 0, \tag{6.41}$$

which vanish only for the Diff case ($a = 1 = b$). We have omitted the terms with only one covariant derivative and used the fact that covariant derivatives do not commute[2]. The last equation means we need to take into account the terms $\nabla_\nu g^{\alpha\beta} E_{\alpha\beta}$ and $\nabla_\nu\Phi$, where Φ is the equation of motion of (6.38) with respect to φ, given by

$$\Phi = (a-1)\nabla^\mu\nabla^\nu h_{\mu\nu} + (a-b)\Box h + (2a-b-1)\Box\varphi + \mathbf{o}(h,\varphi) = 0, \tag{6.42}$$

with

$$\mathbf{o}(h,\varphi) = 2b_1 R\varphi + b_2 R h + b_3 \tilde{R}_{\mu\nu} h^{\mu\nu} - m^2(2a-b-1)\varphi - m^2(a-b)h. \tag{6.43}$$

Then, the most general, up to third order in derivatives, vector term is a linear combination

$$\nabla^\mu E_{\mu\nu} + A\nabla_\nu g^{\alpha\beta} E_{\alpha\beta} + B\nabla_\nu\Phi = \Delta_1\nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + \Delta_2\nabla_\nu\Box h + \Delta_3\nabla_\nu\Box\varphi + \cdots = 0, \tag{6.44}$$

where we are from now on omitting the $\mathbf{o}(h,\varphi)$ terms, and the coefficients are given by

$$\Delta_1 = a - 1 + A(aD-2) + B(a-1); \tag{6.45}$$

$$\Delta_2 = a - b + A(1 + a - bD) + B(a-b); \tag{6.46}$$

$$\Delta_3 = 2a - b - 1 + A(a - 1 + aD - bD) + B(2a-b-1). \tag{6.47}$$

So, if we are looking for a vector constraint we must impose that each one of these coefficients is zero. This is uniquely solvable for the parameters A and B since $\Delta_3 = \Delta_1 + \Delta_2$. Thus, $\Delta_1 = 0 = \Delta_2 \Rightarrow$

[2] More specifically, we have used $\nabla^\mu\nabla^\alpha(\nabla_\nu h_{\mu\alpha} + \nabla_\mu h_{\nu\alpha}) = \nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + \nabla^\mu\Box h_{\mu\nu} + 2R_{\mu\nu}\nabla_\alpha h^{\mu\alpha} - h_{\alpha\beta}\nabla_\mu R^{\alpha\mu\beta}{}_\nu + h_{\mu\alpha}\nabla^\mu R^\alpha{}_\nu$, and $\Box\nabla_\mu h = \nabla_\mu\Box h + R_{\mu\nu}\nabla^\nu h$.

$A = 0$ and $B = -1$. Hence, we have found the vector constraint

$$\phi_\nu = \nabla^\mu E_{\mu\nu} - \nabla_\nu \Phi. \tag{6.48}$$

The next step is to find the scalar constraint. We start by writing

$$\Omega = \nabla^\mu \nabla^\nu E_{\mu\nu} + A\, m^2 g^{\mu\nu} E_{\mu\nu} + c_1 R g^{\mu\nu} E_{\mu\nu} + c_2 \tilde{R}^{\mu\nu} E_{\mu\nu} + \kappa_1 m^2 \Phi + \kappa_2 R\Phi + \kappa_3 \Box\Phi = 0. \tag{6.49}$$

We observe that the only terms containing four derivatives are in $\nabla^\mu\nabla^\nu E_{\mu\nu} + \kappa_3\Box\Phi$. The only combination that gets rid of such terms is given exactly by $\nabla^\nu\phi_\nu$. So, according to (6.48), $\kappa_3 = -1$.

Now, we start collecting terms $\sim \nabla^2 h + \nabla^2\varphi$, either multiplied by m^2 or curvature quantities. From these terms, the only one containing a Weyl tensor, as can be seen in (6.40), comes from $\nabla^\mu\nabla^\nu E_{\mu\nu}$:

$$2a_3 C_{\mu\alpha\nu\beta}\nabla^\nu\nabla^\mu h^{\alpha\beta}, \tag{6.50}$$

whence we conclude that either $a_3 = 0$ or the Weyl tensor is zero. The latter option is more radical once it drastically restricts the possible manifolds in which the massive field theory can be written. Therefore, we assume henceforth that $a_3 = 0$.

Next, we have to get rid of the following terms

$$\tilde{R}_{\mu\nu}\Box h^{\mu\nu}(a_5 - b_3 + c_2) + \tilde{R}_{\mu\nu}\nabla^\mu\nabla^\nu h(a + a_5 + a\, c_2) + \tilde{R}_{\mu\nu}\nabla^\mu\nabla^\nu\varphi[a + b_3 + c_2(a-1) - 1] = 0. \tag{6.51}$$

From the first two parentheses we have $(a + b_3) + c_2(a-1) = 0$. But now compare it with the brackets in the third term above. We cannot simultaneously set those terms to zero unless

$$\tilde{R}_{\mu\nu} = 0. \tag{6.52}$$

Consequently, we are forced to work with **Einstein spaces**, like in the FP case. Terms a_4, a_5, b_3 and c_2 can be set to zero, once they are multiplying $\tilde{R}_{\mu\nu}$. We still have to determine A, c_1, k_1 and k_2 in order to determine Ω. Resuming the process, let us now consider the terms

$$m^2\nabla^\mu\nabla^\nu h_{\mu\nu}\Big[-1 + A(aD-2) + k_1(a-1)\Big] + m^2\Box h\Big[-a + A(1+a-bD) + k_1(a-b)\Big] = 0. \tag{6.53}$$

As each bracket must vanish, we conclude that

$$A = \frac{a^2 - b}{d_0}, \qquad k_1 = \frac{a - 1 + D(b - a^2)}{d_0}. \tag{6.54}$$

where d_0 is given by (1.7). Remember that $d_0 \neq 0$. The term $m^2\Box\varphi$ does not contribute since its coefficient vanishes for these values of A and k_1. Additionally, the last terms are

$$R\nabla^\mu\nabla^\nu h_{\mu\nu}\left[(2a_1-\frac{2}{D})+c_1(aD-2)+k_2(a-1)\right]+R\Box h\left[(2a_2+\frac{a}{D}-b_2)+c_1(1+a-bD)+k_2(a-b)\right]+$$
$$+R\Box\varphi\left[(\frac{a-1}{D}+b_2-2b_1)+c_1(a-1+D(a-b))+k_2(2a-b-1)\right]=0. \tag{6.55}$$

It follows that $b_2=a_1+a_2+b_1-\frac{1}{2D}$, $c_1=\frac{f_1}{d_0}$, and $k_2=\frac{f_2}{d_0}$, where $(2D)f_1(a,b,a_1,a_2,b_1)$ and $(2D)f_2(a,b,a_1,a_2,b_1)$ are polynomial functions. Thus, we have completely determined the scalar constraint $\Omega=0$. On its turn, the Lagrangian (6.38) finally reads as

$$\mathcal{L}_m=\mathcal{L}_{TDiff}-\frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu}-b\,h^2)+\frac{a-1}{2}\varphi\nabla_\mu\nabla_\nu h^{\mu\nu}+\frac{a-b}{2}h(\Box-m^2)\varphi+\frac{2a-b-1}{4}\varphi(\Box-m^2)\varphi+$$
$$+\frac{a_1}{2}Rh_{\mu\nu}h^{\mu\nu}+\frac{a_2}{2}Rh^2+\frac{b_1}{2}R\varphi^2+\frac{(a_1+a_2+b_1-1/2D)}{2}Rh\varphi, \tag{6.56}$$

which is valid in Einstein space backgrounds. Again, minimal coupling does not work. Notice that a_1, a_2 and b_1 are not required to assume any special values, and can be set to zero. On the other hand, the last term does not vanish. That is, it is not possible to have a minimal action (obtained only replacing usual by covariant derivatives) propagating the correct number of degrees of freedom for the curved version of the massive extension obtained from TDiff via dimensional reduction.

Let us explore the massless limit of (6.56). Looking for gauge symmetry, in this case, we set $x=1-a$, $y=a-b$, and $z=2a-b-1$ in (6.30), whose solution is $A_1=\frac{1}{D}$, $A_2=\frac{B_2D-a}{2D}$, $b_1=\frac{B_2D+a-1}{2D}$, $c=0$ and $k=-2$:

$$\mathcal{L}^\nabla_{(3.52)}=-\frac{1}{4}\nabla_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho}+\frac{1}{2}\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu}-\frac{a}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu}+\frac{b}{4}\nabla_\mu h\nabla^\mu h+\frac{a-1}{2}\varphi\nabla_\mu\nabla_\nu h^{\mu\nu}+\frac{a-b}{2}h\Box\varphi+$$
$$+\frac{2a-b-1}{4}\varphi\Box\varphi+\frac{R}{2D}h_{\mu\nu}h^{\mu\nu}+\frac{B_2D-a}{4D}Rh^2+\frac{B_2D+a-1}{4D}R\varphi^2+\frac{B_2}{2}Rh\varphi. \tag{6.57}$$

Notice that we would have to impose only $A_1=1/D$ to have TDiff symmetry. Consequently, by choosing (A_2,b_1,B_2) not satisfying the above relation, we would have solely a TDiff invariant model. Likewise, when $m\to 0$ in (6.56), we can have a massive action whose massless limit is invariant under TDiff, but not generalized Diff. This is a consequence of the existence of non-minimal coupling terms.

6.4 ON CONSTRAINTS VERSUS SYMMETRY

In general, the curved spacetime version of a linearized scalar-tensor model is written as $\mathcal{L}=\mathcal{L}_\nabla+$ mass terms + non-minimal couplings. All the terms which determine the vector constraint come necessarily from $\mathcal{L}_\nabla$, given by (6.24). In fact, as we are going to show below, such a constraint depends on the validity of a subset of equations in (6.30). On the other hand, we have seen that the requirement of an Einstein space follows from the existence of a scalar constraint. But could it be avoided if we had carried out the last section analysis by using arbitrary coefficients?

Let us then finish this chapter generalizing, once and for all, the analysis of constraints for the most general massive scalar-tensor coupled to an external gravitational field. By using (6.24) we start writing

$$\mathcal{L} = \mathcal{L}_{\nabla} + \frac{a_1}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{a_2}{2}Rh^2 + \frac{a_3}{2}C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \frac{a_4}{2}\tilde{R}^{\mu\beta}h_{\mu\nu}h^{\nu}{}_{\beta} + \frac{a_5}{2}\tilde{R}^{\mu\nu}h_{\mu\nu}h + \\ + \frac{b_1}{2}R\varphi^2 + \frac{b_2}{2}Rh\varphi + \frac{b_3}{2}\tilde{R}^{\mu\nu}h_{\mu\nu}\varphi - \frac{m^2}{4}h^{\mu\nu}h_{\mu\nu} + M\frac{m^2}{4}h^2 + M_y\frac{m^2}{2}h\varphi + M_z\frac{m^2}{4}\varphi^2. \quad (6.58)$$

The vector constraint is a linear combination given by

$$\phi_\nu = \nabla^\mu E_{\mu\nu} + A\nabla_\nu g^{\alpha\beta}E_{\alpha\beta} + B\nabla_\nu\Phi = 0. \quad (6.59)$$

By generalizing the results of last section we can show that

$$\phi_\nu = \nabla_\nu\Box h\left[(a-b) + A(1+a-bD) + By\right] + \nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha}\left[(a-1) + A(aD-2) + Bx\right] + \\ + \nabla_\nu\Box\varphi\left[(x+y) + A(x+yD) + Bz\right] + \cdots, \quad (6.60)$$

where the dots denote terms that contain less than two covariant derivatives. Then we have to solve the following system of equations:

$$\begin{cases} (a-b) + A(1+a-bD) + By = 0, \\ (a-1) + A(aD-2) + Bx = 0, \\ (x+y) + A(x+yD) + Bz = 0. \end{cases} \quad (6.61)$$

On the other hand, we can rewrite the last three equations of (6.30) as

$$\begin{cases} (a-b) + \frac{c}{2}(1+a-bD) + \frac{k}{2}y = 0, \\ (a-1) + \frac{c}{2}(aD-2) + \frac{k}{2}x = 0, \\ (x+y) + \frac{c}{2}(x+yD) + \frac{k}{2}z = 0. \end{cases} \quad (6.62)$$

This result shows us that the existence of a vector constraint necessarily constrains the structure of $\mathcal{L}_{\nabla}$, which depends only upon a, b, x, y and z, in the same way as invariance under generalized Diff does. Consequently, for flat spaces, **we have proven that the massless limit of a healthy massive scalar-tensor model is necessarily a member of the generalized Diff family.** For curved spaces, however, we have to further investigate the non-minimal coupling structure of $\mathcal{L}$.

By solving (6.61) we obtain

$$A = \frac{(b-a)x + (a-1)y}{x(1+a-bD) + y(2-aD)}, \qquad B = \frac{d_0}{x(1+a-bD) + y(2-aD)}, \quad (6.63)$$

$$z = \frac{[b(D-1)-1]x^2 + (D-2)y^2 + 2[(D-1)a-1]xy}{d_0}. \quad (6.64)$$

Note that the denominator in (6.63) avoids x, y to assume a certain value. We are not considering this case. Now we look for the scalar constraint:

$$\Omega = \nabla^\nu \phi_\nu + (Cm^2 + c_1 R) g^{\mu\nu} E_{\mu\nu} + c_2 \tilde{R}^{\mu\nu} E_{\mu\nu} + (Km^2 + k_1 R)\Phi = 0. \tag{6.65}$$

We have the combination $\nabla^\nu \phi_\nu$ because it is the only one which gets rid of four-derivative terms. The next steps, then, are all about collecting terms containing two derivatives applied on the fields. Firstly, the only Weyl tensor term is $2a_3 C_{\mu\alpha\nu\beta} \nabla^\mu \nabla^\nu h^{\alpha\beta}$ which implies $a_3 = 0$. Next, we collect $\tilde{R}^{\mu\nu}$ terms:

$$\begin{aligned} &\tilde{R}_{\mu\nu} \Box h^{\mu\nu} \left[a_5 + b_3 B + (2a_4 + a_5 D)A + c_2\right] + \tilde{R}_{\mu\nu} \nabla^\mu \nabla^\nu h \left(a + a_5 + ac_2\right) + \\ &\quad + \tilde{R}_{\mu\nu} \nabla^\mu \nabla^\nu \varphi \left[x(1 + c_2) + b_3\right] + \tilde{R}_{\mu\nu} \nabla^\mu \nabla_\alpha h^{\alpha\nu} \left[2(a_4 - 1) - 2c_2\right]. \end{aligned} \tag{6.66}$$

Now, it comes an important result. By replacing (6.63) above, it is straightforward to show that it is impossible to simultaneously get rid of these four terms! We are forced to use **Einstein spaces**:

$$\tilde{R}_{\mu\nu} = 0. \tag{6.67}$$

The following terms constrain the non-minimal couplings and determine the scalar constraint. We have

$$\begin{aligned} &R \nabla^\mu \nabla^\nu h_{\mu\nu} \left[2a_1 - \frac{2}{D} + c_1(aD - 2) + k_1 x\right] + R\Box\varphi \left[b_2 + \frac{x}{D} + Ab_2 D + 2Bb_1 + c_1(x + Dy) + k_1 z\right] + \\ &\quad + R\Box h \left[2a_2 + \frac{a}{D} + A(2a_2 D + 2a_1) + Bb_2 + c_1(1 + a - bD) + k_1 y\right], \end{aligned} \tag{6.68}$$

and

$$\begin{aligned} &m^2 \nabla^\mu \nabla^\nu h_{\mu\nu} \left[-1 + C(aD - 2) + Kx\right] + m^2 \Box h \left[M + A(MD - 1) + BM_y + C(1 + a - bD) + Ky\right] + \\ &\quad + m^2 \Box\varphi \left[M_y + ADM_y + BM_z + C(x + Dy) + Kz\right]. \end{aligned} \tag{6.69}$$

From these terms, we can determine the four coefficients C, K, c_1 and k_1 and fully determine the scalar constraint. Besides, we can determine two out of the following coefficients $a_1, a_2, b_1, b_2, M, M_y$ and M_z. If we compare (6.68) with the first three terms in (6.30), we realize that the coefficients would be the same, were it not for the presence of c_1 and k_1. Thus, in principle, it is possible to solve (6.68) in such a way that in the massless limit the resulting Lagrangian is still not gauge invariant!

To better understand the last statement, let us consider $x = 0$ and $a = 1$, which is a case always achievable through field redefinitions.

Simplification:

For $a = 1$ and $x = 0$, we have $C = \dfrac{1}{D-2}$, $K = \dfrac{M_y(1-b)(D-2) + y(M(2-D) + bD - 2)}{(D-2)y^2}$, $M = \dfrac{y(y + M_y(2-2b)) - M_z(b-1)^2}{y^2}$, $b_2 = \dfrac{b_1(1-b)}{y} + \dfrac{y - 2(a_1 + a_2)Dy}{2(b-1)D}$, and , $k_1 = \dfrac{b_1(b-1)^2}{y^3} + \dfrac{-2 + D(-3 + 4a_2 + 4b - 2a_2 D + 2a_1(2 + D - 2bD))}{2(D-2)Dy}$, $c_1 = -\dfrac{2(a_1 D - 1)}{D(D-2)}$ for $y \neq 0$. In addition, if $y = 1 - b$, $b_2 = a_1 + a_2 + b_1 - \dfrac{1}{2D}$, as in (6.56).

For the vector constraint we have $A = 0$, $B = \dfrac{b-1}{y}$ and $z = \dfrac{y^2}{1-b}$. Explicitly, we have

$$\phi_\nu = \left[2\left(a_1 - \frac{1}{D}\right)R - m^2\right]\nabla^\mu h_{\mu\nu} + \left[\left(2a_2 + \frac{1}{D} + Bb_2\right)R + (M + BM_y)\,m^2\right]\nabla_\nu h + \\ + \left[(b_2 + 2Bb_1)\,R + (M_y + BM_z)\,m^2\right]\nabla_\nu\varphi = 0. \quad (6.70)$$

Now, if we want to get rid of the terms $R\nabla^\mu h_{\mu\nu}$, $R\nabla_\nu h$ and $R\nabla_\nu\varphi$ - in the sense that $m = 0$ makes (6.70) trivial - we have to demand $a_1 = \dfrac{1}{D}$, $b_1 = \dfrac{(1+2a_2D)y^2}{2(b-1)^2D}$, and $b_2 = \dfrac{(2a_2D+1)y}{(1-b)D}$. Comparison to (6.30) shows that this is equivalent to demand generalized Diff symmetry. Only in this case the massless limit is invariant under such a gauge symmetry. Therefore, we end this section by pointing out the following result, due to the existence of non-minimal terms: if we choose $a_1 = \dfrac{1}{D}$, but b_1 and b_2 not satisfying the previous relation, the massless limit of the massive model (6.58) will be TDiff invariant (as $\mathcal{L}_\nabla$ is) but not invariant under generalized Diff.

6.5 REMARK ON PARTIALLY MASSLESS CASES

How about partially massless theories? We shall look for a gauge symmetry, which takes away some dof. We are not going to discuss all possible partially massless models, but only some of them.

Our start point here is the most general massive action that possibly[3] propagates 6 dof in Einstein spaces. Accordingly to the last subsection,

$$S_{\text{TDiff}} = \int d^Dx\sqrt{-g}\,\Bigg[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h + \\ -\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \frac{\gamma}{2D}R\,h_{\mu\nu}h^{\mu\nu} + \frac{1-2\gamma}{4D}R\,h^2 + \frac{1-a}{2}\nabla_\mu\varphi\nabla_\nu h^{\mu\nu} + \frac{1-a}{2}h(\Box - m^2)\varphi + \\ + \frac{2a-b-1}{4}\varphi(\Box - m^2)\varphi + \frac{b_1}{2}R\varphi^2 + \frac{b_2}{2}Rh\varphi\Bigg]. \quad (6.71)$$

Only as an application, consider the case $R = \dfrac{D(D-1)m^2}{D-2}$. For this value, we have learned[4] that the FP curved action has an extra gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu\nabla_\nu\lambda + \frac{1}{(D-2)}m^2 g_{\mu\nu}\lambda, \quad (6.72)$$

propagating one less dof than the usual $\dfrac{(D+1)(D-2)}{2}$ massive spin-2 dof. Under such transformation, from (6.71), we have

$$\delta\mathcal{L} = \left(\frac{a-1}{2}\right)m^2\left[\varphi\Box\lambda + \frac{D}{D-2}m^2\varphi\lambda\right], \quad (6.73)$$

3 "Possibly" because we can find some cases describing fewer dof, depending on the values of γ, b_1, b_2 and R.

4 If the Weyl tensor vanishes, $R = \dfrac{D(D-1)m^2}{D-2}$ is a dS space, corresponding to a mass $m = \dfrac{2\Lambda}{3}$, which is the upper limit for the forbidden mass range, see (HIGUCHI, 1987).

so we immediately see that if $a = 1$ ($a = 1 = \gamma, b_2 = 0$ in (6.71)), we have a partially massless theory. In $D = 4$, instead of 6, we have 5 dof: 4 coming from the FP action and 1 coming from the decoupled scalar part.

On the other hand, for $a \neq 1$, it is not possible to find x, y such that $\delta\varphi = x\Box\lambda + y\, m^2\lambda$, together with (6.72), makes (G.10) invariant. It is necessary to deform FP ($\gamma \neq 1$), giving up (6.72), or consider $R = km^2$, for some arbitrary real parameter k. A more rigorous analysis, however, requires an investigation on the scalar constraint constructed in the last subsections by requiring it to identically vanish without making the vector constraint trivial. We are still investigating these possibilities.

We finish this chapter pointing out some singular cases when we let $R = k\, m^2$ assume some special values. Firstly, we can use b_1 and b_2 to eliminate the mass of φ in (6.71), for $\gamma = 1$. If $a \neq 1$, $\dfrac{b_1}{b_2} = \dfrac{2a-b-1}{2(1-a)}$ and $R = \dfrac{1-a}{b_2} m^2$:

$$S = \int d^D x\sqrt{-g}\left\{\mathcal{L}_{FP} + \sqrt{-g}\left[\left(\frac{a-1}{2}\right)\varphi(\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h) + \frac{2a-b-1}{4}\varphi\Box\varphi\right]\right\} \tag{6.74}$$

describes the same 6 dof in $D = 4$. Secondly, from (6.71), we can get rid of all the masses by imposing $b_1 = \dfrac{2(2a-b-1)}{5D}$, $b_2 = \dfrac{4(1-a)}{5D}$, $\gamma = \dfrac{2}{5}$ and $R = \dfrac{5D}{4} m^2$:

$$\begin{aligned} S = \int d^D x\sqrt{-g}\left[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h\nabla^\mu h\right] + \\ + \sqrt{-g}\left[\left(\frac{a-1}{2}\right)\varphi(\nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h) + \frac{2a-b-1}{4}\varphi\Box\varphi\right]. \end{aligned} \tag{6.75}$$

Written in this way, the reader could suggest that this action corresponds to a curved version of (3.63) obtained via minimal coupling, describing spin-2 massless particles. However, this action is not invariant under $\delta h_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$. It is massive, why? Because mass appears when manipulating the equations of motion, since $R \sim m^2$. Indeed, it corresponds to a special case of (6.71) and propagates 6 degrees of freedom in $D = 4$! This is a reminder to the fact that minimal couplings, in general, do not work.

7 CONCLUSION AND PERSPECTIVES

We conclude this thesis writing a few words about our work. At its beginning, one of our driving questions was the following: on the perspective of a classical field theory, how much can we modify the weak field limit of GR, and even so have spin-2 particles? Taking into account the remarkable result that transverse diffeomorphisms **TDiff** is the minimal symmetry for massless spin-2 particles (BIJ; DAM; NG, 1982), our starting point was the following TDiff Lagrangian

$$\mathcal{L} = -\frac{1}{4}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{1}{2}\partial^\alpha h^{\mu\beta}\partial_\mu h_{\alpha\beta} - \frac{a}{2}\,\partial^\mu h\partial^\nu h_{\mu\nu} + \frac{b}{4}\partial_\mu h\partial^\mu h, \tag{7.1}$$

invariant under $\delta h_{\mu\nu} = \partial_\mu\xi^T_\nu + \partial_\nu\xi^T_\mu$, where $\partial^\mu\xi^T_\mu = 0$. Then, in chapter 2, we have recovered the result of (ALVAREZ et al., 2006), that TDiff describes 3 physical degrees of freedom if

$$d_0 = 1 - 2a + a^2(D-1) - b(D-2) > 0.$$

The case $d_0 = 0$ corresponds to the two enhancements of TDiff symmetry: Diff and WTDiff, both describing massless spin-2 particles. We have also verified these results in chapter 4 by analysing the unitarity of these models. In chapter 3, we have looked for a massive theory obtained from a Kaluza-Klein dimensional reduction of TDiff. Our motivations behind such approach are related to the fact that it is not possible to add mass terms into the TDiff Lagrangian without introducing ghosts, i.e. we cannot deform TDiff into a massive theory by simply adding a general mass term $(a_1h^2_{\mu\nu} + a_2h^2)m^2$ to the massless Lagrangian, as shown in (ALVAREZ et al., 2006). By performing the dimensional reduction, we have obtained a massive scalar-tensor model propagating 6 physical degrees of freedom if

$$D_0 = 1 - 2a + a^2D - b(D-1) > 0. \tag{7.2}$$

It turns out that such model is a massive extension of Diff. We highlight: even starting from a TDiff invariant massless model, we achieve a massive version whose massless limit is Diff invariant. Together with the massive model obtained from the dimensional reduction of WTDiff (BONIFACIO; FERREIRA; HINTERBICHLER, 2015) (which turned out to be Fierz-Pauli, the massive extension of Diff), it makes us wonder whether or not Diff is the only symmetry that supports mass deformation. This seems to be the case for the flat spacetime as we have argued.

In order to give a more general answer to that question, we have moved our discussion to curved spacetime backgrounds. In chapter 5, we have reviewed the consistency of linearized massive spin-2 propagation in these backgrounds and concluded that Einstein spaces are required (BUCHBINDER; KRYKHTIN; PERSHIN, 1999). Then, in chapter 6, we have started with a rather general second order scalar-tensor model, adding arbitrary mass terms (including different masses for the spin-0 and spin-2 particles), and looking for restrictions on the arbitrary coefficients that assure a healthy particle content (massive spin-2 and spin-0 particles). We have checked that it is possible to have a massive model such that its massless limit is solely TDiff invariant for some choice of the parameters of non-minimal terms. We stress that this result can only be obtained in curved backgrounds. In the flat space, the massive version is necessarily an extension of Diff (or generalized Diff). In this chapter, we have also verified that Einstein spaces are always required in order to have physical particles, regardless of mass or symmetry.

Such results generalize previous conclusions about TDiff (ALVAREZ et al., 2006) and WTDiff (BONIFACIO; FERREIRA; HINTERBICHLER, 2015). Having said that, we believe that we have improved our understanding on the role of the Diff and TDiff symmetries, and their symmetry breakings in the description of massive gravitons models in flat and curved spacetime backgrounds.

We finish this thesis pointing out some possible extensions for the present work. Regarding curved spacetime calculations, it is possible to explore partially massless models. Those can be obtained from the expressions we have derived in the chapter 6. It is also natural to ask for non-linear completions of TDiff. For example, (LOPEZ-VILLAREJO, 2011; CRISTÓBAL, 2014) explore the geometrical description of TDiff, by establishing a connection between such symmetry and scalar-tensor gravity. The reader can also explore non-linear WTDiff and unimodular theories of gravity, as well as their quantum gravity approaches, see for example (PERCACCI, 2018). We can also investigate non-linearities on the curved metric background by considering a series in curvature terms, as carried out in (BUCHBINDER; KRYKHTIN; PERSHIN, 1999) for the Diff case, and whose conclusion was that Einstein spaces were not any more required to consistently describe massive spin-2 particles after the non-linear completion. Finally, we can also ask whether there is a consistent TDiff generalization of the dRGT massive gravity (RHAM; GABADADZE; TOLLEY, 2011), or of the Hassan and Rosen bimetric model (HASSAN; ROSEN, 2012a).

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APÊNDICE A – CONSTRAINED SYSTEMS AND HAMILTONIAN ANALYSIS

Here, we briefly review constrained Hamiltonian systems and the Dirac-Bergmann algorithm applied to the context of field theories. We start from a Lagrangian and its Lagrangian density[1]

$$L = \int_V d^3x \, \mathcal{L}(\varphi_\alpha, \dot{\varphi}_\alpha, \overrightarrow{\nabla}\varphi_\alpha, \vec{x}, t) \tag{A.1}$$

defined in the configuration space (let us suppose it containing N coordinates and their N velocities). We define the canonically conjugated momenta by

$$\pi^\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\alpha(x)}. \tag{A.2}$$

The transition from the configuration space to the phase space (coordinates and momenta) is allowed if all the N canonical momenta π^α are linearly independent, otherwise it will not be possible to write all the N velocities $\dot{\varphi}_\alpha$ in terms of the momenta. In other words, if the canonical phase space has dimension smaller than $2N$, the system is constrained and the Hamilton equations, evaluated from the canonical Hamiltonian

$$H_c = \int_V d^3x \, \mathcal{H}_c(\varphi_\alpha, \pi^\alpha, \partial_i\varphi_\alpha, \partial_i\pi^\alpha), \tag{A.3}$$

do not supply the same (and correct) Euler-Lagrange equations of motion. It turns out that all the models we are studying are constrained[2], and we need to be careful with their Hamiltonian analysis. This is the reason to introduce the Dirac-Bergmann algorithm. Basically, after defining a new Hamiltonian, we demand that each constraint be preserved in time as a consistency condition. This imposition can be trivially satisfied or generate new constraints. At the end, when there is no longer any new constraint, we can set all of them to zero, and obtain the final result - the reduced Hamiltonian. Moreover, we can check how many degrees of freedom are indeed propagating!

A.1 THE DIRAC-BERGMANN ALGORITHM

- The constraints are fundamental relations[3] $\phi_\mu[\varphi_\alpha, \pi^\alpha, \partial\varphi, \partial\pi] \approx 0$. They are named **primary constraints** (PC) because they depend exclusively on the Lagrangian and are not function of any acceleration.

[1] From now on, we refer to Lagrangian densities simply as Lagrangians. See, for instance, the books (DIRAC, 1964; SUNDERMEYER, 1982; HENNEAUX; TEITELBOIM, 1994; LEMOS, 2007) for an introduction. We are following the notation of (LINO DOS SANTOS, 2018; SANTOS, Unpublished; BENNDORF, 2016).

[2] If we remember that spin-s (massless) massive particles should have only (2) $2s + 1$ propagating degrees of freedom, it is natural that the systems are constrained, since we are using tensor objects to describe these particles, and the constraints are required to reduce the dimension of the phase space.

[3] The weak equality is used to highlight that a dynamical variable $F(q, p) \approx 0$ vanishes only in the subspace Γ_c of the phase space defined by the constraints. We will be writing the functional $\phi_\mu[\varphi_\alpha, \pi^\alpha, \partial\varphi, \partial\pi]$ as $\phi_\mu(x)$.

- The primary Hamiltonian is given by $H_P = \int d^3x\, \mathcal{H}_p$

$$\mathcal{H}_p = \mathcal{H}_c + \sum_\mu \lambda^\mu(x)\phi_\mu(x), \tag{A.4}$$

where $\mathcal{H}_c$ is the canonical Hamiltonian density. We have considered the PC with the help of Lagrange multipliers λ^μ, a priori arbitrary functions.

- We apply the **consistency condition** $\dot{\phi}_\mu \approx 0$, where[4]

$$\dot{\phi}_\mu(x) = \{\phi_\mu(x), H_P\}, \tag{A.6}$$

or then

$$\dot{\phi}_\mu \approx \{\phi_\mu(x), H_c\} + \int d^3y\, \lambda_\nu(y)\{\phi_\mu(x), \phi_\nu(y)\} \approx 0. \tag{A.7}$$

- The equations (A.7) can determine some Lagrange multipliers as well as yield new constraints, these ones called **secondary constraints** (SC). The application of the consistency condition to the SC can determine other Lagrange multipliers or give rise to new constraints[5]. The process is repeated until no new constraints arise. This is the so-called **Dirac-Bergmann algorithm**.

- After a finite number of steps, the process ends with j constraints (n primary and $j-n$ secondary). We shall verify whether they are **first class constraints** (FCC) or **second class constraints** (SCC). A function $F(q,p)$ is first class (FC) if

$$\{F, \phi_\rho\} \approx 0 \qquad\qquad (\rho = 1, 2, ..., j), \tag{A.8}$$

for all ρ, otherwise the function is second class.

- It is postulated that every gauge transformation is due to a FCC. Since the Poisson brackets among FCC are zero, the Lagrange multipliers associated to first class PC remain unknown until the end of the process. They are arbitrary functions v_a that, however, do not alter the physical state of the system[6]. So, the number of unknown Lagrange multipliers is equal to the number of gauge transformations parameters.

- We eliminate the ambiguity by imposing restrictions upon the canonical variables, without altering the observable properties. In order to do so, we introduce a set of j_1 functions $\Omega_\alpha(\varphi, \pi)$,

4 Recall that the Poisson brackets $\{F(x), G(y)\}$ are defined in terms of functional derivatives as

$$\{F(y), G(z)\} = \int d^3x \left[\frac{\delta F(y)}{\delta\varphi_\alpha(x)}\frac{\delta G(z)}{\delta\pi^\alpha(x)} - \frac{\delta F(y)}{\delta\pi^\alpha(x)}\frac{\delta G(z)}{\delta\varphi_\alpha(x)}\right], \tag{A.5}$$

and notice that $\{\varphi_\alpha(x), \varphi_\beta(y)\} = 0 = \{\pi^\alpha(x), \pi^\beta(y)\}$, and $\{\varphi_\alpha(x), \pi^\beta(y)\} = \delta^\beta_\alpha\, \delta^{(3)}(x-y)$.

5 The secondary, tertiary, quaternary, ... , constraints are usually called SC, in the sense that they are not directly obtained from the Lagrangian.

6 All the $\{\varphi_a, \pi^\alpha\}$ whose dynamical evolutions depend on the set of arbitrary functions v_a belong to a same equivalence class $[(\bar{\varphi}, \bar{\pi})]$: $(\varphi_0, \pi_0) \longrightarrow (\varphi, \pi)_1 \sim (\varphi, \pi)_2 \sim \cdots \sim (\bar{\varphi}, \bar{\pi}) \in [(\bar{\varphi}, \bar{\pi})]$, which reflects the gauge freedom property of the problem. According to this standpoint, fixing the gauge is nothing but choosing one element of this equivalence class.

called **gauge fixing conditions** in a problem that has j_1 $\psi_\alpha \approx 0$ FCC, transforming them in SCC:

$$\begin{aligned} &\psi_\alpha \approx 0\ ;\ \Omega_\alpha(\varphi,\phi) \approx 0, \qquad (\alpha = 1, 2, ..., j_1) \\ &det\ [\psi_\alpha, \Omega_\beta] \neq 0 \Rightarrow \text{SCC}. \end{aligned} \tag{A.9}$$

We shall regard these functions Ω_α as constraints that cannot be obtained from the Lagrangian. By fixing the gauge, the functions v_a are no longer arbitrary, and a solution can be found. Such gauge fixing choice - we stress - does not affect the physical content of the system. After this, we can set every constraint to zero and write the **reduced Hamiltonian**. It must be positive if the system is classically stable, which we claim to be equivalent to the absence of ghosts[7] (BUONINFANTE, 2016).

- Finally, the number (n) of physical degrees of freedom in the Hamiltonian formulation is

$$n = PS - SCC - 2(FCC), \tag{A.10}$$

where PS is the dimension of the (full) phase space Γ, and the factor 2 is due to the fact that to eliminate the arbitrariness, we impose a gauge fixing condition for each FCC. In the Lagrangian formalism, the number of degrees of freedom is $n_L = (n/2)$.

This prescription allow us to apply the Dirac-Bergmann method to the field theories we are interested in. It is worth highlighting that this method, per si, allows us to identify when there is a gauge symmetry as well as to count the number of physical degrees of freedom; however, the calculations are not manifestly covariant once the role of the time coordinate is unique when evaluating the momenta.

[7] At a quantum level, we say that ghosts are responsible for breaking the unitarity of the model, and then ghostly states do not have their norm preserved. For an instructive introduction to this subject, see the appendix of (BUONINFANTE, 2016).

APÊNDICE B – KALUZA-KLEIN DIMENSIONAL REDUCTION

In this subsection, we review what we call off a truncated Kaluza-Klein (KK) dimensional reduction[1] (HINTERBICHLER, 2012; LINO DOS SANTOS, 2018), in order to relate massless theories in $D+1$ dimensions to their massive counterparts in D, following the procedure of (BONIFACIO; FERREIRA; HINTERBICHLER, 2015; BONIFACIO, 2017; KHOUDEIR; MONTEMAYOR; URRUTIA, 2008; GRACIA, 2015). An important observation about this procedure is the fact that a given spin-s massless particle in $D+1$ dimensions has the same number of degrees of freedom as a spin-s massive particle in D. Such "conservation of degrees of freedom"has a strong connection with the gauge symmetries of each analysed model, and can be checked by using, for example, a Dirac-Bergmann Hamiltonian analysis.

B.1 HOW THE REDUCTION IS PERFORMED

In our procedure, we assume that the extra dimension is compactified so that[2]

$$T^{A,B,\cdots}(x,y) = T^{A,B,\cdots}(x,y+L), \tag{B.1}$$

where y runs along a circumference of fixed length L. A rather natural Ansatz for this sort of condition is to suppose that the dependence on y is given in terms of sines or cosines.

Recalling that in the natural system of units, $[M]=[L]^{-1}$, let us assume that the inverse of the circle radius defined in the extra dimension can be identified with a mass m. Such choice allows us to identify m as the mass of the massive terms obtained after the dimensional reduction. Our procedure is thus a prescription of how a massive model can be obtained from a massless model in $D+1$ dimensions. For simplicity, we are considering only one massive mode given by $m=\frac{2\pi}{L}$ (BONIFACIO; FERREIRA; HINTERBICHLER, 2015; KHOUDEIR; MONTEMAYOR; URRUTIA, 2008). It is possible to consider all the massive modes and, in this case, one can show that a infinite tower of massive modes will be produced (HINTERBICHLER, 2012). Consequently, our procedure is called a truncated KK dimensional reduction in the sense we are using solely the first mode. Regarding parity, if the number of y indices is odd or even, our choice is

$$\Phi^{yy\mu\nu\cdots} = \sqrt{\frac{m}{\pi}}\phi^{\mu\nu\cdots}(x)\cos(my), \tag{B.2}$$

$$\Phi^{y\mu\nu\cdots} = \sqrt{\frac{m}{\pi}}\phi^{\mu\nu\cdots}(x)\sin(my), \tag{B.3}$$

[1] As a brief historical note (KALUZA, 1921; KLEIN, 1926), it brings us back to the 1920s when Kaluza employed a 5D metric as an attempt to unificate gravitation and electromagnetism. Later O. Klein suggested that the fifth dimension should be compact, fact which we are using in our calculations. Although they had not achieved the desired unification, their idea was renewed with the rise of the string theories and their higher dimensions. See, for example, the review (OVERDUIN; WESSON, 1997).

[2] In this subsection, Latin labels are used for $D+1$ dimensions whereas Greek labels for D dimensions. The extra dimension is labelled by y. The components of the metric, stil diagonal, are given by $\eta_{AB}=(-1,1,\cdots,1,\eta_{yy}=1)$.

where the constant $\sqrt{\frac{m}{\pi}}$ arises as a normalization factor and reminds us that fields in different dimensions have different units. The integration in the $(D+1)$-action is written as

$$S_{D+1} = \int d^{D+1}x \, \mathcal{L}^{m=0}_{(D+1)} = \int_0^L dy \int dx_0 \cdots dx_{D-1} \, \mathcal{L}^{m=0}_{(D+1)} = \int_0^{\frac{2\pi}{m}} dy \int d^D x \, \mathcal{L}^{m=0}_{(D+1)}(x,y). \tag{B.4}$$

Integrating in y, we get the D-action

$$S_D = \int d^D x \, \mathcal{L}^m_D \tag{B.5}$$

from which we obtain the reduced massive model.

For instance, the KK dimensional reduction of the Maxwell spin-1 model produces the Maxwell-Proca model; of LEH yields FP, which is the same result (after redefinitions) if we perform the reduction of WTDiff (BONIFACIO; FERREIRA; HINTERBICHLER, 2015; LINO DOS SANTOS, 2018). In this work, we obtained the KK dimensional reduction of the TDiff model in $D+1$ dimensions.

APÊNDICE C – PROJECTION OPERATORS

For vector fields we can define the **projection operators** according to the decomposition on their longitudinal and transverse components. They are given by

$$\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}, \tag{C.1}$$

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}. \tag{C.2}$$

In general, projection operators decompose a given mathematical object in irreducible parts, are idempotent ($P^2 = P$), do not have an inverse P^{-1}, and project in a subspace whose dimension is determined by the their traces (TrP). Additionally, there is a closure relation (completeness: $\sum P = \mathbb{1}$). Explicitly for vectors, we have the idempotent property

$$\theta_{\mu\alpha}\theta^{\alpha\nu} = \theta_\mu^\nu \qquad \text{and} \qquad \omega_{\mu\alpha}\omega^{\alpha\nu} = \omega_\mu^\nu, \tag{C.3}$$

the completeness one

$$\delta_\nu^\mu = \theta_\nu^\mu + \omega_\nu^\mu, \tag{C.4}$$

and the orthogonality between the operators

$$\omega_{\mu\nu}\theta^{\nu\alpha} = 0. \tag{C.5}$$

The transverse operator is $\theta_{\mu\nu}$ given by

$$\partial_\mu \theta^{\mu\nu} = 0. \tag{C.6}$$

By defining (C.1) and (C.2) we are assuming a decomposition in a spin basis. It is possible to use them as building blocks for higher-spin representations, see (BITTENCOURT, 2019).

Quantum mechanics teaches us that spin s eigenvalues can only occur in integer or semi-integer quantities, and the eigenvalues of the S_z operator are $m_s = -s, -s+1, \ldots, s-1, s$. As a result of this, there are $2s+1$ quantum states between $-s$ and s for a given spin s. By studying one particle states, see (WEINBERG, 2013), particles are seen as irreducible representations of the Poincaré group. They are characterized by their mass and spin. Through Wigner's little group techniques (WIGNER, 1939) we can show that spin-s massive particles are represented by operators that satisfy the same properties of the angular momentum operator, which means that they are characterized by $2s+1$ degrees of freedom. However, if the particles are massless and we require parity invariance, they are described by only 2 degrees of freedom $\pm s$ (1, if $s = 0$). These results are true for $D = 3+1$.

As the trace of the projector operators informs us about the dimension of the projected subspaces, we can relate degrees of freedom (and therefore spin) to these operators. From (C.1) and (C.2) we notice, for $D = 3+1$, that $Tr\omega \equiv \eta^{\mu\nu}\omega_{\mu\nu} = 1$ and $Tr\theta \equiv \eta^{\mu\nu}\theta_{\mu\nu} = D-1 = 3$. Then,

$$\omega : (2s+1) = 1 \Rightarrow s = 0,$$
$$\theta : (2s+1) = 3 \Rightarrow s = 1.$$

From $\partial_\mu = iP_\mu$ and $\omega^\nu_\mu A_\nu = \partial_\mu(\frac{\partial^\nu A_\nu}{\Box})$, we note that $\omega^\nu_\mu A_\nu$ points to the direction of the momentum P_μ whereas (C.6) implies that $\theta^\nu_\mu A_\nu$ is orthogonal to it. Thus, from (C.4), $A_\mu = \delta^\nu_\mu A_\nu \Rightarrow A_\mu = \theta^\nu_\mu A_\nu + \omega^\nu_\mu A_\nu$, we conclude that every quadrivector can be decomposed in a spin-0 projection (one longitudinal direction, $\omega^\nu_\mu A_\nu$) and a spin-1 projection (three transverse directions $\theta^\nu_\mu A_\nu$).

From the vector projection operators we can construct a basis for rank-2 tensors which allows us to decompose them in orthogonal subspaces. In this work we are using the projection operators of (NIEUWENHUIZEN, 1973). Notice that a slightly different basis was employed before by (RIVERS, 1964). Let us limit ourselves to **rank-2 symmetric tensors**[1]:

$$(P^{(2)}_{ss})^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\left(\theta^{\mu\alpha}\theta^{\nu\beta} + \theta^{\mu\beta}\theta^{\nu\alpha}\right) - \frac{1}{D-1}\theta^{\mu\nu}\theta^{\alpha\beta}, \tag{C.7}$$

$$(P^{(1)}_{ss})^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\left(\theta^{\mu\alpha}\omega^{\nu\beta} + \theta^{\mu\beta}\omega^{\nu\alpha} + \theta^{\nu\alpha}\omega^{\mu\beta} + \theta^{\nu\beta}\omega^{\mu\alpha}\right), \tag{C.8}$$

$$(P^{(0)}_{ss})^{\mu\nu\alpha\beta} \equiv \frac{\theta^{\mu\nu}\theta^{\alpha\beta}}{D-1} \quad , \quad (P^{(0)}_{ww})^{\mu\nu\alpha\beta} \equiv \omega^{\mu\nu}\omega^{\alpha\beta}. \tag{C.9}$$

In this case, it is necessary to include transition operators

$$(P^{(0)}_{sw})^{\mu\nu\alpha\beta} \equiv \frac{\theta^{\mu\nu}\omega^{\alpha\beta}}{\sqrt{D-1}} \quad , \quad (P^{(0)}_{ws})^{\mu\nu\alpha\beta} \equiv \frac{\omega^{\mu\nu}\theta^{\alpha\beta}}{\sqrt{D-1}}. \tag{C.10}$$

Both projection and transition operators satisfy the following algebra

$$P^{(r)}_{ij}P^{(q)}_{kl} = \delta^{rq}\delta_{jk}P^{(r)}_{il}, \tag{C.11}$$

and the completeness relation is given by

$$S^{\mu\nu\alpha\beta} = \left(P^{(2)}_{ss} + P^{(1)}_{ss} + P^{(0)}_{ss} + P^{(0)}_{ww}\right)^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta}\right), \tag{C.12}$$

where $S^{\mu\nu\alpha\beta}$ is symmetric part of the identity operator. Again, the trace of these operators are related to spin s through $dim = 2s+1$. The value of s shows in which spin the decomposition is done and such value is indicated inside the upper parentheses that identify the operators. In this way, $(P^{(r)}_{ss})$ projects in a sector associated with spin-r.

Reminder: In the Hamiltonian analysis we have used these operators only for spatial indices. In this case, if the spacetime dimension is D, replace $D-1 \longrightarrow D-2$ in (C.7-C.10). Additionally, replace the D'Alembert operator $\Box$ by the Laplacian ∇^2 and the metrics $\eta_{\mu\nu}$ by δ_{ij}.

[1] For a generalization see (BITTENCOURT, 2019). In the same way we can define projection operators for the antisymmetric part. Therefore any tensor can be decomposed in its symmetric and antisymmetric parts through these operators.

APÊNDICE D – MATHEMATICAL DEFINITIONS

In the chapters 5 and 6, we have been using geometrical quantities defined on differential manifolds with non-vanishing curvature. Their derivation or motivations can be found in any general relativity book, e.g. (WEINBERG, 2013). In the following, we are describing some of those quantities[1].

Given a metric $g_{\mu\nu}$, we can define a derivative tensor operation through the affine connection,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}), \tag{D.1}$$

symmetric in its lower indexes (torsionless), and such that $\nabla_\mu g_{\alpha\beta} = 0$ (metricity - the metric is compatible). Then, the covariant derivative acts on contravariant and covariant labels according to

$$\nabla_\mu A^{\lambda}{}_{\nu} = \partial_\mu A^{\lambda}{}_{\nu} + \Gamma^{\lambda}_{\mu\kappa} A^{\kappa}{}_{\nu} - \Gamma^{\kappa}_{\mu\nu} A^{\lambda}{}_{\kappa}. \tag{D.2}$$

As an effect of the manifold curvature, we have the Riemann tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\mu}_{\lambda\beta} \Gamma^{\lambda}_{\nu\alpha}, \tag{D.3}$$

and

$$[\nabla_\mu, \nabla_\nu] h^{\lambda}{}_{\rho} = R^{\lambda}{}_{\sigma\mu\nu} h^{\sigma}{}_{\rho} - R^{\sigma}{}_{\rho\mu\nu} h^{\lambda}{}_{\sigma}, \tag{D.4}$$

The Riemann tensor has the following properties:

- 1. $R^{\beta}{}_{\lambda\mu\nu} = -R^{\beta}{}_{\lambda\nu\mu}$;
- 2. $R_{\beta\lambda\mu\nu} = -R_{\lambda\beta\mu\nu}$, where the metric $g_{\mu\nu}$ lowered the first index;
- 3. $R_{\lambda[\nu\alpha\mu]} = 0$, algebraic Bianchi identity, where the brackets denote sum over a cyclic permutation of the last three indexes;
- 4. $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$;
- 5. $\nabla_{[\sigma} R_{\lambda\rho]\mu\nu} = 0$: Bianchi identity.

Moreover, we have the Ricci tensor

$$R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu}, \tag{D.5}$$

and the Ricci (curvature) scalar

$$R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{D.6}$$

[1] Be careful with the conventions assumed by each book. In this work, we are using Maggiore's convention (MAGGIORE, 2008).

The Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \qquad \Rightarrow \qquad \nabla^{\mu} G_{\mu\nu} = 0, \tag{D.7}$$

as a consequence of the Bianchi identity.

One can show that in D dimensions, the Riemann tensor has $\dfrac{D^2(D^2-1)}{12}$ independent components and we can decompose it as

$$\begin{aligned} R_{\lambda\mu\nu\alpha} = C_{\lambda\mu\nu\alpha} - \frac{R}{(D-1)(D-2)}(g_{\lambda\nu} g_{\mu\alpha} - g_{\lambda\alpha} g_{\mu\nu}) + \\ - \frac{1}{(D-2)}(g_{\lambda\alpha} R_{\mu\nu} + g_{\mu\nu} R_{\lambda\alpha} - g_{\lambda\nu} R_{\mu\alpha} - g_{\mu\alpha} R_{\lambda\nu}), \end{aligned} \tag{D.8}$$

where $C_{\lambda\mu\nu\alpha}$ is the Weyl (**conformal**) tensor, which satisfies all the algebraic properties of the Riemann tensor, but it is completely traceless, i.e. vanishes for any contraction like $g^{\mu\nu} C_{\mu\alpha\nu\beta} = 0$. Under a Weyl, or conformal, transformation $\tilde{g} = e^{2\phi} g$, for any scalar function $\phi(x)$, the Weyl tensor transforms simply as $\tilde{C}_{\lambda\mu\nu\alpha} = e^{2\phi} C_{\lambda\mu\nu\alpha}$, which justifies its name. Other remarkable property is the fact that, for $D \geq 4$, the Weyl tensor do vanish if and only if the metric tensor is locally conformally flat.[2]

Rewriting the Ricci tensor with its traceless part $R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{R}{D} g_{\mu\nu}$, we finally obtain the irreducible parts of the Riemann tensor for $D > 3$:

$$\begin{aligned} R_{\lambda\mu\nu\alpha} = \; & C_{\lambda\mu\nu\alpha} - \frac{1}{(D-2)}(g_{\lambda\alpha} \tilde{R}_{\mu\nu} + g_{\mu\nu} \tilde{R}_{\lambda\alpha} - g_{\lambda\nu} \tilde{R}_{\mu\alpha} - g_{\mu\alpha} \tilde{R}_{\lambda\nu}) + \\ & + \frac{R}{D(D-1)}(g_{\lambda\nu} g_{\mu\alpha} - g_{\lambda\alpha} g_{\mu\nu}). \end{aligned} \tag{D.9}$$

Einstein spaces are a special type of manifold, defined by

$$\tilde{R}_{\mu\nu} = 0. \tag{D.10}$$

In this case, from (D.7), we conclude that the curvature is constant, i.e. $\nabla_{\mu} R = 0$ and $\nabla_{\mu} R_{\alpha\nu} = 0$. From the Bianchi identity, we also have $\nabla^{\nu} R_{\nu\mu\alpha\beta} = 0 = \nabla^{\nu} C_{\nu\mu\alpha\beta}$. The Weyl tensor, more precisely its transverse part, is arbitrary. The maximally symmetric spaces are well-known Einstein spaces, when $\tilde{R}_{\mu\nu} = 0$ and $C_{\lambda\mu\nu\alpha} = 0$.

For example, maximally symmetric spaces with (negative) positive constant curvature, are called (A)dS, or (anti-)de Sitter. A global zero curvature manifold is simply the Minkowski space. These three cases, AdS, dS and Minkowski, are the solutions of the Einstein equations (1.11) for an empty universe ($T_{\mu\nu} = 0$) with negative, positive, and zero cosmological constants, respectively. Another relevant maximally symmetric space is the spatial part of the FLRW metric (standard cosmology), where space (here, space does not mean spacetime) is assumed to be homogeneous and isotropic.

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[2] Notice from the Einstein equations that the Weyl tensor is the only irreducible part of the Riemann tensor which is not ruled by the energy-momentum tensor. This means that even in vacuum, where $R_{\mu\nu} = 0$, we could have non-vanishing Weyl tensors. However, if we assume that spacetime becomes curved in the presence of gravitational fields, and in their absence there always exists a coordinate system where $g_{\mu\nu} = \eta_{\mu\nu}$, then in vacuum the Weyl tensor will vanish (zeroth law of gravitation, (STRAUMANN, 2013)).

APÊNDICE E – CURVED BACKGROUND PASSAGES: IDENTITIES

Let us deduce here some mathematical identities. Those were largely used in chapters 5 and 6. Basically they are consequence of (D.4). We will be using the notation $\doteq$ to indicate that we are using Einstein spaces properties.

$$[\nabla_\mu, \nabla_\nu]\xi^\mu = R^{\mu\lambda}{}_{\mu\nu}\xi_\lambda = R_{\lambda\nu}\xi^\lambda. \tag{E.1}$$

$$\begin{aligned}\nabla^\alpha\Box\psi_\nu &= \nabla^\alpha\nabla^\beta\nabla_\beta\psi_\nu = \nabla^\beta\nabla^\alpha\nabla_\beta\psi_\nu + R_\nu{}^{\rho\alpha\beta}\nabla_\beta\psi_\rho + R_\beta{}^{\rho\alpha\beta}\nabla_\rho\psi_\nu \\ &= \nabla^\beta\nabla_\beta\nabla^\alpha\psi_\nu + \nabla^\beta R_\nu{}^{\rho\alpha}{}_\beta\psi_\rho + R_\nu{}^{\rho\alpha\beta}\nabla_\beta\psi_\rho - R^{\rho\alpha}\nabla_\rho\psi_\nu \\ &\doteq \Box\nabla^\alpha\psi_\nu + 2R_\nu{}^{\rho\alpha\beta}\nabla_\beta\psi_\rho - R/D\nabla^\alpha\psi_\nu.\end{aligned} \tag{E.2}$$

$$\begin{aligned}\nabla^\alpha\Box\psi_\alpha &= \Box\nabla^\alpha\psi_\alpha + \nabla^\beta R_\alpha{}^{\rho\alpha}{}_\beta\psi_\rho + R_\alpha{}^{\rho\alpha\beta}\nabla_\beta\psi_\rho - R^{\rho\alpha}\nabla_\rho\psi_\alpha \\ &= \Box\nabla^\alpha\psi_\alpha + \nabla_\beta R^{\rho\beta}\psi_\rho + R^{\rho\beta}\nabla_\beta\psi_\rho - R^{\rho\alpha}\nabla_\rho\psi_\alpha \\ &= \Box\nabla^\alpha\psi_\alpha + \nabla_\beta R^{\rho\beta}\psi_\rho \doteq \Box\nabla^\alpha\psi_\alpha + R/D\nabla^\alpha\psi_\alpha.\end{aligned} \tag{E.3}$$

$$\begin{aligned}\Box\nabla_\alpha\varphi &= \nabla^\beta\nabla_\beta\nabla_\alpha\varphi = \nabla^\beta\nabla_\alpha\nabla_\beta\varphi = \nabla_\alpha\nabla^\beta\nabla_\beta\varphi - R^\lambda{}_\beta{}^\beta{}_\alpha\nabla_\lambda\varphi \\ &= \nabla_\alpha\Box\varphi + R_{\lambda\alpha}\nabla^\lambda\varphi \doteq \nabla_\alpha\Box\varphi + R/D\nabla_\alpha\varphi.\end{aligned} \tag{E.4}$$

$$\begin{aligned}\nabla^\mu\nabla^\alpha\Box\xi_\mu &= \nabla^\alpha\nabla^\mu\Box\xi_\mu + R^{\lambda\alpha}\Box\xi_\lambda = \nabla^\alpha\Box\nabla^\mu\xi_\mu + \nabla^\alpha\nabla_\beta R^{\rho\beta}\xi_\rho + R^{\lambda\alpha}\Box\xi_\lambda \\ &\doteq \nabla^\alpha\Box\nabla^\mu\xi_\mu + R/D\nabla^\alpha\nabla^\rho\xi_\rho + R/D\Box\xi^\alpha.\end{aligned} \tag{E.5}$$

$$\begin{aligned}\nabla^\mu\nabla^\alpha\nabla^\nu\nabla_\mu\xi_\nu &= \nabla^\mu\nabla^\alpha\nabla_\mu\nabla\cdot\xi + \nabla^\mu\nabla^\alpha R_{\lambda\mu}\xi^\lambda \\ &\doteq \Box\nabla^\alpha\nabla\cdot\xi + R/D\nabla^\mu\nabla^\alpha\xi_\mu = \Box\nabla^\alpha\nabla\cdot\xi + R/D\nabla^\alpha\nabla\cdot\xi + R^2/D^2\xi^\alpha.\end{aligned} \tag{E.6}$$

$$\begin{aligned}\nabla^\mu\nabla^\alpha\nabla_\nu h_{\mu\alpha} &= \nabla^\mu\nabla_\nu\nabla^\alpha h_{\mu\alpha} + \nabla^\mu R_\mu{}^{\lambda\alpha}{}_\nu h_{\lambda\alpha} + \nabla^\mu R_\alpha{}^{\lambda\alpha}{}_\nu h_{\mu\lambda} \\ &= \nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + R_\mu{}^{\lambda\mu}{}_\nu\nabla^\alpha h_{\lambda\alpha} + \nabla^\mu R_{\mu\lambda\alpha\nu}h^{\lambda\alpha} + \nabla_\mu R_{\lambda\nu}h^{\mu\lambda} \\ &= \nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + R_{\lambda\nu}\nabla_\alpha h^{\lambda\alpha} + \nabla^\mu R_{\mu\lambda\alpha\nu}h^{\lambda\alpha} + \nabla_\mu R_{\lambda\nu}h^{\mu\lambda} \\ &\doteq \nabla_\nu\nabla^\mu\nabla^\alpha h_{\mu\alpha} + R_{\mu\lambda\alpha\nu}\nabla^\mu h^{\lambda\alpha} + 2R/D\nabla^\mu h_{\mu\nu}.\end{aligned} \tag{E.7}$$

$$\begin{aligned}\nabla^\mu\nabla^\alpha\nabla_\mu h_{\nu\alpha} &= \nabla^\alpha\nabla^\mu\nabla_\mu h_{\nu\alpha} + R_\mu{}^{\lambda\mu\alpha}\nabla_\lambda h_{\nu\alpha} + R_\nu{}^{\lambda\mu\alpha}\nabla_\mu h_{\lambda\alpha} + R_\alpha{}^{\lambda\mu\alpha}\nabla_\mu h_{\nu\lambda} \\ &= \nabla^\mu\Box h_{\mu\nu} - R_{\mu\lambda\alpha\nu}\nabla^\mu h^{\lambda\alpha} + R^{\lambda\alpha}\nabla_\lambda h_{\nu\alpha} - R^{\mu\lambda}\nabla_\mu h_{\nu\lambda} \\ &= \nabla^\mu\Box h_{\mu\nu} - R_{\mu\lambda\alpha\nu}\nabla^\mu h^{\lambda\alpha}.\end{aligned} \tag{E.8}$$

APÊNDICE F – TDIFF - CONSISTENT WITH FLAT LIMIT

In this Appendix, we derive the TDiff action in curved backgrounds, without starting from a general Lagrangian. Here, and in the next Appendix, we follow exactly the approach of (BUCHBINDER; KRYKHTIN; PERSHIN, 1999) when it comes to consistency with flat limit. Recalling that TDiff is given by (2.7), we write the starting action $S \equiv \int d^D x\sqrt{-g}\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_{TDiff} + \frac{a_1}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{a_2}{2}Rh^2 + \frac{a_3}{2}R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \frac{a_4}{2}R^{\mu\beta}h_{\mu\nu}h^{\nu}{}_{\beta} + \frac{a_5}{2}R^{\mu\nu}h_{\mu\nu}h, \tag{F.1}$$

where $\mathcal{L}_{TDIFF}$ is written with the minimal coupling as

$$\mathcal{L}_{TDIFF} = -\frac{1}{4}\nabla_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho} + \frac{1}{2}\nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\alpha\nu} - \frac{a}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{b}{4}\nabla_\mu h\nabla^\mu h. \tag{F.2}$$

Under transverse diffeomorphisms (TDiff) $\delta h_{\mu\nu} = \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu$, let us find the five coefficients above such that the action becomes invariant. The calculation is straightforward, and a little bit tedious, so we are omitting some steps. Noting that $\delta h = 0$, the transformation only affects the traceless part of $h_{\mu\nu}$. Consequently, from the arbitrary coefficients a and b that define the TDiff model, b plays no role here. Writing the infinitesimal transverse vector as $\chi_\mu = \xi^T_\mu$, and from the commutation relation (D.4), we have, up to first order in χ_μ,

$$2\delta\mathcal{L}_{TDIFF} = 2\nabla_\alpha h_{\mu\nu}\nabla^\mu\nabla^\nu\chi^\alpha + 2R_{\nu\lambda\mu\alpha}\chi^\lambda\nabla^\alpha h^{\mu\nu} - a\ \nabla_\mu h(R^{\lambda\mu}\chi_\lambda + \Box\chi^\mu), \tag{F.3}$$

and for the non-minimal terms,

$$\delta(R^{\mu\beta}h_{\mu\nu}h^{\nu}{}_{\beta}) = 2R^{\mu\beta}h_{\mu\nu}(\nabla_\beta\chi^\nu + \nabla^\nu\chi_\beta), \tag{F.4}$$

$$\delta(R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}) = 2R^{\mu\alpha\nu\beta}h_{\mu\nu}(\nabla_\alpha\chi_\beta + \nabla_\beta\chi_\alpha), \tag{F.5}$$

$$\delta(h_{\mu\nu}h^{\mu\nu}) = 4h_{\mu\nu}\nabla^\mu\chi^\nu, \tag{F.6}$$

$$\delta(R^{\mu\nu}h_{\mu\nu}h) = 2R^{\mu\nu}\nabla_\mu\chi_\nu h. \tag{F.7}$$

According to (D.4), we also have

$$\nabla_\alpha h_{\mu\nu}\nabla^\mu\nabla^\nu\chi^\alpha = \nabla^\mu h_{\mu\nu}R^{\nu\lambda}\chi_\lambda + R^{\lambda\nu\mu\alpha}h_{\mu\lambda}\nabla_\nu\chi_\alpha - R^{\lambda\alpha}h_\lambda{}^\nu\nabla_\nu\chi_\alpha, \tag{F.8}$$

$$\nabla_\mu h(R^{\lambda\mu}\chi_\lambda + \Box\chi^\mu) = -2h(\nabla_\mu R^{\mu\nu})\chi_\nu - 2hR_{\mu\nu}(\nabla^\mu\chi^\nu). \tag{F.9}$$

Combining these results in (F.3), we obtain the following terms

$$R^{\lambda\nu\mu\alpha}h_{\mu\lambda}(\nabla_\nu\chi_\alpha) + R^{\nu\lambda\mu\alpha}(\nabla_\alpha h_{\mu\nu})\chi_\lambda = (\nabla_\nu R^{\nu\lambda\mu\alpha})h_{\mu\lambda}\chi_\alpha, \tag{F.10}$$

after integration by parts. Consequently,

$$2\delta\mathcal{L}_{TDIFF} = 2(\nabla_\nu R^{\nu\lambda\mu\alpha})h_{\mu\lambda}\chi_\alpha - 2(\nabla^\mu R^{\nu\lambda})h_{\mu\nu}\chi_\lambda - 4R^{\nu\lambda}h_\nu{}^\mu(\nabla_\mu\chi_\lambda) + 2a(\nabla_\mu R^{\mu\lambda})h\chi_\lambda + 2aR_{\mu\lambda}h(\nabla^\mu\chi^\lambda). \tag{F.11}$$

On the other hand, the only non-minimal term containing the Riemann (and the Weyl) tensor gives

$$\delta(a_3 R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}) = 4a_3(\nabla_\alpha R^{\alpha\mu\nu\beta})h_{\mu\nu}\chi_\beta - 4a_3 R^{\mu\alpha\nu\beta}(\nabla_\alpha h_{\mu\nu})\chi_\beta. \tag{F.12}$$

It is a wise choice to start by analysing the Weyl tensor terms:

$$2\delta\mathcal{L} = (2+4a_3)(\nabla_\nu C^{\nu\lambda\mu\alpha})h_{\mu\lambda}\chi_\alpha - 4a_3 C^{\mu\alpha\nu\beta}(\nabla_\alpha h_{\mu\nu})\chi_\beta + \dots . \tag{F.13}$$

To assure $\delta\mathcal{L} = 0$, each coefficient must vanish. Hence, from the second term above, $a_3 = 0$, otherwise the Weyl tensor would have to be null - a solution more restrictive than we find below. Furthermore, it is necessary that $\nabla_\alpha C^{\alpha\nu\lambda\mu} = 0$, which means that the curved space background cannot be arbitrary.

Next, we shall consider the traceless Ricci tensor $\tilde{R}_{\mu\nu}$ (without derivatives upon it) terms:

$$2\delta\mathcal{L} = (2a+2a_5)\tilde{R}_{\mu\alpha}h(\nabla^\mu\chi^\alpha) + 2a_4\tilde{R}_{\mu\nu}h^{\mu\alpha}(\nabla^\nu\chi_\alpha) + (2a_4-4)\tilde{R}_{\mu\nu}h^{\mu\alpha}(\nabla_\alpha\chi^\nu) + \cdots . \tag{F.14}$$

Now, it follows that $\tilde{R}_{\mu\nu} = 0$ is required for $\delta\mathcal{L} = 0$, and we are back to Einstein spaces.

According to the Appendix, for Einstein spaces, the curvature is constant, the Ricci tensor is proportional to the Ricci scalar, the Weyl tensor is arbitrary, and these quantities are transverse. In addition, the coefficients a_4 and a_5 are redundant. Then, by using (F.6) and (F.11), we rewrite

$$\delta\mathcal{L} = \left(\frac{2R}{D} - 2a_1\right)\chi^\mu\nabla^\nu h_{\mu\nu}, \tag{F.15}$$

and conclude that $a_1 = R/D$. Noticing that h^2 is kept invariant under TDiff, a_2 is arbitrary. Therefore the action of the Lagrangian

$$\mathcal{L}_{TDIFF} = -\frac{1}{4}\nabla_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{a}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{b}{4}\nabla_\mu h\nabla^\mu h + \frac{R}{2D}h_{\mu\nu}h^{\mu\nu} + \gamma R h^2 \tag{F.16}$$

is invariant under TDiff in Einstein spaces. Comparing (F.16) with (5.24), we notice that if $a = 1 = b$, $S_{\text{TDiff}}[\gamma = -R/4D] = S_{\text{Diff}}$, showing that Diff is also a natural enhancement of the TDiff symmetry in curved space backgrounds.

Alternatively, we can formulate an Einstein space extension of Diff propagating 3 degrees of freedom by using the expression (3.63). Although it describes 3 degrees of freedom, its symmetry is still Diff. Recalling that (5.24) is invariant under Diff, in order to find which non-minimal terms must be added, it suffices to vary $\mathcal{L}_\phi$

$$\mathcal{L}_\phi = \frac{1-a}{2}\nabla_\nu\phi\nabla_\mu h^{\mu\nu} + \frac{1-a}{2}h\Box\phi + \frac{2a-b-1}{4}\phi\Box\phi, \tag{F.17}$$

so, up to integration by parts, under $\delta h_{\mu\nu} = \nabla_\mu \psi_\nu + \nabla_\nu \psi_\mu$,

$$\delta \mathcal{L}_\phi = (a-1)\phi R_{\mu\nu} \nabla^\mu \psi^\nu , \tag{F.18}$$

which clearly suggests that we shall include a term $\sim R_{\mu\nu}\phi h^{\mu\nu}$ in the action. Recall that $R_{\mu\nu} = R/D g_{\mu\nu}$ and note that $R\phi^2$ is invariant. Hence,

$$S = \int d^Dx\sqrt{-g}\left[-\frac{1}{4}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} + \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\nu h^{\mu\alpha} - \frac{1}{2}\nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h \nabla^\mu h + \right. \\ \left. + \frac{R}{2D}\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2}h^2\right) + \frac{1-a}{2}\nabla_\mu\phi\nabla_\mu h^{\mu\nu} + \frac{1-a}{2}h\Box\phi + \frac{2a-b-1}{4}\phi\Box\phi + \frac{1-a}{2D}R\phi h + \gamma R\phi^2 \right] \tag{F.19}$$

is another massless spin-2 and spin-0 model in Einstein space backgrounds but, differently from (F.16), it is now invariant under linearized Diff.

APÊNDICE G – MASSIVE SCALAR-TENSOR - CONSISTENT WITH FLAT LIMIT

Here we have chosen to deal with (3.64), instead of (3.16) or (3.17). We have done it for two reasons: first, a part of the problem is already done (just recover the FP equations of motion for arbitrary backgrounds) and secondly, for the sake of simplicity, it is prudent[1] to consider kinetic terms invariant by Diff and mass terms being the responsible for breaking such invariance. Differently from the FP case, we now have a scalar field, coupled to $h_{\mu\nu}$. So, in $D = 4$, we have a set of 11 independent variables $h_{(\mu\nu)}$,ϕ that must be reduced to 6 (5 for the massive graviton and 1 for the massive scalar); thus, like in the FP case, we have to find out 5 constraints from the equations of motion.

Our input Ansatz is

$$\mathcal{L}_m = \mathcal{L}_{FP} + \frac{1-a}{2}\nabla_\mu\varphi\nabla_\nu h^{\mu\nu} + \frac{1-a}{2}h(\Box - m^2)\varphi + \frac{2a-b-1}{4}\varphi(\Box - m^2)\varphi + \frac{b_1}{2}R\varphi^2 + \frac{b_2}{2}Rh\varphi + \frac{b_3}{2}R_{\mu\nu}\varphi h^{\mu\nu}, \tag{G.1}$$

where $\mathcal{L}_{FP}$ is given by (5.7). Again, we emphasize that we could have written (5.7) in terms of the irreducible parts of the Riemann tensor, as well as have admitted higher derivative terms.

We start evaluating the equations of motion. From $\delta h_{\mu\nu}$ and $\delta\varphi$, we respectively have

$$E_{\mu\nu} = E^{FP}_{\mu\nu} + (a-1)\nabla_\mu\nabla_\nu\varphi + (1-a)(\Box - m^2)\varphi g_{\mu\nu} + b_2 R\varphi g_{\mu\nu} + b_3 R_{\mu\nu}\varphi = 0, \tag{G.2}$$

$$\Phi = (a-1)\nabla_\mu\nabla_\nu h^{\mu\nu} + (1-a)(\Box - m^2)h + (2a-b-1)(\Box - m^2)\varphi + 2b_1 R\varphi + b_2 Rh + b_3 R_{\mu\nu}h^{\mu\nu} = 0, \tag{G.3}$$

where $E^{FP}_{\mu\nu}$ is given by (5.8). From (G.2), we have

$$\nabla^\mu E_{\mu\nu} = (\nabla^\mu E^{FP}_{\mu\nu}) + (a-1)R_{\lambda\nu}\nabla^\lambda\varphi + (a-1)m^2\nabla_\nu\varphi + b_2 R\nabla_\nu\varphi + b_3 R_{\mu\nu}\nabla^\mu\varphi + [\varphi(\nabla R)], \tag{G.4}$$

where $[\varphi(\nabla R)]$ denotes terms with derivatives of the curvature quantities. Since (5.9) is first order in derivatives, it follows that the previous equation is also. Thus,

$$\phi_\nu = \nabla^\mu E_{\mu\nu} = 0 \tag{G.5}$$

is the vector constraint. Regarding the scalar one, we start with

$$\Omega = A\, m^2 g^{\mu\nu}E_{\mu\nu} + B\nabla^\mu\nabla^\nu E_{\mu\nu} + c_1 g^{\mu\nu}RE_{\mu\nu} + c_2 R^{\mu\nu}E_{\mu\nu} + \kappa_1 m^2\Phi + \kappa_2 R\Phi. \tag{G.6}$$

Following the FP case, we decompose the Riemann and Ricci tensor in its irreducible parts. By factoring out each different term coming from Ω, we immediately obtain $C^{\mu\alpha\nu\beta}\nabla_\mu\nabla_\nu h_{\alpha\beta}\,(2a_3 B) = 0$. Therefore $a_3 = 0$, otherwise the Weyl tensor would be null, or $B = 0$ would imply that $\Omega = 0$

[1] Notice that $\mathcal{L}(1,1,1,1)$ is essential for $\nabla^\mu E_{\mu\nu} = 0$ to be a constraint.

identically[2]. Moreover, from the terms proportinal to $\tilde{R}_{\mu\nu}$:

$$\tilde{R}^{\mu\nu}\nabla_\mu\nabla^\alpha h_{\alpha\nu}\left[2B\left(a_4-1-c_2\right)\right]+\tilde{R}_{\mu\nu}\nabla^\mu\nabla^\nu\varphi\left[B(b_3+a-1)+c_2(a-1)\right]+ \\ +\tilde{R}^{\mu\nu}\Box h_{\mu\nu}\left[Ba_5+c_2\right]+\tilde{R}^{\mu\nu}\nabla_\mu\nabla_\nu h\left[B\left(a_5+1\right)+c_2\right]+\cdots, \tag{G.7}$$

where $\cdots$ denote terms with derivatives upon $R_{\mu\nu}$, we see that it is not possible to get rid simultaneously of the second line brackets unless $\tilde{R}_{\mu\nu}=0$. Again, we have Einstein spaces. So, this is a **necessary condition** for the curved action to be consistent with the correct number of degrees of freedom. The coefficients a_4, a_5, c_2 and b_3 become redundant, and can all be set to zero. Now, note that $\Omega=0$ can be written as

$$m^2(\nabla_\mu\nabla_\nu h^{\mu\nu}-\Box\varphi)\left[A(D-2)-B+\kappa_1(a-1)\right]+m^2\Box h\left[A(2-D)+B+\kappa_1(1-a)\right]+ \\ +R\Box\varphi\left[B\left(b_2+\frac{b_3}{D}+\frac{a-1}{D}\right)+c_1(D-1)(1-a)+\kappa_2\left(2a-b-1\right)\right]+ \\ +R\nabla^\mu\nabla^\nu h_{\mu\nu}\left[B\left(2a_1-\frac{2}{D}\right)+c_1(D-2)+\kappa_2(a-1)\right]+ \\ +R\Box h\left[B\left(2a_2+\frac{1}{D}\right)+c_1(2-D)+\kappa_2(1-a)\right]+\cdots=0, \tag{G.8}$$

where the $\cdots$ denote terms without derivatives, and bear in mind that any derivative of Riemann or Ricci tensors vanishes. To ensure $\Omega=0$ has no second derivatives, each bracket must vanish. So, from the last two lines, we have

$$2a_1+2a_2-\frac{1}{D}=0. \tag{G.9}$$

This is the same condition that has emerged in the FP case, when we have defined $a_1=\dfrac{\gamma}{D}$, and then $a_2=\dfrac{1-\gamma}{2D}$. This is also the necessary and sufficient condition for Ω to be a constraint. Indeed, from the first line, we can solve it to obtain $A=B\dfrac{(a^2-b)}{d_0}$ and $k_1=B\dfrac{(a-1)}{d_0}$, whilst from the other lines, $c_1=\dfrac{Bf_1}{d_0}$ and $k_2=\dfrac{Bf_2}{d_0}$, where f_1 and f_2 are functions only of the Lagrangian coefficients. B can be set to one, without loss of generality. Hence, Ω is completely determined in terms of a,b,γ,b_1 and b_2, and we have an authentic constraint. Recall that the condition $d_0\neq 0$, given by (1.7), again shows up.

Notice that b_1 and b_2 are not required to assume any special values, and can be set to zero. On the other hand, (G.9) states that it is not possible to have a minimal action (obtained only replacing usual by covariant derivatives) propagating the correct number of degrees of freedom. Finally, if $a_1=\dfrac{\gamma}{D}$, and $a_2=\dfrac{1-\gamma}{2D}$, the massive "TDiff" model can be written in a curved background as

$$S_m=S_{FP}+\int d^Dx\sqrt{-g}\left[\left(\frac{1-a}{2}\right)\left(\nabla_\mu\varphi\nabla_\nu h^{\mu\nu}+h(\Box-m^2)\varphi\right)+\frac{2a-b-1}{4}\varphi(\Box-m^2)\varphi\right], \tag{G.10}$$

where S_{FP} is given by (5.14), and the required manifold is an Einstein space.

[2] The solution $B=0$ would imply that Ω is identically zero, and we would not have a scalar constraint, for any pair (a,b). This would also be the FP case if we had started with arbitrary coefficients in (5.10).