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Time Defects in Floquet Topological Phases

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Dedico esta dissertação aos meus pais,
Boris e Fátima.

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"Tudo o que eu fiz não vale uma sinfonia de Mozart."

César Lattes[1]

[1] Relato de Henrique Linz de Barros à revista "Pesquisa FAPESP"

Resumo

Sistemas topológicos de Floquet são generalizações de sistemas topológicos estáticos para hamiltonianas periódicas no tempo. Recentemente, aqueles modelos receberam especial atenção em parte devido a novas propriedades topológicas dinâmicas que estendem seus análogos estáticos. Nesta dissertação, avançamos nesse sentido ao tentar definir defeitos extrínsecos da simetria da periodicidade do tempo, os quais chamamos de defeitos temporais. Depois de revisar a literatura sobre fases estáticas e topológicas de Floquet, discutimos por que a construção de defeitos em sistemas estáticos não pode ser estendida diretamente para defeitos temporais. Isso é mostrado explicitamente por meio de simulações numéricas que indicam inconsistências da abordagem estática no modelo inerentemente dinâmico de Rudner-Lindner-Berg-Levin.

Palavras Chaves: Fases topológicas da matéria; Sistemas de Floquet; Isolantes topológicos de Floquet; Simetria temporal; Defeitos temporais.

Áreas do conhecimento: Teoria quântica; Matéria condensada; Informação quântica.

Abstract

Floquet topological systems are generalizations of static topological systems to time-periodic Hamiltonians. They have recently drawn attention partly due to novel dynamical topological properties that extend their static counterparts. In this essay, we continue this trend by attempting to define extrinsic defects of the time periodicity symmetry, which we call time defects. After reviewing the literature on static and Floquet topological phases, we discuss why the construction of defects in static systems cannot be directly extended to time defects. This is shown explicitly via numerical simulations that point to inconsistencies of the static approach in the inherently dynamical Rudner-Lindner-Berg-Levin model.

Keywords: Topological phases of matter; Floquet systems; Floquet topological insulators; Temporal symmetry; Time defects.

Subject areas: Quantum theory; Condensed matter; Quantum information.

Contents

Chapter 1

Introduction

At every scale, the universe is composed of elementary systems that interact with each other in complex ways, from the gravitational pull of galaxies to the social communication between people to the strong force that holds atoms together. In all of these systems, new properties of the whole *emerge* from the interaction between its parts [1]. In physics, condensed matter broadly studies the principles that govern these patterns of emergence.

One of the ways to organize physical systems based on their macroscopic properties is classify them into different phases of matter. These include everyday materials, such as liquids and solids, but also exotic systems, such as superconductors and *topological systems*. The latter has gained much research attention in the last decades and is responsible for a paradigm shift from Landau's theory of phase transitions [2] based on spontaneous symmetry breaking (SSB) to a more general framework that will be thoroughly described in this essay.

One of the key features of topological phases is the stability against generic local perturbations [3]. Consequently, they do not generally support local quantities such as an order parameter or a bulk current. Instead, nonlocal analogs may appear, such as integer-valued topological invariants that require global information of the system and conducting currents localized at the material's boundary. Not only are these properties of fundamental theoretical relevance, as they showcase unusual behaviors of many-body systems, but of experimental too. Two classical examples are the precise and robust measurement of the fine-structure constant via the quantum Hall effect [4]; and the possibility of topological quantum computing, which promises stability against decoherence due to nonlocal excitations [5].

The majority of the developments in the area of topological systems were made in the static regime. However, the last decade has brought extensions of the static formalism to Hamiltonians that are periodic in time [6–8], also called *Floquet Hamiltonians*. The interest in them naturally arises when a periodic drive is perturbing a system that would otherwise be at rest, e.g. a photoreceptive material under the external influence of a light wave. Since it contains the static case as

a trivial limit, Floquet topological phases exhibit a richer set of phenomena than their static partners. At the same time, many concepts of the latter have already been generalized to the former [9].

One concept of static topology has not been explored before in the literature of Floquet topology: *extrinsic defects*. These are external modifications of symmetric systems that can probe nontrivial topological features in the bulk. For example, one can alter a two-dimensional Hamiltonian along a line in such a way that their endpoints, where the defects lie, host a symmetry "flux" that is detectable via an adiabatic loop that passes through the line and contains one of the defects [10]. For Floquet systems, time translation by one period is their defining symmetry, but the characterization of their associated extrinsic defects, which we will call *time defects*, is still lacking.

The research goal of this essay is, then, to explore the construction of time defects, focusing on the special role of the time symmetry. One immediate problem hinders a direct extension of the static case: it is assumed by the definition of extrinsic defects that the symmetry is realized by an on-site operator, i.e. that can be decomposed into parts that act on a single site of a lattice. On the contrary, the operator of time evolution by one period U_F need not be on-site. In fact, for some models, U_F is identity in the bulk and not on-site at the edge, so we would expect that their time defects would either be trivial or ill-defined, respectively. One such model is due to Rudner, Lindner, Berg, and Levin (RLBL) [8], which will be studied in detail due to its inherently dynamical properties and will serve as the basis for numerical simulations that illustrates the aforementioned issue.

The text is organized as follows: we begin with a literature review in Chapter 2 with static topological phases, followed by a general discussion of Floquet systems in Chapter 3, and more specifically Floquet topological phases in Chapter 4. At this point, we will have enough background material for the discussion of time defects in Chapter 5. We end with a summary of the discussion of time defects in Chapter 6.

Chapter 2

Static topological phases

In this section, we will discuss the general theory of static topological phases of matter, paving the way for the study of (dynamical) Floquet topological phases in Section 4.

We begin with a general discussion of what is a phase of matter in Section 2.1, to then define and characterize static topological phases and related concepts in 2.2. The focus here will be on the microscopic properties of topological phases and on the Hamiltonian formalism, as they will be the foundations and the appropriate language for time defects in Section 5. This comes at the cost of not discussing the important connection between topological phases and topological quantum field theories.

The bulk-edge correspondence for static topological systems will be treated in Section 4.4. Finally, extrinsic defects will be presented in Section 2.4.

2.1 Phases of matter

One of the ultimate goals of condensed matter is to classify all phases of physical systems. Then, a natural first question is: what is a *phase*, or *order*, in this context?

Roughly, a phase of matter is an emergent macroscopic organization of the collective behavior of several subsystems, such as electrons and nuclei, that is also robust to certain perturbations[1] [12, 13]. Even if we could know and probe everything about macroscopic systems, there would still be no universal way of classifying them into different phases because there is freedom in selecting which properties characterize them.

Historically, the paradigm has been to use the classification proposed by Landau [2] based on spontaneous symmetry breaking. It differentiates quantum systems by the symmetry group G_H of the Hamiltonian H and by the remaining

[1] As Hoi Chun Po *et al.* put it, "To define a phase, one should demonstrate that the defining properties are robust, i.e. they are stable under all small physical perturbations." [11].

symmetry subgroup $G_\rho \leq G_H$ of one of its equilibrium states[2] ρ after SSB [14, 15]. Thus, systems with the same symmetry group pair (G_H, G_ρ), with G_H a subgroup of G_ρ, belong to the same phase in Landau's theory [12]. A collection of familiar systems that undergo SSB can be found in Table 2.1.

Broken system	Unbroken system	$G_H \xrightarrow{\text{broken to}} G_\rho$
$\mathbb{Z}_2$ Ising ferromagnet	Paramagnet	$\mathbb{Z}_2 \to$ Trivial
Superfluid	Normal fluid	$U(1) \to$ Trivial
Crystalline solid	Liquid / Gas	$\mathbb{R}^3 \rtimes O(3) \to$ Space group [16]

Table 2.1: Examples of systems with relevant Hamiltonian symmetry group G_H that is spontaneously broken to the symmetry subgroup $G_\rho \leq G_H$ of a equilibrium state ρ of H, and their unbroken counterparts. More examples can be found in the Chapter 6 of [14].

2.2 Microscopic definition

Landau's classification scheme works well when symmetry is an essential feature of a system. However, beginning with the discovery [17] and the study [18] of the fractional quantum Hall effect, long-range entanglement structures, and other remarkable features were found in systems at zero-temperature with no symmetry at all. Now we call these properties *topological* and their corresponding non-trivial phases of matter are said to have topological or quantum order.

The use of the word topological in this context is closely related to the homonymous concept in mathematics: *properties of spaces invariant under continuous transformations*. Using these, we can define equivalence classes of spaces connected by continuous transformations, which in our case will be topological phases. To strengthen the analogy, we only have to define what we mean by *spaces* and by *continuous transformations* in condensed matter[3]. We will briefly present these concepts now by closely following [3] and will leave the details such as the thermodynamic limit and the precise definitions of local and of gapped Hamiltonians to [12].

Spaces: Since we are interested in low energy or even zero-temperature properties of quantum systems, our spaces will be the ground subspaces of local gapped Hamiltonians. The gap sets a reference energy value in relation to which we can

[2]These can be, for example, a thermal state with temperature $T > 0$ or a ground state.

[3]This nomenclature, however, is not widely used. We will translate them to more common terms later.

talk about low energies[4]; and (quasi-)local roughly means that the Hamiltonian H is a sum of terms $H = \sum_i H_i$ that only act non-trivially in regions whose size that does not scale with the size of the system (apart from exponential decays)[5]. Locality is a feature present in almost all many-body systems of interest, given that the elementary interactions of matter – mostly electromagnetic – are usually local in nature. If not explicitly stated otherwise, we will assume from now on that all Hamiltonians are local.

The reader must be warned, though, that the locality of the interactions does not imply that all subsequent properties are also localized in space. One does not need to go far to see this, as any local system with SSB has a long-range order parameter. In fact, other nonlocal features, such as long-range entanglement and effects at the boundary, are hallmark properties of topological systems that we are going to see later.

Continuous transformations: It remains to define what we mean by continuous transformations of ground states $|\psi\rangle$ of local gapped Hamiltonians, assumed to be non-degenerate for simplicity. Taking the analogy with mathematical topology seriously, we define a continuous transformation from $|\psi(0)\rangle$ to $|\psi(s)\rangle$ to be a path of ground states $s \in [0,1] \mapsto |\psi(s)\rangle$ of local gapped Hamiltonians $H(s)$ that depend smoothly on s [3]. Hence, two states $|\psi(0)\rangle$ and $|\psi(1)\rangle$ are in the same topological phase if there is a continuous transformation connecting the two. It still remains, however, to connect this definition to what we physically expect from a zero-temperature phase of matter.

2.2.1 Physical motivation

It is believed that this rather mathematical definition is equivalent to saying that two states can be continuously transformed from one to the other if there is a path that does not pass through a quantum phase transition [3, 22]. Here, a quantum phase transition happens if the expectation value of *any* local observable is non-smooth at some point, not just the free energy density, as in [23]. In simple terms, we want to prove that, along a path of Hamiltonians $H(s)$,

$$\text{Closing of the gap} \Leftrightarrow \text{Quantum phase transition.}$$

For the $(\Leftarrow)$ implication, we use the fact that if $H(s)$ has a gap bigger than

[4] However, there is a large class of gapless topological phases, such as topological semimetals [19] and even intrinsically gapless topological phases [20].

[5] A more precise definition can be found in [21].

$\Delta > 0$ for all $s \in [0,1]$, then the expectation value of any local operator $\langle O \rangle (s) := \langle \psi(s)|O|\psi(s)\rangle$ is a smooth function of t. This is due to the non-zero gap Δ and to the Lieb-Robinson bounds [21], from which it can be shown that the unitary evolution $U(s)$ that performs the adiabatic evolution $|\psi(s)\rangle = U(s)\,|\psi(0)\rangle$ is a *local unitary* (LU) [3]. By definition, a LU is generated by a local path-dependent effective Hamiltonian $H_{\text{eff}}(s)$ via $U(s) = \mathcal{P}e^{-i\int_0^s H_{\text{eff}}(r)\,\mathrm{d}r}$, where $\mathcal{P}$ is the path-ordering operator. Hence, the locality of $H_{\text{eff}}(s)$ implies that the evolved operator $U^\dagger(s)OU(s)$ is also local. In other words, U is *locality-preserving*. This in turn implies that $\langle \psi(s)|O|\psi(s)\rangle = \langle \psi(0)|U^\dagger(s)OU(s)|\psi(0)\rangle$ is not singular [24, 25].

In a lattice, LUs are approximated arbitrarily well by *finite depth local unitaries* (FDLUs)[6]. Hence, they can replace the LUs in the discussion above, with the advantage of displaying more evidently the causal structure due to locality [26].

The other implication ($\Rightarrow$) is not true in general, as there are gapless Hamiltonians that share the same ground state of topologically ordered systems [27], but we will follow [26] and assume that it is true for our systems of interest.

To complete the physical motivation of topological phases, we mention that it is indeed a phase in the sense described in Section 2.1, since it can be proved for a large class of topologically ordered systems that their topological order is stable under local perturbations [24, 25, 28]. This is but a consequence of the invariance under local unitaries.

2.2.2 Local unitaries and entanglement

In the middle of the argument above, we found a more operational definition of continuous transformations. Namely, a continuous transformation is a LU U that connects two gapped states $|\psi\rangle$ and $|\phi\rangle$ via $|\psi\rangle = U\,|\phi\rangle$. It has the advantage of being independent of the parent Hamiltonians of the ground states, apart from the gap condition. With this new definition in mind, we can connect the discussion so far to concepts from quantum information theory, such as *short* and *long-range entanglement*. This is because a LU transformation preserves the overall entanglement structure of states, as will be detailed now.

Suppose we have a product state $|\psi\rangle = \bigotimes_{x\in\Lambda} |\psi_x\rangle$ in a lattice Λ. Applying a LU to it can only entangle different sites *locally*, i.e., within a finite range. This can be

[6] A FDLU U is a sequence of N piece-wise local unitaries $U = U_{\text{pwl}}^{(N)} U_{\text{pwl}}^{(N-1)} \cdots U_{\text{pwl}}^{(1)}$, with N independent of the system size. Each $U_{\text{pwl}}^{(i)}$ is a product of non-overlapping k-local terms $U_{\text{pwl}}^{(i)} = \prod_x U_x^{(i)}$, i.e. $U_x^{(i)}$ acts only on k sites, with k constant.

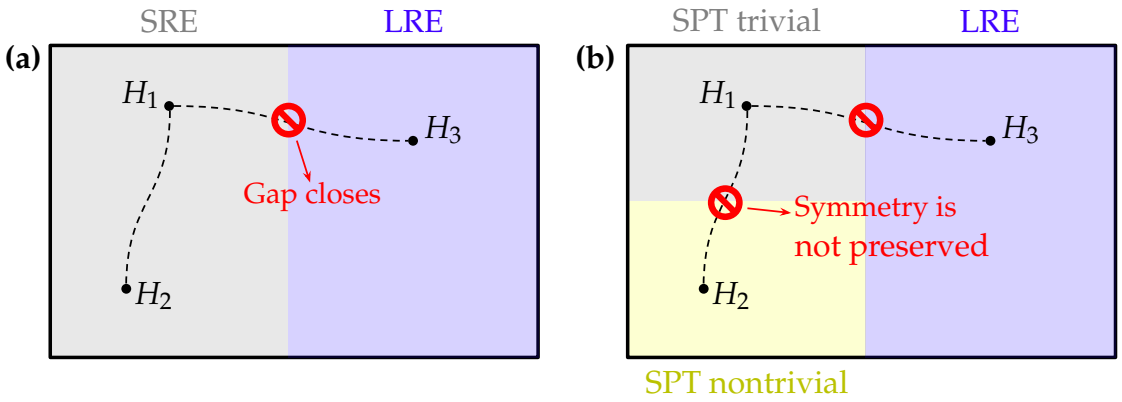

Figure 2.1: Topological phases **(a)** with and **(b)** without symmetry. The dashed lines represent local unitaries between Hamiltonians or, equivalently, their ground states. In red are obstructions to continuous transformations, either because of **(a)** gap closing or because **(b)** the symmetry is broken or the LU is not symmetric, although this is not exhaustive (see main text).

probed, for example, by the entanglement entropy $S(\rho_A)$ associated to the reduced density matrix $\rho_A = \mathrm{Tr}_{A^C}(|\psi\rangle\langle\psi|)$ of a compact region A with boundary ∂A. Since only the sites near the boundary of A are entangled with the complement $A^C = \Lambda - A$, $S(\rho_A)$ scales linearly with the size of ∂A, also called *area-law entanglement*.

Those states that are connected to a product state via a LU are *short-range entangled* (SRE) or even topologically trivial[7]. On the other hand, states that cannot be disentangled via LU transformations are said to be *long-range entangled* (LRE) (see Figure 2.1(a)). Their entanglement entropy still scales with the size of ∂A, but with an additional constant factor γ, called the topological entanglement entropy, such that $S(\rho_A) \sim c|\partial A| + \gamma$ for big enough regions A. Surprisingly, there are many LRE phases (not shown in Figure 2.1) that differ by their pattern of long-range entanglement [12].

Having the connection between topological phases and entanglement in mind, we can understand how a phase transition between topological phases is different from other phase transitions, such as those captured by Landau's formalism. The latter can be described by a local order parameter that is zero in one phase and nonzero on the other, but the former cannot. This is due to the invariance of topological phases under *any* local unitary, with which we can disentangle a small portion of a system, thus trivializing any meaningful local order parameter there. In contrast, a topological phase may be characterized by a *topological invariant*,

[7] Another definition of SRE systems are those that have non-degenerate ground states if put in any closed manifold. This is not equivalent to the definition stated [11], although the difference will not be relevant here.

which is a continuous map from the space of Hamiltonians to a discrete space, such as the integers [29]. Because of the discreetness of its image, topological invariants are constant under continuous transformations of Hamiltonians.

2.2.3 Symmetry-protected topological phases

Implicit in our arguments above was the fact that many properties were kept constant under the continuous transformations described, such as the dimension of the physical space or whether it had bosonic or fermionic degrees of freedom. This is justified, as they dramatically alter the structure of the Hilbert space in question. However, another important constraint that we may impose is symmetry, such that we only consider Hamiltonians and LUs compatible with a given symmetry group.

The subsequent equivalence classes of symmetric states under symmetric LUs are called *symmetry-protected topological* (SPT) phases, if they don't posses topological order, or *symmetry-enriched topological* (SET) phases if they do. Figure 2.1(b) illustrates the splitting of a trivial SRE phase into two SPT phases: one whose Hamiltonians are still connected to a product state via a symmetric LU and another whose Hamiltonians are not, meaning it is a nontrivial SPT phase. Phases with different SSB ground states are also separated after the inclusion of symmetry.

Two important subclasses of SPT systems is that of *topological insulators* (TIs) and of *topological superconductors* (TSCs). They correspond to free-fermion Hamiltonians that are (gapped) insulators and superconductors in the bulk, respectively. Not only is the calculation of their spectrum simplified when compared to interacting systems, but their symmetry classification is simplified too, because the only effect of any internal unitary symmetry that commutes with the Hamiltonian is to further block diagonalize it and, hence, they are not responsible for nontrivial TIs or TSCs. Therefore, only symmetries that are antiunitary or that anticommute with the Hamiltonian matter to the nonspatial SPT classification of free-fermion systems. Three of this kind are of fundamental relevance: the time-reversal symmetry $\mathcal{T}$, the particle-hole symmetry $\mathcal{P}$ and the chiral symmetry $\mathcal{C} = \mathcal{T}\mathcal{P}$, all squaring to ± 1. The first two are antiunitary and the last two anticommute with the Hamiltonian. As such, we can classify TIs and TSCs according to if they have each symmetry and, if they have, if it squares to $+1$ or -1, giving a total of 10 possibilities, called Altland-Zirnbauer (AZ) classes [30].

2.2.4 Classification of topological phases

The definition of topological phases presented above is impractical to use in most situations, given that we need to work directly at the level of the wavefunction and of the microscopic Hamiltonian. To mitigate this difficulty, many classification schemes for topological phases have been developed. Generally, they work by separating systems with different topological invariants into different phases. Some of the techniques for the classification of SPT phases are K-theory [31], group cohomology [32], and cobordisms [33, 34]. For these examples, the invariants are elements of an algebraic structure, like the additive group of integers, $\mathbb{Z}$, or of integers module n, $\mathbb{Z}_n$.

We will leave their discussion to Section 4.3, where we compare Floquet topological classification schemes with static ones.

2.3 Static bulk-edge correspondence

Although a topological system can be identified solely from bulk properties, some of their most prominent features reside at the boundary. Furthermore, there is a direct mapping between topological invariants in the bulk and those at the edge called the *bulk-edge correspondence* [29].

One of the possible consequences of this correspondence that is also experimentally relevant is the appearance of topologically protected currents at the boundary. They were first studied through the integer quantum Hall effect, by which current can flow at the boundary of a two-dimensional material perpendicular to an applied electric field as a result of a magnetic field normal to the surface, despite the bulk being an insulator[8]. The associated nontrivial electromagnetic response function is the Hall conductance. Even though it corresponds to an edge current, it can be calculated using a bulk quantity: the total Chern number of occupied bands. This topological invariant can be viewed as the total flux of the Berry curvature in the Brillouin zone [35].

In general, gapless modes appear at the boundary between topological systems with different invariants. Such is the case considered in the previous paragraph, if one treat empty space as a trivial insulator. For one-dimensional systems, they appear as modes with zero energy, as there is no extra dimension to allow for

[8] Although the current in the integer quantum Hall effect can flow in the bulk. This can be seen from the famous Laughlin argument (see, e.g., [14]). Thanks to Dominic Else and Chong Wang for pointing this out to me in my defense.

Figure 2.2: Unpaired Majorana fermions (blue) in the solvable limit of the topological Kitaev chain (2.2). The Majorana operators that come from the same site are joined by black horizontal lines and the ones that interact through the Hamiltonian are connected in red.

a momentum-dependent dispersion relation. One example of this is illustrated below.

2.3.1 Example: Kitaev chain

A simple toy model that has topologically induced edge states is the Kitaev chain [36]. It is a one-dimensional chain of spinless fermions with p-wave superconducting pairing [37]:

$$H = -\mu \sum_{i=1}^{L} c_i^\dagger c_i - \sum_{i=1}^{L-1} \left(t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \text{h.c.} \right), \tag{2.1}$$

where L is the length of the chain and μ, t and Δ are the chemical potential, the hopping strength and the superconductor order parameter, respectively. We may assume $\Delta > 0$ for simplicity.

This system is a nontrivial topological superconductor for $2|t| > |\mu|$ [36]. One can see this in the special case $\Delta = t$ and $\mu = 0$, for which (2.1) becomes (see Figure 2.2):

$$\begin{aligned} H &= -t \sum_{i=1}^{L-1} (c_i + c_i^\dagger)(c_{i+1} - c_{i+1}^\dagger) \\ &= -2it \sum_{i=1}^{L-1} \gamma_{B,i} \gamma_{A,i+1}, \end{aligned} \tag{2.2}$$

where γ_A and γ_B represent *Majorana fermions*, which are self-adjoint fermionic annihilation and creation operators, $\gamma_{A(B)} = \gamma_{A(B)}^\dagger$. In this model, they are linear combinations of c and $c^\dagger$:

$$\gamma_{B,i} = \frac{1}{\sqrt{2}}(c_i + c_i^\dagger), \quad \gamma_{A,i} = \frac{1}{i\sqrt{2}}(c_i - c_i^\dagger), \tag{2.3}$$

such that there are two Majorana fermions $\gamma_{A,i}$ and $\gamma_{B,i}$ for each site i.

If we regroup the Majorana fermions into new Dirac fermions

$$d_i = \frac{1}{\sqrt{2}}(\gamma_{A,i+1} + i\gamma_{B,i}),$$

then (2.2) becomes a homogeneous chain of independent d-fermions with an exception: since the Majorana operators at the both ends of the chain are not present in (2.2), then we can combine them into a nonlocal Dirac fermion $d = \frac{1}{\sqrt{2}}(\gamma_{A,1} + i\gamma_{B,L})$ which can be shown to create an orthogonal state to the ground state with zero energy, called a *Majorana zero mode* [36]. This is consistent with the general discussion of this section: a topological phase host gapless or zero modes at the boundary.

2.4 Extrinsic defects

We saw in the previous section that topological features in the bulk are manifest at the boundary, which can be thought of as a codimension one surface that separates a trivial topological phase – the vacuum – from a nontrivial one. Defects are a generalization of this for higher codimensions [30].

Defects were already studied in the context of spontaneous symmetry breaking. For example, if we remove a point (zero-dimensional) from a two-dimensional system with a broken $U(1)$ order parameter $\psi(x) \in U(1)$ then we can wind ψ around it, thus creating a point defect called a vortex. Defects in topological phases, however, do not require SSB and can trap zero or gapless modes that are topologically protected [38].

The specific type of defect we are going to focus on in this essay is called an *extrinsic defect*, because it is created by externally modifying the Hamiltonian itself instead of being an internal excitation. Another denomination for them are *symmetry twist defects* [39], since they also involve the addition of a "flux" of the symmetry group, akin to how a magnetic flux twists the Berry phase of a state that goes around it. These are not to be confused with topological defects, which follow another construction and does not require symmetry [40].

From now on, we will refer to extrinsic or symmetry twist defects as just *defects*. Our main reference will be [10], which describes point defects in two-dimensional topological systems.

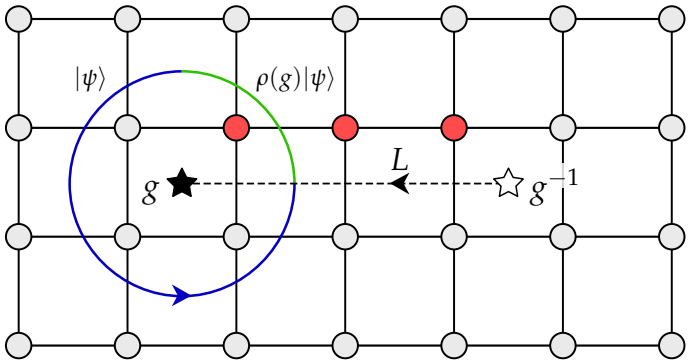

Figure 2.3: Defect-antidefect pair (black and white stars) with opposite fluxes g and $g^{-1} \in G$ connected by an oriented branch line L (dotted). The red sites are the ones conjugated due to the defect. The loop around the defect g represents an adiabatic evolution of localized quasiparticle $|\psi\rangle$. When it passes through L, $|\psi\rangle$ gets acted on by $\rho(g)$. Based on [10].

2.4.1 Definition

Given a Hamiltonian H_0 with a suitable symmetry group G, a defect is a modification of H_0 labeled by an element $g \in G$ such that charged quasiparticles get acted on by g when adiabatically transported around it. For this reason, defects are also called g-fluxes. More precisely, suppose for simplicity that the real space is a square lattice and the Hamiltonian is of the form[9]

$$H_0 = \sum_i h_i + \sum_{\langle ij \rangle} h_{ij}, \tag{2.4}$$

where each h_i only acts on the site i and h_{ij}, on the pair of neighboring sites $\langle ij \rangle$.

In the bulk, a defect carrying flux $g \in G$ is connected to an antidefect carrying flux g^{-1} by an oriented segment L, called a *branch line*. To create those in H_0, we conjugate terms in the Hamiltonian that straddles L with the symmetry action constrained to the right side of L (see Figure 2.3). For that, we need the unitary symmetry representation $\rho : G \to U(\mathcal{H})$ to be on-site, i.e., $\forall h \in G$, $\rho(h) = \bigotimes_i \rho_i(h)$, where $\rho_i : G \to U(\mathcal{H}_i)$ only acts on site i. In the case of the Hamiltonian (2.4), the terms h_{ij} in (2.4) are modified in the following way:

$$h_{ij} \to \rho_j(g) h_{ij} \rho_j(g)^\dagger, \tag{2.5}$$

for links $\langle ij \rangle$ that straddle L with j to the right of L and i, to the left (if the opposite occurs, just exchange i with j). The single site terms h_i do not get altered because

[9] The definition of defects for more general spaces and Hamiltonians can be found in [10].

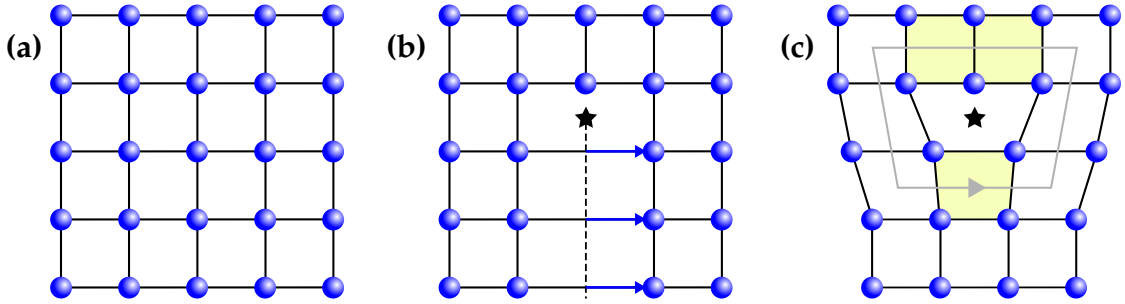

Figure 2.4: Lattice edge dislocation created in three steps. **(a)** First, we start with a regular square lattice, then **(b)** we move every lattice site to the right of the branch line (dashed) of the defect (star) one lattice unit to the right. Lastly, **(c)** we continuously deform the lattice so that it is regular far from the defect, even though it can be probed by a loop around it (solid gray).

they do not straddle L.

By asymmetrically conjugating terms of the Hamiltonian only to the right of the branch line L, it can be proven that a charged quasiparticle is acted on by $\rho(g)$ if adiabatically transported around a counterclockwise loop that passes through L and contains the defect [10]. The opposite happens if the loop is traveled in the clockwise direction.

We emphasize that the defect construction given above requires the symmetry to be on-site. This will be important for our later discussion of time defects in Section 5 as the unitary operator corresponding to the time periodicity symmetry may not be on-site at the edge. Even though the same formalism can be adapted to familiar symmetries that don't act on-site, such as the translation symmetry, whose defect is an edge dislocation (see Figure 2.4) [10], the author is not aware of any extension that systematically encompasses any symmetry that is not on-site.

Chapter 3

Floquet systems

The study of time-independent or static quantum systems is vast and shares its roots with quantum mechanics itself. If we allow for time-dependent or dynamical systems, we get an even larger set of phenomena. Studying them in all generality is a daunting task, so demanding restrictions on the Hamiltonian $H(t)$ is a first step. One possible restraint is requiring periodicity of the Hamiltonian $H(t) = H(t+T)$, for all $t \in \mathbb{R}$, with period $T > 0$. These are called *Floquet systems*, named after French mathematician Gaston Floquet, and are not only of fundamental theoretical relevance, but also of experimental too in areas such as quantum optics.

Even though this essay is focused on the generalization of the formalism of topological phases introduced in Section 2 to Floquet systems, we will present now a fundamental result valid for any Floquet Hamiltonian – topological or not.

3.1 Floquet's theorem

For time-dependent quantum systems, the unitary evolution operator $U(t)$ can be formally written using the time-ordered Dyson series ($\hbar = 1$)

$$\begin{aligned} U(t) &= \mathcal{T} \exp\left(-i \int_0^t H(t')\,\mathrm{d}t'\right) \\ &= \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, H(t_1)H(t_2)\cdots H(t_n) \\ &= \mathbb{1} + (-i)\int_0^t \mathrm{d}t_1\, H(t_1) + (-i)^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, H(t_1)H(t_2) + \cdots, \end{aligned} \tag{3.1}$$

such that $U(t)$ satisfies Schrödinger equation $i\frac{d}{dt}U(t) = H(t)U(t)$, with initial condition $U(0) = \mathbb{1}$.

Using (3.1) can be cumbersome for explicit calculations because of the time ordering of the Hamiltonian operators, which becomes necessary if $H(t)$ does not commute with itself at other times $H(t')$, $t' \neq t \in \mathbb{R}$. However, the time periodicity

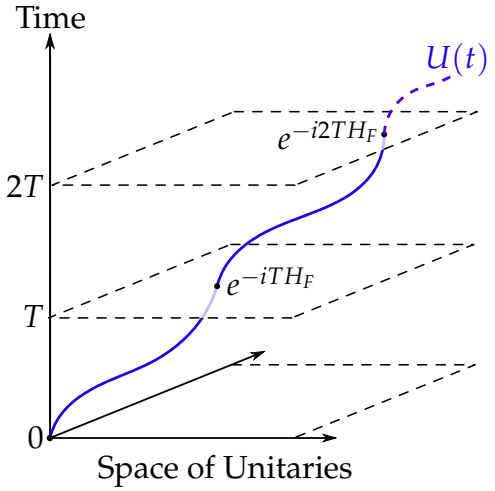

Figure 3.1: Illustration of the path traced by the time evolution operator $U(t)$ in the space of unitary operators. Looking stroboscopically at integer multiples of the period T, the evolution is generated by the Floquet Hamiltonian H_F.

of the Hamiltonian simplifies the form of the unitary evolution operator. This simplification is encoded in *Floquet's theorem* below, which we prove in Appendix A for Bochner integrable Hamiltonians.

Theorem 1 (Floquet's theorem)**.** *Let H be a T-periodic Hamiltonian and U be its time evolution operator. Then there exists a Hermitian operator H_F and a T-periodic unitary operator function Φ such that*

$$\forall t \in \mathbb{R}, \quad U(t) = \Phi(t)e^{-itH_F} \tag{3.2}$$

This theorem is the application to quantum mechanics of the theory of linear ordinary differential equations with periodic coefficients due to Gaston Floquet [41].

H_F is called the *Floquet Hamiltonian*, Φ the *micromotion loop*, and the time evolution operator over a period $U(T) = e^{-iTH_F}$, the *Floquet evolution operator*, *Floquet unitary* or *stroboscopic evolution operator*, denoted by U_F (see Figure 3.1).

Since H_F is defined by taking a logarithm of $U(T)$, it is not unique. We can resolve this by demanding that its spectrum $\sigma(H_F)$ satisfies $\sigma(H_F) \subseteq [-\pi, \pi)$. In close analogy with Bloch's theorem, we call the elements of $\sigma(H_F)$ *quasienergies*, which are only defined mod 2π and thus live in S^1.

Two limits are important: the *static limit* $H(t) = H =$ constant, in which $\Phi(t) = \mathbb{1}$ if we define $H_F = H$; and the *unitary loop* limit, in which $U(T) = \mathbb{1}$, $H_F = 0$, and thus $U(t) = \Phi(t)$.

Because of this, H_F might be regarded as the static part and Φ the inherently dynamical part of the periodic system. Indeed, we will see in Section 4.3 that some classification schemes of Floquet topological phases include the static case by treating H_F as a static Hamiltonian.

We close this section by mentioning that the stroboscopic evolution of a Floquet system is equivalent to the evolution under the (constant) Floquet Hamiltonian H_F (see Figure 3.1). One consequence of this fact is that we may engineer the evolution under a static Hamiltonian H_s of interest by tuning a periodic drive $H(t)$ in such a way that H_F matches H_s. This gives us more control over the design of Hamiltonians H_s that would otherwise be experimentally difficult to manufacture in the static setting. This technique is called *Floquet engineering* and has been widely used to study nontrivial interacting and topological systems [42].

Chapter 4

Floquet topological phases

The main theme of this section will be to compare Floquet topological phases with static topological phases. After we introduce the generalities of the theory in Section 4.1, we discuss in Sections 4.3 and 4.4 the Floquet generalization of the topological phase classification and of the bulk-edge correspondence, respectively.

For reasons that will become clearer later, we are not going to follow the systematic approach of Section 2.2 here. Instead, we will use static topology tools to gradually develop the formalism in the Floquet case.

For a good review of the topic, the interested reader is referred to [9].

4.1 Introduction

As a start, let us consider a Hamiltonian $H(t) = H_s$ static in time as a Floquet system with any period $T > 0$. In that case, we take the Floquet Hamiltonian H_F to be equal to H_s, since $U(T) = e^{-iTH_s}$. Now imagine that we continuously deform $H(t)$, maintaining constant its periodicity $H(t+T) = H(t)$ and its Floquet unitary

$$U(T) = \mathcal{T} e^{-i\int_0^T H(t)\,\mathrm{d}t} = e^{-iTH_s}.$$

Under some suitable definition of a continuous deformation, we expect that this procedure should not change the Floquet topological phase of $H(t)$, by the same principles stated in the beginning of Section 2.2. Hence, we conclude that Floquet topology must at least contains static topology, even for non-constant Hamiltonians.

This procedure captures only a fraction of all possible Floquet topological phases because there are micromotion loops $\Phi(t) = U(t)e^{itH_F}$ that are not topologically connected to the identity $\Phi(t) = \mathbb{1}$. Remember that to continuously vary $H(t)$ from a static Hamiltonian H_s while maintaining $U(T)$ constant is the same as deforming $\Phi(t)$ from $\Phi(t) = \mathbb{1}$. Similar to how there are different static topological phases that cannot be connected to each other via a continuous transformation that

respects locality and symmetry constraints, there are obstructions on the space of unitary loops that separate different Floquet systems with the same H_F (see the following sections for a more detailed discussion).

One simple demonstration of the incompleteness of the bulk's Floquet Hamiltonian H_F is the emergence of edge modes that are inconceivable for static topological systems due to the periodicity of the quasienergy spectrum. As an example, suppose a single-particle translationally-invariant static Hamiltonian H is topologically characterized solely by the Chern numbers $\mathcal{C}_i \in \mathbb{Z}$, $i \in \{1, 2, \ldots, N\}$, of its N bulk bands, in order of increasing energy. Then it can be shown that there are n_i gapless edge modes with energy dispersion between bands i and $i+1$, such that $n_i - n_{i-1} = \mathcal{C}_i$, where negative n_i mean $|n_i|$ edge modes with opposite chirality [43]. The same is true for analogous Floquet topological systems if we look at bands in the quasienergy spectrum [44].

However, due to the quasienergy periodicity $\epsilon + 2\pi = \epsilon \pmod{2\pi}$, we could have edge modes that wrap around the quasienergy spectrum, going from the highest quasienergy band to the lowest. Moreover, even if the Chern numbers of all bands were zero, there could still be $n_i = n = \text{const.}$ edge modes between each band, which is not possible for static systems. There, the energy spectrum cannot have edge modes with energy lower than the lowest energy band and higher than the highest band [8]. From this example, we conclude that bulk Floquet Hamiltonians are not enough to completely describe Floquet topology. Moreover, we are going to see in Section 4.4 that not even the full bulk-and-edge Floquet Hamiltonian is enough.

Since different definitions of Floquet topological phases are used in the literature [45], we are not going to follow the systematic approach of Section 2.2 here. Instead, we will now closely study one representative example that showcases inherently dynamical topological properties.

4.2 RLBL model

As a good example of an inherently dynamical Floquet topological system, the RLBL model will be studied extensively in this and in the forthcoming sections. It was first introduced by Rudner, Lindner, Berg, and Levin (RLBL) in [8]. Although we will not consider the exact same model as the one in the RLBL paper, we will follow [9] and call it the "RLBL model". Another name for it is "Anomalous Floquet-Anderson insulator" [11].

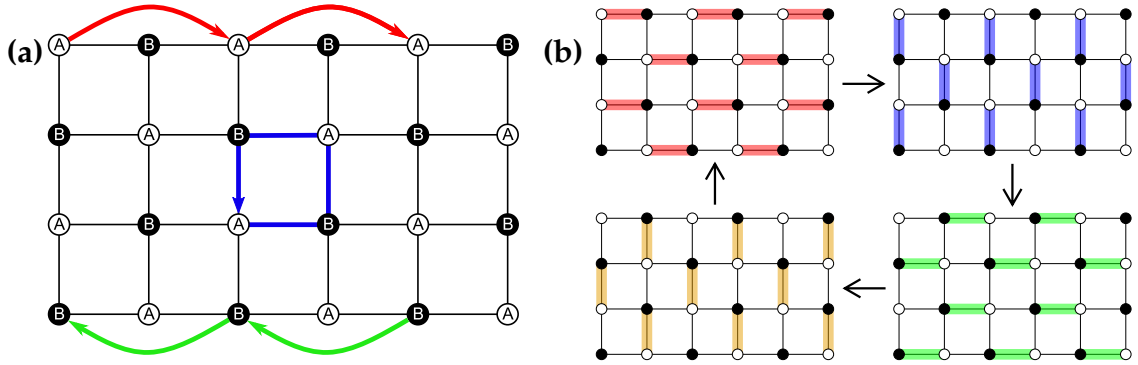

Figure 4.1: RLBL model in an open 6 by 4 lattice. **(a)** The arrows show the movement of a localized particle starting at different positions. In the bulk (blue) it returns to its initial position, but at the edge (red and green) there are edge modes depending on the sublattice (A or B). **(b)** shows the 4-step sequence of bonds, beginning in the top-left corner. Based on [8].

4.2.1 Definition

The RLBL model is an exactly solvable quadratic (non-interacting) fermionic Floquet system in a 2D square lattice $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_m$, including open and closed, finite and infinite ($\mathbb{Z}_\infty := \mathbb{Z}$) lattices. The time-dependent Hamiltonian H is divided into five consecutive steps H_n, $n = 1, 2, \ldots, 5$, each of duration $T/5$, where T is the period, such that $H(t) = H_n$ for $t \in [(n-1)T/5, nT/5)$ and is periodically extended for all $t \in \mathbb{R}$. The first four steps are given by [46]

$$H_n = -J \sum_{\substack{r \in A: \\ r+b_n \in \Lambda}} (c_r^\dagger c_{r+b_n} + \text{h.c.}), \tag{4.1}$$

where $J = \frac{5\pi}{2T}$ is a coupling constant, A is a sublattice of the checkerboard bipartition $\Lambda = A \sqcup B$ (see Figure 4.1(a)) and b_n rotates 90 degrees clockwise in each step:

$$b_1 = -b_3 = (1, 0), \quad b_4 = -b_2 = (0, 1). \tag{4.2}$$

Since the Hamiltonian is piecewise constant, the Floquet unitary operator can be divided into five steps $U_F = U_5 U_4 \cdots U_1$, where $U_n = \exp(-i\frac{T}{5} H_n)$. In the n-th step, a site $r \in A$ interacts with only one of its nearest neighbors, $r + b_n$, via a hopping term (see Figure 4.1(b)). We will call the pairs of interacting sites at any given step *bonds*.

The fifth step Hamiltonian is $H_5 = \sum_{r \in \Lambda} W_r c_r^\dagger c_r$ and can be used in simulations to add disorder by choosing W_r randomly. In the original RLBL paper, the

authors used it to set up an equal but opposite sublattice potential to lift sublattice degeneracy. However, for analytical simplicity, we will set $H_5 = 0$ if not stated otherwise.

We prove in Appendix B that for the specific value of $JT/5 = \frac{\pi}{2}$, the stroboscopic dynamics is greatly simplified: a state $|r\rangle$ localized at site $r \in \Lambda$ is swapped with its bond partner after each step. By accompanying how $|r\rangle$ moves after all the steps, we can easily conclude that if $|r\rangle$ is in the bulk, it comes back to itself after one period, and if it is at the edge, it will get translated by one lattice unit depending on the sublattice (see Figure 4.1(a)).

In a closed lattice or in the bulk of an open lattice, the Floquet evolution operator is trivial, $U_F = \mathbb{1}$ albeit $H(t) \neq 0$. This means that the nontrivial dynamics comes solely from the micromotion evolution $\Phi(t) = U(t)$ inside one period, which reveals internal loops in space under close inspection. As we said at the beginning of the section, the RLBL model is *inherently dynamical*. However, if the lattice is open, there is a nontrivial stroboscopic dynamics at the edge, which is the topic of the next point.

The RLBL model has a variety of topological properties, which will serve as a good example for more general discussions later. They can be summarized in the following way:

4.2.2 Anomalous robust chiral edge modes

Because of the lack of bonds at the edge for some steps, states at one sublattice get translated by one lattice unit, whereas states at the other sublattice returning to their original site after one period. Consequently, a counterclockwise dynamical current emerges at the boundary. Accordingly, a gapless mode with linear dispersion in the quasienergy spectrum appears when we take periodic boundary conditions in just one direction (see Figure 4.2).

Let us explain the adjectives used above to describe these edge modes:

Chiral: the direction of the movement at the edge is determined only by the chirality of the Hamiltonian evolution, encoded in the rotation direction of b_n. Reversing the latter has the same effect as reversing time.

Robust: RLBL constructs a bulk topological invariant given by

$$W[U] = \frac{1}{8\pi^2} \int_{T^3} \mathrm{d}t\mathrm{d}k_x\mathrm{d}k_y \,\mathrm{Tr}\left(U^{-1}\partial_t U\left[U^{-1}\partial_{k_x}U, U^{-1}\partial_{k_y}U\right]\right) \tag{4.3}$$

where $U(\mathbf{k}, t)$ is a single-particle bulk evolution operator of a translationally invari-

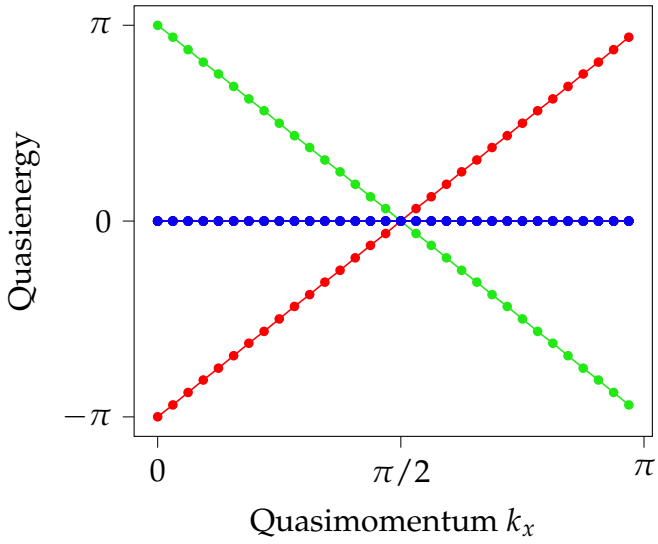

Figure 4.2: Quasienergy spectrum of the RLBL model in a cylindrical 64 by 4 grid by the quasimomentum k_x on the horizontal axis. The lines correspond to the eigenstates illustrated in Figure 4.1(a).

ant system that is an unitary loop ($U(T) = \mathbb{1}$) and $T^3 = S^1 \times (S^1)^2$ is the 3-torus in (periodic) time and quasimomentum spaces. $W[U]$ is a topological invariant because, being the degree (winding number) of the unitary evolution mapping $U : T^3 \to U(\mathcal{H})$ with non-trivial de Rham cohomology group $H^3(U(\mathcal{H})) = \mathbb{Z}$ [47, 48], it is an integer invariant under homotopic equivalences of U. For the RLBL model, $W[U] = 1$ [8].

Furthermore, RLBL proved that $W[U]$ is equal to the number of edge modes, which is a clear example of the bulk-edge correspondence for Floquet systems. As long as U has nontrivial topology in the bulk probed by index (4.3), an open system will host chiral edge modes.

Anomalous: Contrary to static topological insulators, all the Chern numbers for the RLBL model's phase are zero [8], even though it hosts chiral edge modes. This is yet another reason why the RLBL model has no static counterpart.

A different sense in which the edge modes are anomalous is that the edge unitary evolution Y, which is a 1D translation operator for one of the sublattices, is not a local unitary or, equivalently, a finite depth quantum circuit [11]. This means that, although the full bulk-and-edge unitary evolution operator $U(T) = \mathcal{T} e^{-i \int_0^T H(t)\, \mathrm{d}t}$ is generated by a local Hamiltonian H, the corresponding edge evolution operator Y is not (see Section 4.4 for further discussion).

4.2.3 SWAP models

There is a family of systems that generalize the RLBL model collectively known as *SWAP models* [11]. The difference is that they do not constrain the Hilbert space of each site $\mathcal{H}_r = \mathcal{H}_{\text{site}}$, for all $r \in \Lambda$, to be part of a fermionic Fock space, as is in the RLBL model, but they maintain the SWAP operations for each step. Let us illustrate with the following example: the Floquet unitary is made of four steps $U_F = U_4 U_3 U_2 U_1$, each of which are made of SWAP operations following Figure 4.1(b) and Equation 4.2:

$$U_n = \prod_{\substack{r \in A: \\ r+b_n \in \Lambda}} \chi_{r,r+b_n}, \tag{4.4}$$

where $\chi_{r,r'} : \mathcal{H}_r \otimes \mathcal{H}_{r'} \to \mathcal{H}_r \otimes \mathcal{H}_{r'}$ is the SWAP operator between sites r and r': $\chi_{r,r'} |\psi_r \psi_{r'}\rangle = |\psi_{r'} \psi_r\rangle$. With this evolution, the dynamics will be almost the same as in the RLBL model, with the difference that the transport at the edge will not be of charge, but only of *quantum information* [11]. This will be further discussed in Section 4.4.

4.2.4 Simulations

The author made simulations of the dynamics of single-particle states evolving under the RLBL model Hamiltonian (4.1). They were made primarily for the tentative inclusion of time defects (See Section 5.2), but they also serve illustration purposes.

The code essentially used exact diagonalization methods of the many-body physics Python package QuSpin [49, 50]. Since the RLBL Hamiltonian is quadratic, the particle number is conserved. As such, it suffices to calculate the unitary evolution restricted to the single-particle state space, thus saving computer time and memory.

For purposes of visualization and to validate our simulations, two videos were made showing the time evolution of single-particle localized states in a 6 by 10 open rectangular lattice. One video [51] shows a state that starts in the bulk and returns to its original position after one period ($T = 1$) and another [52] shows an edge state that travels the entire 30-site perimeter in a time duration of $\frac{30}{2}T$. The results shown in the videos conform to our expectations from the theoretical model.

We also computed the quasienergy spectrum of the Floquet Hamiltonian for a 64 by 4 cylindrical lattice periodic in the x axis in terms of the quasimomentum k_x

(see Figure 4.2). There, the two chiral edge modes that travel in opposite directions are clearly seen in the two diagonal lines. The flat bands represents the bulk modes. If disorder or other perturbations were added, these bands would acquire a band width, but the topological properties would stay intact.

4.3 Floquet classification schemes

Many of the classification schemes for static topology have been generalized to Floquet topology. We will not elaborate on the details, but we will describe below two methods that, in spite of employing different mathematical techniques and having different domains of validity, both associate to the set of Floquet topological phases a group $\mathcal{G} = \mathcal{G}_s \times \mathcal{G}_d$, such that $\mathcal{G}_s \times \{e_d\}$ reproduces the static classification if $H(t) = H_s = \text{const.}$ and $\{e_s\} \times \mathcal{G}_d$ classifies inherently dynamical phases, which are not connected to any static phase, where e_s and e_d are the identity elements of $\mathcal{G}_s$ and $\mathcal{G}_d$, respectively.

4.3.1 K-theory

Roy and Harper [53] proved that all Floquet unitaries pertaining to single-particle fermionic systems with gapped quasienergy spectrum can be continuously deformed without closing the gap to a unitary loop[1] L followed by an evolution C under the (static) Floquet Hamiltonian H_F. Moreover, they proved that L and C are unique up to homotopy. After that, they were able to use K-theory to independently classify the homotopy classes of the "static evolutions" C into a group $\mathcal{G}_s$ and the ones of the unitary loops into $\mathcal{G}_d$ for all AZ classes.

In particular, the K-theory classification for class A, of asymmetric systems, is $\mathcal{G} = \mathbb{Z}^N$, where N is the number of quasienergy bands. This agrees with the discussion of Section 4.1, since $\mathcal{G}$ can be separated into $\mathcal{G} = \mathcal{G}_s \times \mathcal{G}_d$, with $\mathcal{G}_s = \mathbb{Z}^{N-1}$ representing all possible independent Chern numbers of static topology[2] and $\mathcal{G}_d = \mathbb{Z}$ representing the anomalous edge states [53]. For example, the RLBL model is classified by element $(0, 1) \in \mathcal{G}_s \times \mathcal{G}_d$

[1] Not to be confused with the micromotion loop Φ of Section 3.1.

[2] The Chern numbers are constrained by $\sum_{i=1}^{N} \mathcal{C}_i = n_N - n_0 = 0$ in both static and Floquet settings.

4.3.2 Group cohomology

It has been conjectured by [54] and proved for dimensions one [55] and two [56] that interacting bosonic Floquet SPT phases with symmetry group G that exhibit many-body localization (MBL) are classified by the cohomology group $H^{d+1}[\tilde{G}, U(1)]$, where d is the spatial dimension and $\tilde{G} = \mathbb{Z} \rtimes G$ is the extended symmetry group. It includes the symmetry group of discrete time translations $t \mapsto t + nT$, $n \in \mathbb{Z}$, represented by $\mathbb{Z}$, alongside the original symmetry group G. The semidirect product is relevant for antiunitary elements $g \in G$, that reverse the Floquet unitary $U_F \in \mathbb{Z}$ as $gU_Fg^{-1} = U_F^{-1}$ [54]. In other words, the time periodicity is incorporated as a symmetry that enters the classification in the same manner as G.

The separation into a static $\mathcal{G}_s$ part and a inherently dynamical part $\mathcal{G}_d$ becomes explicit if G has a unitary representation, so $\tilde{G} = \mathbb{Z} \times G$, and if we use the Künneth formula from group cohomology (see Corollary 5.8 of [57]):

$$H^{d+1}[\mathbb{Z} \times G, \mathrm{U}(1)] = H^{d+1}[G, \mathrm{U}(1)] \times H^{d}[G, \mathrm{U}(1)].$$

The inherently dynamical group $\mathcal{G}_d = H^d[G, \mathrm{U}(1)]$ is the same as the static cohomology classification group $\mathcal{G}_s = H^{d+1}[G, \mathrm{U}(1)]$ but one dimension lower. The interpretation of this is that U_F "pumps" a SPT phase to the $((d-1)$-dimensional) edge of the system each period [54].

4.4 Floquet bulk-edge correspondence

Similar to static topological insulators and superconductors, their Floquet analogs can host edge modes in their quasienergy spectrum that are related to topological invariants in the bulk, the main difference being the periodicity of the quasienergy [8, 44], whose effects were exemplified in Section 4.1.

In addition to these currents, Floquet topology allows for a new type of edge modes, which are independent of any charge flow or symmetry and originate in the non-locality of the edge evolution operator[3] [11, 58] (see Anomalous point in Section 4.2). We have already seen this in the Anomaly point and the SWAP models section of Section 4.2. More generally, for any compact connected region R in the bulk of a d-dimensional MBL Floquet system, one can define a boundary operator

[3] They have, however, been generalized to $U(1)$ symmetric Floquet topological phases [45]

Y that only acts near the edge of R, which is a $d-1$-dimensional surface, and evolves it by one period in a manner compatible with the rest of bulk system. Since the bulk is generated by a local Floquet Hamiltonian, then Y is locality-preserving, i.e. for any local operator A, YAY^{-1} is another local operator. However, the key point is that it is *not* guaranteed that Y is local. If it is not, then Y is said to be *anomalous* [11], as it cannot be realized as the Floquet unitary of a local periodic Hamiltonian living solely at the boundary.

Furthermore, the set of all localization-preserving unitaries (LPUs) in $d-1$ dimensions form a group under function composition with the set of all finite-depth local unitaries (FDLUs) as a normal subgroup. Then, a natural subsequent step is to describe the quotient group FDLUs/LPUs, whose nontrivial unitaries are anomalous. This question in $d=2$, i.e. for unitaries living on one-dimensional chains, was explored by Gross, Nesme, Vogts, and Werner (GNVW) in [59]. They proved that the quotient group is isomorphic to $\mathbb{Q}$, the set of rational numbers, and devised a way to compute the rational number $\mathrm{ind}(U) \in \mathbb{Q}$ for any LPU U, called the chiral unitary index or GNVW index [11].

The index is *chiral* because it measures the flow of quantum information when evolved with U. For example, the edge unitary of the SWAP model described in Section 4.2 has $\mathrm{ind}(Y) = \dim(\mathcal{H}_{\text{site}})$, where Y is its edge unitary, because it translates states in $\mathcal{H}_{\text{site}}$ by one lattice unit[4]. One can get rational indices if $\mathcal{H}_{\text{site}}$ is a tensor product $\mathcal{H}_p \otimes \mathcal{H}_q$, $\dim(\mathcal{H}_{p\,(q)}) = p\,(q)$, and Y translates states in $\mathcal{H}_p$ forwards but the ones in $\mathcal{H}_q$, backwards.

Finally, the GNVW index can be used to classify Floquet topological phases if we use a definition based not on the homotopy classes of Hamiltonians or Floquet unitaries, but on the many-body localizability of the edge unitary operator. That is, two Floquet systems are in the same phase if, when they share a boundary, the edge unitary there is MBL and, thus, has trivial GNVW index. It is easy to see that this definition leads to a classification isomorphic to the quotient group FDLUs/LPUs.

[4]Here, we are either ignoring the sublattice that is not transported along the edge or including it in internal degrees of freedom.

Chapter 5

Time defects

By definition, a time-dependent Hamiltonian H is T-periodic if, and only if, it commutes with the Floquet evolution operator $U_F = U(T)$, since $[U_F, H(t)] = 0 \Leftrightarrow U_F^\dagger H(t) U_F = H(t+T) = H(t)$. Hence, U_F is the generator of the symmetry group of discrete time translations[1] $t \mapsto t + nT$, $n \in \mathbb{Z}$, which we shall call *Floquet symmetry group*.

In Section 2.4 we showed how to create extrinsic defects for any unitary on-site symmetry group. So one natural question arises: *Can we define extrinsic defects for the Floquet symmetry group?*. This is the central question of this essay. Answering it would not only further our understanding of static topological phases to dynamical Floquet ones, but could reveal unique properties of symmetries related to time not shared by other symmetries, such as spatial and internal ones. To the best of the author's knowledge, this question has not been asked before in a published work.

Unfortunately, we still have not fully answered this question. The most natural thing to do would be to try to replicate the construction given in Section 2.4. This is what numerical simulations do in the RLBL model in Section 5.2; however, our results do not fully match what we expect from a time defect. More generally, we will present in Section 5.1 reasons for and against the expectation that a time defect would have to be constructed similarly to the other type of defects.

We stress that our concept of a time defect is not the same as a spacetime topological line defect with an extension in time. The latter can be in principle associated with any symmetry group, while the former is associated only to the Floquet $t \mapsto t + T$ symmetry. As a matter of fact, spacetime topological defects of arbitrary dimension were studied in the context of Floquet topological phases in [60], in which the authors construct a periodic table of Floquet topological defects, extending the static classification of [38].

[1] It is a well-known fact that all discrete subgroups of $(\mathbb{R}, +)$ are of the form $T \cdot \mathbb{Z}$, for $T \in \mathbb{R}$. Thus, it is sufficient to refer to the group generated by U_F by just discrete time translation group.

5.1 Is time special?

The problem of defining a time defect is but an instance of the following general question: Does time play a special role that space and internal degrees of freedom do not? Without delving into the philosophical debate regarding time, we will show here two arguments against and two for the exceptionality of the time periodicity symmetry, before going over the details of the numerical simulations.

5.1.1 Arguments against the exceptionality of time

A similar problem that has been extensively studied is to incorporate space group symmetries into the formalism of SPT phases, called topological crystalline phases. Surprisingly, Thorngren and Else showed [61] the crystalline equivalence principle, which states that the classification of crystalline topological liquids protected by a space group G is the same as the one for an *internal* group G. Spatial defects, such as lattice dislocations (see Figure 2.4), are then viewed as gauge fluxes of the corresponding symmetry.

Similarly to how spatial symmetries had no special treatment in the SPT classification, we saw in Section 4.3 that the cohomology classification conjectured by Else and Nayak in [54] involved enlarging the symmetry group G of the Floquet system to $\tilde{G} = \mathbb{Z} \rtimes G$, where the $\mathbb{Z}$ part represents the Floquet group. The time periodicity symmetry is, then, included in the classification on the same level as the non-temporal symmetry group G (apart from the nontrivial conjugation of U_F by an antiunitary element of G).

5.1.2 Arguments for the exceptionality of time

A large class of Floquet systems is of unitary loops in the bulk, i.e., such that $U_F = \mathbb{1}$ in an infinite open system. In those cases, the Floquet symmetry is trivialized and any time defect construction in the bulk that involves conjugation by U_F would amount to doing nothing to the system. The only nontrivial action would be at the edge. This is by itself, if not an indication that the definition of extrinsic defects for time symmetry needs to be modified, at least an example of the unusualness of such construction.

We conjecture that the problem with unitary loops described above and more generally with any Floquet topological system with nontrivial GNVW index at the edge is connected with the non-on-site character of the Floquet evolution at the

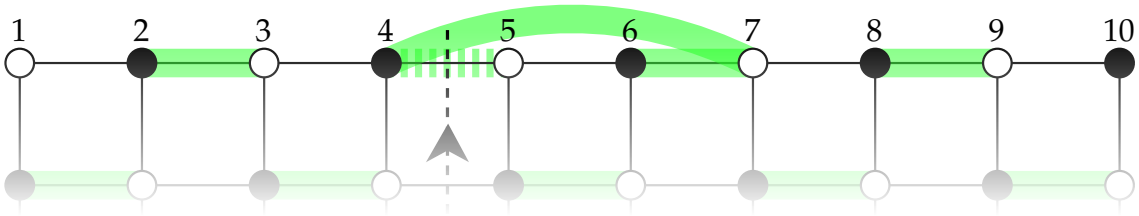

Figure 5.1: The time defect, whose branch line is indicated by the black dashed line, evolves by one period all sites directly to the right of its branch line (dashed), which passes between sites 4 and 5. This amounts to moving the original bond (striped) that straddles the line to a link between sites 4 and 7, since site 5 evolves to 7 after one period.

edge. The reason being that the extrinsic defect construction of Section 2.4 requires an on-site symmetry action.

Finally, the Hamiltonian formalism explicitly treats time differently from space and internal degrees of freedom. As such, even if one can give a reasonable definition of time defects, it may be hard to express it in terms of a modification of the Hamiltonian.

5.2 Time defect in the RLBL model

In this section we will describe numerical simulations of the RLBL model with a time defect, following as close as possible the construction described in Section 2.4. The setup is the same as in Section 4.2.4, with modifications that will be explained now.

There are two main problems, already discussed in the previous sections: U_F is identity in the bulk and is not on-site at the edge. Thus, either the time defect is really trivial, or it has a nontrivial action only at the edge, where we do not know exactly how to proceed. Nevertheless, we investigated the second option, extending the time defect branch line from the bulk to the edge. More precisely, we conjugated (evolved in time) the sites directly at the right of a vertical line that begins anywhere in the bulk and ends at the space between sites 4 and 5, if one starts counting from the top-left corner. The only bond at the edge that straddles the branch line is in the third step, and so it is the only one to get modified by the time defect, as indicated in Figure 5.1.

In the bulk, there would be a defect with flux U_F^{-1}. Therefore, we expect that a state at site 1, which would normally rotate around the entire lattice clockwise in a time period of $\frac{5\times 6}{2}T$ (see Video [52]), will get acted upon by U_F, thus shortening

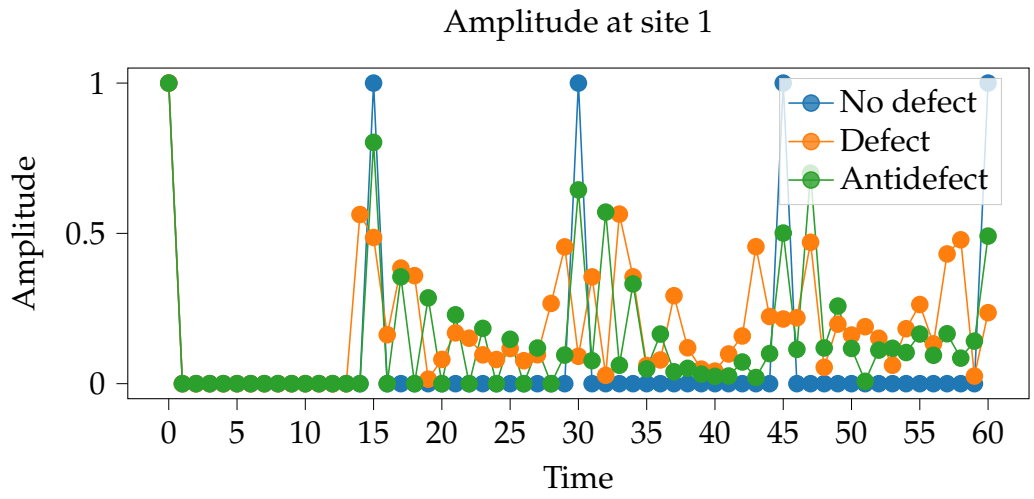

Figure 5.2: Amplitude at site 1, in the top-left corner, of a single-particle state that started localized at site 1 over the time evolution of the $T = 1$ RLBL model with and without a time (anti)defect.

the time it takes to do a full rotation by one period. The opposite would occur for a time antidefect, for which there would be a flux U_F in the bulk and the edge state would arrive at site 1 one period later. In both cases, there will be a site with two bonds connected to it in some step. Because of this, we lose the simplicity of the dynamics that the original model has, and as such we have to do numerical simulations.

The time evolution of the edge mode in the model with a time defect and antidefect can be seen in videos [62] and [63], respectively. Furthermore, the amplitude of the state at site 1 for each period is plotted in Figure 5.2. From the videos and the plot, we can see that the state gets scrambled because of the (anti)defect and that, for the case of the defect, the edge mode does partly arrive one period earlier than the case with no defect, but the opposite does not occur for the antidefect. This is not what we expected from the usual effect of defects on particles that travel around it. Thus, we cannot decisively conclude that the edge mode was acted upon by the Floquet evolution operator.

5.3 Discussion and outlook

We have seen that extending the previous formalism of extrinsic defects to the Floquet symmetry group led us to inconclusive results. This is not entirely unexpected, since the edge Floquet operator is not on-site and the time evolution

of an edge state is not adiabatic. However, if we tried to reach the adiabatic limit by making the period T larger, the system might not be in the same phase as the original RLBL model [8]. Nevertheless, we cannot reject this method entirely yet, and future work will be done to verify if the introduction of disorder in the fifth step might dampen the scrambling and reveal a clearer picture, in which the edge modes do arrive one period earlier or later on average.

Another fact is that the time defect does nothing in the bulk. Whether one views this as a problem depends on the assumption that this would only occur if the system itself was trivial. One further direction if one accepts this assumption is to modify the definition of time defect. To that effect, we propose two alterations: conjugate the terms in the Hamiltonian $H(t)$ at time $t \in \mathbb{R}$ with $U(t)$, instead of U_F; or conjugate them with the Floquet unitary generated by the Hamiltonian with the time defect itself in a self-consistent manner. In contrast, if one does not accept the assumption above, systems with non-trivial topological Floquet Hamiltonians could still be explored.

Chapter 6

Conclusion

In this essay, we explored the generalization of an extrinsic defect to the time periodicity symmetry of Floquet systems. We called the resulting objects *time defects* and presented arguments in Section 5.1 and numerical simulations in Section 5.2 that show that a direct extension of the construction of defects in the static setting to time defects leads to results inconsistent with the expected behaviors of such objects.

The main reason for the inconsistency is that the definition of extrinsic defects presented in Section 2.4 assumes on-site symmetries, while the Floquet unitary U_F, which implements the symmetry of time translation by one period, is not on-site at the edge for any Floquet topological system with an anomalous edge unitary (see Section 4.4). One example that was studied in detail in Section 4.2 is the RLBL model.

The lack of the on-site property leads to further issues illustrated in the numerical simulations of Section 4.2.4. With the tentative time defect introduced, the system is no longer analytically solvable (see Figure 5.1), which complicates its evolution (see Videos [62, 63]). Most importantly and contrary to the expected signatures from extrinsic defects, the edge mode does not get simply acted on by the Floquet evolution when going around the defect, as Figure 5.2 shows.

Therefore, further work is needed to give a precise definition of time defects that exhibits behavior compatible with the expectations. This includes adding disorder into the simulations and exploring different interpretations of time defects outlined in Section 5.3.

The solution to the problems of time defects will continue the trend of extending the existing ideas of static topology to Floquet topology. In the end, this may prove to be very similar to the former case, but the inconsistencies shown above point to another picture, in which the dynamical features of Floquet systems reveal yet another surprising property of quantum matter.

Appendix A

Proof of Floquet's theorem

Floquet's theorem as stated in Theorem 1 concerns the time evolution operator of periodic quantum systems. We will now prove it for Bochner integrable (bounded) periodic Hamiltonians, which includes the finite-dimensional ones.

In what follows, let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of linear operators $A : \mathcal{H} \to \mathcal{H}$ that are bounded, that is

$$\|A\| := \sup_{\langle\psi|\psi\rangle=1} \langle\psi|A|\psi\rangle < \infty.$$

Definition 1. A *time-dependent Hamiltonian* $H : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ is a function that assigns a bounded self-adjoint operator $H(t) : \mathcal{H} \to \mathcal{H}$ to each time $t \in \mathbb{R}$. H is *integrable* if it is Bochner integrable from 0 to t for all $t \in \mathbb{R}$, i.e. $\int_0^t \|H(t)\| \, \mathrm{d}t < \infty$[1].

The boundedness and integrability conditions are needed for the proofs below.

Definition 2. The *time evolution operator* of a time-dependent Hamiltonian H is a solution $U : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ to *Schrödinger equation*

$$i\frac{d}{dt}U(t) = H(t)U(t), \tag{A.1}$$

with initial condition $U(0) = \mathbb{1}$.

Equation (A.1) is equivalent to the usual Schrödinger equation for states $|\psi\rangle \in \mathcal{H}$ if we define its time evolution as $|\psi(t)\rangle = U(t)\,|\psi\rangle$.

Lemma 1. *Given a time evolution operator U, $U(t)$ is an unitary operator for all $t \in \mathbb{R}$.*

Proof. We have that

$$\begin{aligned} i\frac{d}{dt}\left(U(t)U(t)^\dagger\right) &= i\frac{dU(t)}{dt}U(t)^\dagger + iU(t)\frac{dU(t)^\dagger}{dt} \\ &= [H(t), U(t)U(t)^\dagger] \end{aligned}$$

[1] If $t < 0$, we swap the limits of integration to keep the integral positive.

has a unique solution $U(t)U(t)^\dagger = \mathbb{1}$, as the initial condition is $U(0)U(0)^\dagger = \mathbb{1}$. The uniqueness comes from the fact that it is a linear first-order ordinary differential equation (ODE) [64].

Analogously, $U(t)^\dagger U(t) = \mathbb{1}$, thus proving $U(t)$ is unitary. □

In Section 3.1, we gave a formal expression for the solution to Schrödinger equation (A.1) in terms of the Dyson series, which we repeat below:

$$\begin{aligned} U(t) &= \mathcal{T} \exp\left(-i \int_0^t H(t')\,\mathrm{d}t'\right) \\ &= 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, H(t_1)H(t_2)\cdots H(t_n) \\ &= \mathbb{1} + (-i)\int_0^t \mathrm{d}t_1\, H(t_1) + (-i)^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, H(t_1)H(t_2) + \cdots, \end{aligned} \tag{3.1 revisited}$$

In the following Lemma, we prove that this is the unique solution to Schrödinger equation (A.1) if H is integrable in the sense of Definition 1.

Lemma 2. *The time evolution operator U exists and is unique for any integrable time-dependent Hamiltonian H. An explicit solution is given by* (3.1).

Proof. For the existence, we will use the time-ordered exponential series expansion of (3.1). The series converges because it absolutely converges:

$$\begin{aligned} &1 + \sum_{n=1}^{\infty} (-i)^n \left\| \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, H(t_1)H(t_2)\cdots H(t_n) \right\| \\ &\leq 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, \|H(t_1)\|\,\|H(t_2)\|\cdots\|H(t_n)\| \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \int_0^t \mathrm{d}t'\, \|H(t')\|\right)^n \\ &= \exp\left(-i \int_0^t \mathrm{d}t'\, \|H(t')\|\right), \end{aligned}$$

which is well-defined and finite for any $t \in \mathbb{R}$ if H is integrable.

Using Fubini's theorem to exchange the series with the first integral of (3.1), we get the recursive relation $U(t) = \mathbb{1} - i\int_0^t \mathrm{d}t' H(t')U(t')$. Hence $U(t)$ also satisfies Schrödinger equation: $i\frac{d}{dt}U(t) = 0 - iH(t)U(t)$, with initial condition $U(0) = \mathbb{1}$.

The uniqueness follows from the theory of ODEs as mentioned in Lemma 1. □

Definition 3. A *T-periodic Hamiltonian* H is a time dependent Hamiltonian with period $T > 0$, that is, $H(t+T) = H(t)$ for all $t \in \mathbb{R}$. H is *integrable* if it is Bochner integrable over a period, i.e., $\int_0^T \|H(t)\| \, \mathrm{d}t < \infty$

Bochner integrability over a period guarantees integrability in the sense of Definition 1 because for any $t \in \mathbb{R}$,

$$\int_0^t \|H(t)\| \, \mathrm{d}t < \int_0^{NT} \|H(t)\| \, \mathrm{d}t = N \int_0^T \|H(t)\| \, \mathrm{d}t$$

for $N = \lceil \frac{t}{T} \rceil$.

Lemma 3. *The time evolution operator U of a T-periodic Hamiltonian H satisfies $U(t+T) = U(t)U(T)$.*

Proof. If we prove that $U(t+T)U(T)^\dagger$ is another time evolution operator for H, then from Lemma 2 we conclude that $U(t+T)U(T)^\dagger = U(t)$ and thus $U(t+T) = U(t)U(T)$, using the unitarity result of Lemma 1. Indeed,

$$\begin{aligned} i\frac{d}{dt}\left[U(t+T)U(T)^\dagger\right] &= H(t+T)U(t+T)U(T)^\dagger, \\ &= H(t)U(t+T)U(T)^\dagger, \end{aligned}$$

and for $t = 0$, $U(0+T)U(T)^\dagger = \mathbb{1}$. □

Finally, we give below the proof of Floquet's theorem (Theorem 1):

Proof. For a fixed $T > 0$, we can use the spectral theorem applied to the unitary operator $U(T)$ to define a bounded self-adjoint operator $H_F \in \mathcal{L}(\mathcal{H})$ satisfying $U(T) = \exp(-iTH_F)$ (see proof of Theorem 10.5-4 of [65]).

Now we can define $\Phi(t) := U(t)e^{itH_F}$ and use Lemma 3 to prove that it is T-periodic:

$$\begin{aligned} \Phi(t+T) &= U(t+T)e^{i(t+T)H_F}, \\ &= U(t)U(T)e^{iTH_F}e^{itH_F}, \\ &= U(t)e^{itH_F}, \\ &= \Phi(t). \end{aligned}$$

□

The case of unbounded Hamiltonian operators is much more complicated, and the interested reader is referred to [64, 66]. Fortunately, in a large part of our applications, we will only be interested in the finite-dimensional (and thus bounded) case, e.g. a spin system in a finite lattice.

Appendix B

Stroboscopic dynamics of the RLBL model

The RLBL model discussed in Section 4.2 is exactly solvable under a specific choice of coupling constant. In this section, we show how this is accomplished. First, we will prove that the evolution of a localized state $|r\rangle$ after each step is, apart from a phase factor, just a translation to the neighboring site indicated by the bonds at that step (see Figure 4.1(b)).

Since all the terms in H_n are interactions between non-overlapping sites (see Figure 4.1(b)), they commute with each other. Hence, the time evolution operator of the n-th step can be split into $U_n = \prod_{r\in A} \exp(-i\frac{T}{5}H_{n,r})$, where $H_{n,r} = -J(c_r^\dagger c_{r+b_n} + \text{h.c.})$. The time evolution of the annihilation operator c_r at a site $r \in A$ under U_n is

$$\begin{aligned} U_n c_r U_n^\dagger &= \exp\left(-i\frac{T}{5}H_{n,r}\right) c_r \exp\left(i\frac{T}{5}H_{n,r}\right) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\left[-i\frac{T}{5}H_{n,r}, c_r\right]_m \\ &= \cos(JT/5)c_r - i\sin(JT/5)c_{r+b_n}, \end{aligned} \tag{B.1}$$

where in the second equality we have used Hadamard's lemma, with $[A,B]_m$ being the m-nested commutator: $[A,B]_m = [A,[A,B]_{m-1}]$, $[A,B]_0 = B$. Thus, for $JT/5 = \frac{\pi}{2}$, we have that $c_r \to -ic_{r+b_n}$. Similarly, for $r \in B$, $c_r \to -ic_{r-b_n}$.

By accompanying how c_r moves around the lattice and by noting that the evolution of a state at site $|r\rangle$ is the same as the one for the creation operator at the same site:

$$U_n |r\rangle = U_n c_r^\dagger |0\rangle \overset{(B.1)}{=} ic_{r+b_n}^\dagger U_n |0\rangle = i |r+b_n\rangle ,$$

where $|0\rangle$ is the vacuum, we conclude that, after one whole period, a state in the bulk rotates counterclockwise and returns to its initial position and a state at the edge in one of the sublattices translates clockwise by two lattice units (see Figure 4.1(a)).

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