



Para-orthogonal polynomials on the unit circle satisfying three term recurrence formulas[☆]



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ABSTRACT

When a nontrivial measure μ on the unit circle satisfies the symmetry $d\mu(e^{i(2\pi-\theta)}) = -d\mu(e^{i\theta})$ then the associated orthogonal polynomials on the unit circle, say Φ_n , are all real. In this case, in 1986, Delsarte and Genin have shown that the two sequences of para-orthogonal polynomials $\{z\Phi_n(z) + \Phi_n^*(z)\}$ and $\{z\Phi_n(z) - \Phi_n^*(z)\}$, where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$, satisfy three term recurrence formulas and have also explored some further consequences of these sequences of polynomials such as their connections to sequences of orthogonal polynomials on the interval $[-1, 1]$. The same authors, in 1988, have also provided a means to extend these results to cover any nontrivial measure on the unit circle. However, only recently the extension associated with the para-orthogonal polynomials $z\Phi_n(z) - \Phi_n^*(z)$ was thoroughly explored, especially from the point of view of three term recurrence and chain sequences. The main objective of the present article is to provide the theory surrounding the extension associated with the para-orthogonal polynomials $z\Phi_n(z) + \Phi_n^*(z)$ for any nontrivial measure on the unit circle. As an important application of the theory, a characterization for the existence of the integral $\int_0^{2\pi} |e^{i\theta} - w|^{-2} d\mu(e^{i\theta})$, where w is such that $|w| = 1$, is given in terms of the coefficients $\alpha_{n-1} = -\Phi_n(0)$, $n \geq 1$. Examples are also provided to justify all the results.

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1. Introduction

Given a nontrivial probability measure μ on the unit circle, the associated sequence $\{\Phi_n\}$ of monic *orthogonal polynomials on the unit circle* (OPUC) are those polynomials with the property

$$\int_{\mathbb{T}} \overline{\Phi_m(\zeta)} \Phi_n(\zeta) d\mu(\zeta) = \delta_{mn} \kappa_n^{-2}.$$

The contour integration notation used above, which has also been adopted in the rest of this article, is such that $\int_{\mathbb{T}} f(\zeta) d\mu(\zeta) = \int_0^{2\pi} f(e^{i\theta}) d\mu(e^{i\theta})$.

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The orthonormal polynomials on the unit circle are $\phi_n(z) = \kappa_n \Phi_n(z)$, $n \geq 0$. The monic OPUC satisfy the recurrence

$$\Phi_n(z) = z\Phi_{n-1}(z) - \overline{\alpha_{n-1}} \Phi_{n-1}^*(z), \quad n \geq 1, \quad (1.1)$$

where $\alpha_{n-1} = -\overline{\Phi_n(0)}$ and $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$. Following Simon [33, Chap. 1.5] (see also [35]) we will refer to the numbers α_n , $n \geq 0$, as Verblunsky coefficients. It is well known that these coefficients are such that $|\alpha_n| < 1$, $n \geq 0$. It is also known that OPUC are completely determined in terms of these coefficients.

For some recent contributions on this topic we refer to [5,7,8,10,11,25,26,30–32,37,39] and references therein. Detailed accounts regarding the earlier research on these polynomials can be found, for example, in Szegő [38], Geronimus [20], Freud [17] and Van Assche [40]. However, for recent and more up to date texts on this subject we refer to the two volumes of Simon [33] and [34]. There is also a nice chapter about these polynomials in Ismail [22].

Given the sequence $\{\Phi_n\}$ of monic OPUC, an associated sequence $\{P_n\}$ of monic para-orthogonal polynomials on the unit circle (POPUC) is such that

$$P_n(z) = z\Phi_{n-1}(z) - \rho_n \Phi_{n-1}^*(z), \quad n \geq 1,$$

where $\{\rho_n\}$ is a any sequence of complex numbers such that $|\rho_n| = 1$. These polynomials have also turned out to be important in the theory of OPUC.

It is well known that the zeros of OPUC are all within the open unit disk. POPUCs are interesting because their zeros are all simple and lie on the unit circle $|z| = 1$. Moreover, the interpolatory quadrature rules based on the zeros of these polynomials are those quadrature rules on the unit circle which are analogous to the Gaussian quadrature rules on the real line. These facts were first observed in Jones, Njåstad and Thron [24] and further explored, for example, in [6,12,21] and many other contributions.

When the measure μ on the unit circle satisfies the symmetry $d\mu(e^{i(2\pi-\theta)}) = -d\mu(e^{i\theta})$, then the OPUC Φ_n are all real and, consequently, the real Verblunsky coefficients can be given by $\alpha_n = -\Phi_{n+1}(0)$, $n \geq 0$. In this case, the two sets of para-orthogonal polynomials

$$R_n^{(1)}(z) = z\Phi_{n-1}(z) + \Phi_{n-1}^*(z) \quad \text{and} \quad (z-1)R_n^{(2)}(z) = z\Phi_n(z) - \Phi_n^*(z), \quad n \geq 1,$$

considered even earlier than [24] by Delsarte and Genin [13], are important from the point of view of connecting real OPUC to symmetric orthogonal polynomials on the interval $[-1, 1]$. The importance of this connection has been nicely explored by Zhedanov in [45]. A further use of this connection to the problem of frequency analysis, see [3]. In [13], the polynomials $R_n^{(1)}$ and $R_n^{(2)}$ referred to as the first and second kind *singular predictor polynomials*, respectively, are shown to satisfy the three term recurrence formulas

$$\begin{aligned} R_{n+1}^{(1)}(z) &= (z+1)R_n^{(1)}(z) - 4d_{n+1}^{(1)}zR_{n-1}^{(1)}(z), \\ R_{n+1}^{(2)}(z) &= (z+1)R_n^{(2)}(z) - 4d_{n+1}^{(2)}zR_{n-1}^{(2)}(z), \end{aligned} \quad n \geq 1, \quad (1.2)$$

where $R_0^{(1)}(z) = R_0^{(2)}(z) = 1$, $R_1^{(1)}(z) = R_1^{(2)}(z) = z+1$,

$$d_{n+1}^{(1)} = \frac{1}{4}(1 - \alpha_{n-2})(1 + \alpha_{n-1}) \quad \text{and} \quad d_{n+1}^{(2)} = \frac{1}{4}(1 + \alpha_{n-1})(1 - \alpha_n), \quad n \geq 1.$$

Here, one must take $\alpha_{-1} = -1$. We will refer to $R_n^{(1)}$ and $R_n^{(2)}$ as Delsarte and Genin 1 para-orthogonal polynomials (DG1POP) and Delsarte and Genin 2 para-orthogonal polynomials (DG2POP), respectively.

Since the real Verblunsky coefficients $\alpha_n = -\Phi_{n+1}(0)$ are such that $-1 < \alpha_n < 1$, $n \geq 0$, one can easily verify that the sequences $\{d_{n+1}^{(1)}\}_{n=1}^{\infty}$ and $\{d_{n+1}^{(2)}\}_{n=1}^{\infty}$ are both positive chain sequences.

Following the definition adopted by Chihara [9], we say that $\{a_n\}_{n=1}^{\infty}$ is a positive chain sequence if there exists a second sequence $\{g_n\}_{n=0}^{\infty}$ such that

$$\begin{aligned} \text{i)} \quad & 0 \leq g_0 < 1, \quad 0 < g_n < 1 \quad \text{for } n \geq 1; \\ \text{ii)} \quad & a_n = (1 - g_{n-1})g_n \quad \text{for } n \geq 1. \end{aligned} \quad (1.3)$$

This definition is slightly more restrictive than the definition used for general chain sequences by Wall [43], where g_n , $n \geq 1$ is also allowed to be zero.

The sequence $\{g_n\}_{n=0}^{\infty}$ is called a parameter sequence of the positive chain sequence $\{a_n\}_{n=1}^{\infty}$. In general the parameter sequence of a positive chain sequence is not unique. Every positive chain sequence has a minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ uniquely determined by the condition $m_0 = 0$. Every positive chain sequence $\{a_n\}_{n=1}^{\infty}$ also has a maximal parameter sequence $\{M_n\}_{n=0}^{\infty}$ which is characterized by the condition if $g_0 > M_0$ then $\{g_n\}_{n=0}^{\infty}$ generated by (ii) of (1.3) does not satisfy (i) of (1.3).

Delsarte and Genin, in [14], also have provided a means to extend the results in (1.2), especially those associated with the DG1POP, to include complex Verblunsky coefficients. In their analysis the following three term recurrence formula

$$\tilde{R}_{n+1}(z) = (\bar{\beta}_n z + \beta_n) \tilde{R}_n(z) - z \tilde{R}_{n-1}(z), \quad n \geq 1,$$

plays an important role. In the present article, the studies are based on the three term recurrence formula

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1,$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$, where $\{c_n\}_{n=1}^\infty$ is a real sequence and $\{d_{n+1}\}_{n=1}^\infty$ is a positive chain sequence. We remark that the three term recurrence formula considered by Delsarte and Genin [14] can also be made equivalent to the above three term recurrence formula.

Only recently, in [10] and [8], the extension associated with the DG2POP to accommodate any nontrivial probability measure was thoroughly explored. One of the results established in [10], which we have briefly reproduced in Section 2, is that a family of OPUC can be found that leads to the same DG2POP. The resulting para-orthogonal polynomials are also connected to certain real functions on the interval $[-1, 1]$. These real functions which also satisfy certain orthogonality properties can be viewed as an extension to symmetric orthogonal polynomials on the interval $[-1, 1]$ (see [4]).

The principal objective of the present article is to provide a complete picture of the extension associated with the DG1POP. We show that given any nontrivial measure on the unit circle one can find a whole family of DG1POP and we will give the moment functionals associated with each of these DG1POP.

The article is organized as follows. In Section 2, we briefly provide some results established in [10] and then [8]. These results form the base for the development of the new results presented in Sections 3 and 4. Moreover, the results provided in Section 2 together with the new results are used in the applications presented in Section 5. As a particular application, we establish in Section 5 a characterization for the existence of the integral $\int_{\mathbb{T}} |\zeta - w|^{-2} d\mu(\zeta)$ (i.e., the Geronimus transformation of μ) with $|w| = 1$, in terms of the Verblunsky coefficients of μ . Finally, in Section 6, examples having explicit formulas are provided to illuminate the present work.

2. A brief review of some recent developments

Some of the results obtained in [10] that are relevant for the understanding of the results in the present article can be summarized as follows.

Let the nontrivial probability measure μ on the unit circle be such that it has no positive mass (no pure point) at $z = 1$. Let the family of nontrivial probability measures on the unit circle $\mu(\delta; \cdot)$, $0 \leq \delta < 1$, be given by the Uvarov perturbation (see, for example, [18] and [22, Chap. 2.7])

$$\int_{\mathbb{T}} f(\zeta) d\mu(\delta; \zeta) = (1 - \delta) \int_{\mathbb{T}} f(\zeta) d\mu(\zeta) + \delta f(1).$$

Let $\Phi_n(\delta; \cdot)$ be the OPUC with respect to the nontrivial probability measure $\mu(\delta; \cdot)$ on the unit circle. Then for the (appropriately scaled) POPUC $(z - 1)R_n^{(2)}(z)$, which are independent of the mass δ at $z = 1$, given by

$$R_n^{(2)}(z) = \frac{1}{z - 1} \frac{\prod_{j=0}^{n-1} [1 - \rho_j^{(2)} \alpha_j(\delta)]}{\prod_{j=0}^{n-1} [1 - \text{Re}(\rho_j^{(2)} \alpha_j(\delta))]} [z \Phi_n(\delta; z) - \rho_n^{(2)} \Phi_n^*(\delta; z)], \quad n \geq 0,$$

where $\rho_0^{(2)}(\delta) = \Phi_0(\delta; 1) / \Phi_0^*(\delta; 1) = 1$ and

$$\alpha_{n-1}(\delta) = -\overline{\Phi_n(\delta; 0)}, \quad \rho_n^{(2)} = \frac{\Phi_n(\delta; 1)}{\Phi_n^*(\delta; 1)} = \frac{\rho_{n-1}^{(2)} - \bar{\alpha}_{n-1}(\delta)}{1 - \rho_{n-1}^{(2)} \alpha_{n-1}(\delta)}, \quad n \geq 1, \tag{2.1}$$

the following three term recurrence formula holds.

$$R_{n+1}^{(2)}(z) = [(1 + ic_{n+1}^{(2)})z + (1 - ic_{n+1}^{(2)})]R_n^{(2)}(z) - 4d_{n+1}^{(2)}zR_{n-1}^{(2)}(z), \tag{2.2}$$

with $R_0^{(2)}(z) = 1$ and $R_1^{(2)}(z) = (1 + ic_1^{(2)})z + (1 - ic_1^{(2)})$, where the real sequences $\{c_n^{(2)}\}$ and the positive chain sequence $\{d_{n+1}^{(2)}\}$ are such that

$$c_n^{(2)} = \frac{-\text{Im}(\rho_{n-1}^{(2)} \alpha_{n-1}(\delta))}{1 - \text{Re}(\rho_{n-1}^{(2)} \alpha_{n-1}(\delta))} = i \frac{\rho_n^{(2)} - \rho_{n-1}^{(2)}}{\rho_n^{(2)} + \rho_{n-1}^{(2)}} \quad \text{and} \quad d_{n+1}^{(2)} = d_{1,n}^{(2)} = (1 - g_{1,n-1}^{(2)}(\delta))g_{1,n}^{(2)}(\delta),$$

for $n \geq 1$. Here, the parameter sequence $\{g_{1,n}^{(2)}(\delta)\}_{n=0}^\infty$ is given by

$$g_{1,n}^{(2)}(\delta) = \frac{1}{2} \frac{|1 - \rho_n^{(2)} \alpha_n(\delta)|^2}{[1 - \text{Re}(\rho_n^{(2)} \alpha_n(\delta))]}, \quad n \geq 0.$$

Since $0 < g_{1,0}^{(2)}(\delta) < 1$, one can consider the positive chain sequence $\{d_n^{(2)}\}_{n=1}^\infty$ with the additional element $d_1^{(2)} = d_1^{(2)}(\delta)$ given by $d_1^{(2)} = g_{1,0}^{(2)}(\delta)$. Then the sequence $\{c_n^{(2)}\}_{n=1}^\infty$ together with the minimal parameter sequence $\{m_n^{(2)}\}_{n=0}^\infty$ of the positive chain sequence $\{d_n^{(2)}\}_{n=1}^\infty$, where

$$m_0^{(2)} = 0 \quad \text{and} \quad m_n^{(2)} = g_{1,n-1}^{(2)}(\delta), \quad n \geq 1,$$

can be used to completely characterize the above measure $\mu(\delta; \cdot)$. For example, the associated Verblunsky coefficients can be given as

$$\alpha_{n-1}(\delta) = \frac{1}{\rho_n^{(2)}} \frac{1 - 2m_n^{(2)} - ic_n^{(2)}}{1 + ic_n^{(2)}} \quad \text{and} \quad \rho_n^{(2)} = \frac{1 - ic_n^{(2)}}{1 + ic_n^{(2)}} \rho_{n-1}^{(2)}, \quad n \geq 1,$$

where $\rho_0^{(2)} = 1$. The maximal parameter sequence $\{M_n^{(2)}\}_{n=0}^\infty$ of the positive chain sequence $\{d_n^{(2)}\}_{n=1}^\infty$ is also such that $M_0 = \delta$.

With respect to any of the measures $\mu(\delta; \cdot)$ the polynomials $\{R_n^{(2)}\}$ satisfy the so called L-orthogonality

$$\int_{\mathbb{T}} \zeta^{-n+j} R_n^{(2)}(\zeta) (1 - \zeta) d\mu(\delta; \zeta) = 0, \quad 0 \leq j \leq n - 1.$$

In some literature on biorthogonal Laurent polynomials (see [41,42]) a change from $\mu(\zeta)$ to $(w - \zeta)\mu(\zeta)$ is referred to as the Christoffel transformation. The polynomials $R_n^{(2)}$ are actually constant multiples of the Christoffel–Darboux (CD) kernels $K_n(\delta; z, 1) = \sum_{j=0}^n \phi_j(\delta; 1) \phi_j(\delta; z)$ associated with the probability measure $\mu(\delta; \cdot)$. Here, $\phi_n(\delta; \cdot)$ are the orthonormal polynomials associated with $\mu(\delta; \cdot)$. For those interested in further studies discussing the connection between CD kernels and para-orthogonal polynomials, we recommend the articles by Cantero, Moral and Velázquez [6] and Golinskii [21]. For a recent paper exploring this connection see [44]. We also cite [36], where there is a nice section on para-orthogonal polynomials.

As shown in [8], given the real sequence $\{c_n^{(2)}\}_{n=1}^\infty$ and the positive chain sequence $\{d_n^{(2)}\}_{n=1}^\infty$ one can also recover information about the measure $\mu(\delta; \cdot)$ and the associated OPUC $\Phi_n(\delta; \cdot)$ directly with the use of the three term recurrence formula (2.2). Some of the results found in [8] are briefly stated below. These results will be used in Section 3 to recover another measure from the same three term recurrence formula. These new results give us the means to answer the question about the extension to the DG1POP.

Let the sequences of polynomials $\{Q_n\}$ and $\{R_n\}$ be such that

$$\begin{aligned} Q_{n+1}(z) &= [(1 + ic_{n+1})z + (1 - ic_{n+1})]Q_n(z) - 4d_{n+1}zQ_{n-1}(z), \\ R_{n+1}(z) &= [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \end{aligned} \quad n \geq 1, \tag{2.3}$$

with $Q_0(z) = 0$, $R_0(z) = 1$, $Q_1(z) = 2d_1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$, where we always assume $\{c_n\}_{n=1}^\infty$ to be any real sequence. Any of the coefficients c_n can also be allowed to be zero. However, when $c_n = 0$, $n \geq 1$, then the results will lead to those found in [3,13] and [45].

From the recurrence formula (2.3)

$$R_n^*(z) = z^n \overline{R_n(1/\bar{z})} = R_n(z) \quad \text{and} \quad Q_n^*(z) = z^{n-1} \overline{Q_n(1/\bar{z})} = Q_n(z), \quad n \geq 1.$$

With this property the polynomials R_n and Q_n can be called self-inversive polynomials or, more appropriately, conjugate reciprocal polynomials. Moreover, if $R_n(z) = \sum_{j=0}^n r_{n,j} z^j$ and $Q_n(z) = \sum_{j=0}^{n-1} q_{n,j} z^j$, then

$$r_{n,n} = \bar{r}_{n,0} = \prod_{k=1}^n (1 + ic_k), \quad n \geq 1 \quad \text{and} \quad q_{n,n-1} = \bar{q}_{n,0} = 2d_1 \prod_{k=2}^n (1 + ic_k), \quad n \geq 2.$$

Firstly, assuming $\{d_n\}_{n=1}^\infty$ to be any sequences such that $d_n \neq 0$, $n \geq 1$, we give the following lemma that follows from the series expansions of the rational functions $Q_n(z)/R_n(z)$.

Lemma 2.1. *Given the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, where $d_n \neq 0$, $n \geq 1$, let the sequences of polynomials $\{Q_n\}$ and $\{R_n\}$ be as in (2.3). Then there exist two series expansions*

$$E_0(z) = - \sum_{n=0}^\infty v_{n+1} z^n \quad \text{and} \quad E_\infty(z) = \sum_{n=1}^\infty v_{-n+1} z^{-n}, \tag{2.4}$$

where

$$v_n = -\bar{v}_{-n+1}, \quad n = 1, 2, \dots, \tag{2.5}$$

such that there hold the correspondence properties

$$E_0(z) - \frac{Q_n(z)}{R_n(z)} = \frac{\bar{\gamma}_n z^n}{\bar{r}_{n,n}} + O(z^{n+1}) \quad \text{and} \quad E_\infty(z) - \frac{Q_n(z)}{R_n(z)} = \frac{\gamma_n}{r_{n,n} z^{n+1}} + O((1/z)^{n+2}), \tag{2.6}$$

for $n \geq 0$. Moreover, if the (moment) functional \mathcal{N} on the space of Laurent polynomials is defined by $\mathcal{N}[z^{-n}] = v_n, n = 0, \pm 1, \pm 2, \dots$, then the polynomials R_n satisfy the orthogonality property

$$\mathcal{N}[z^{-n+j} R_n(z)] = \begin{cases} -\bar{\gamma}_n, & j = -1, \\ 0, & j = 0, 1, \dots, n-1, \\ \gamma_n, & j = n \end{cases} \quad n \geq 1.$$

Here, $\gamma_n = \frac{4d_{n+1}}{(1 + ic_{n+1})} \gamma_{n-1}, n \geq 1$, with $\gamma_0 = v_0 = \frac{2d_1}{1 + ic_1}$.

Proof. Applying the respective three term recurrence formulas in

$$U_n(z) = Q_n(z)R_{n-1}(z) - Q_{n-1}(z)R_n(z), \quad n \geq 1,$$

there follows $U_1(z) = 2d_1$ and

$$U_{n+1}(z) = 4d_{n+1}zU_n(z) = 2^{2n+1}d_1d_2 \cdots d_{n+1}z^n, \quad n \geq 1.$$

Hence, considering the series expansions in terms of the origin and infinity,

$$\frac{Q_n(z)}{R_n(z)} - \frac{Q_{n-1}(z)}{R_{n-1}(z)} = \begin{cases} \frac{\bar{\gamma}_{n-1}}{\bar{r}_{n-1,n-1}} z^{n-1} + O(z^n), \\ \frac{\gamma_{n-1}}{r_{n-1,n-1}} \frac{1}{z^n} + O((1/z)^{n+1}), \end{cases} \quad n \geq 1, \tag{2.7}$$

where $\gamma_{n-1} = \frac{2^{2n-1}d_1d_2 \cdots d_n}{r_{n,n}}, n \geq 1$. That is, there exist formal series expansions E_0 and E_∞ , as in (2.4), such that the correspondences in (2.6) hold. Since R_n and Q_n are self-inversive, from the infinity correspondence we also have

$$z^{-1} \overline{E_\infty(1/\bar{z})} - \frac{Q_n(z)}{R_n(z)} = \frac{\bar{\gamma}_n}{\bar{r}_{n,n}} z^n + O(z^{n+1}), \quad n \geq 0.$$

Comparing this with the origin correspondence we then have the symmetry property $E_0(z) = z^{-1} \overline{E_\infty(1/\bar{z})}$, which is equivalent to (2.5). The remaining result of the theorem follows by considering the systems of equations in the coefficients of $Q_n(z) = \sum_{j=0}^{n-1} q_{n,j} z^j$ and $R_n(z) = \sum_{j=0}^n r_{n,j} z^j$, which follow from (2.6). \square

The orthogonality results given by the above lemma can also be recovered from results given in [41]. For other such contribution on three term recurrence formulas of the type (2.3) we also refer to [23, Thm. 2.1].

For the next lemma we restrict the sequence $\{d_n\}_{n=1}^\infty$ in (2.3) to be such that $\{d_{n+1}\}_{n=1}^\infty = \{d_{1,n}\}_{n=1}^\infty$ is a positive chain sequence. Clearly, Lemma 2.1 continues to hold as long as $d_1 \neq 0$. From (2.3) we have $\frac{R_n(1)}{2R_{n-1}(1)} [1 - \frac{R_{n+1}(1)}{2R_n(1)}] = d_{1,n}, n \geq 1$. Hence, $\{m_n\}_{n=0}^\infty$, with

$$m_n = 1 - \frac{R_{n+1}(1)}{2R_n(1)}, \quad n \geq 0, \tag{2.8}$$

is the minimal parameter sequence of the positive chain sequence $\{d_{1,n}\}_{n=1}^\infty$.

Note that in [8] we have assumed the stronger restriction that $\{d_n\}_{n=1}^\infty$ is a positive chain sequence. It is important to note that if $\{d_n\}_{n=1}^\infty$ is a positive chain sequence then so is $\{d_{1,n}\}_{n=1}^\infty$. However, the inverse of the above affirmation is not always true.

If $\{m_n\}_{n=0}^\infty$ is the minimal parameter sequence of $\{d_n\}_{n=1}^\infty$ then $\{m_{1,n}\}_{n=0}^\infty$, where $m_{1,n} = m_{n+1}, n \geq 0$, is also a parameter sequence of $\{d_{1,n}\}_{n=1}^\infty$ and the minimal parameter sequence $\{m_n\}_{n=0}^\infty$ of $\{d_{1,n}\}_{n=1}^\infty$ is such that $m_n < m_{1,n}, n \geq 0$. Moreover, if $\{M_n\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_n\}_{n=1}^\infty$ then $\{M_{1,n}\}_{n=0}^\infty$, where $M_{1,n} = M_{n+1}, n \geq 0$, is exactly the maximal parameter sequence of $\{d_{1,n}\}_{n=1}^\infty$. Hence, only when $\{d_{1,n}\}_{n=1}^\infty$ is a positive chain sequence with its maximal parameter sequence $\{M_{1,n}\}_{n=0}^\infty$ different from its minimal parameter sequence $\{m_n\}_{n=0}^\infty$, i.e. $M_{1,0} > 0$, then the choice of d_1 such that $0 < d_1 \leq M_{1,0}$ makes $\{d_n\}_{n=1}^\infty$ also a positive chain sequence.

A positive chain sequence, for which the maximal parameter sequence is the same as the minimal parameter sequence, is said to determine its parameter uniquely. We will refer to such a positive chain sequence as a *single parameter* positive chain

sequence. By Wall's criteria (see [9, p. 101]) for maximal parameter sequence, the sequence $\{d_n\}_{n=1}^\infty$ is a single parameter positive chain sequence if and only if

$$\sum_{n=1}^\infty \prod_{k=1}^n \frac{m_k}{1 - m_k} = \infty,$$

where $\{m_n\}_{n=0}^\infty$ is the minimal parameter sequence of $\{d_n\}_{n=1}^\infty$.

Lemma 2.2 ([15]). *Let the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$ be such that $\{d_{n+1}\}_{n=1}^\infty$ is also a positive chain sequence. Let $\{R_n\}$ be the sequence of polynomials obtained from (2.3) with the use of $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$. Then the zeros of the polynomial R_n are simple and lie on the unit circle $|z| = 1$. By denoting the zeros of R_n by $z_{n,j} = e^{i\theta_{n,j}}$, $j = 1, 2, \dots, n$, where $\theta_{n,j} < \theta_{n,j+1}$, then there holds the interlacing property*

$$0 < \theta_{n+1,1} < \theta_{n,1} < \theta_{n+1,2} < \dots < \theta_{n,n} < \theta_{n+1,n+1} < 2\pi, \quad n \geq 1.$$

Moreover, if $V_n(z) = R'_n(z)R_{n-1}(z) - R'_{n-1}(z)R_n(z)$, $n \geq 1$, then

$$z_{n,j}^{-(n-2)}(z_{n,j} - 1)^{-1} V_n(z_{n,j}) > 0, \quad j = 1, 2, \dots, n, \quad n \geq 1.$$

The initial part of this lemma was established in [15] with the use of the functions

$$G_n(x) = (4z)^{-n/2} R_n(z), \quad n \geq 0, \tag{2.9}$$

given by the transformation $2x = z^{1/2} + z^{-1/2}$. This transformation, which maps the points $z = e^{i\theta}$ onto the points $x = \cos(\theta/2)$, is referred to in [45] as the DG transformation.

Clearly, the zeros of the function G_n in $[-1, 1]$ are $x_{n,j} = \cos(\theta_{n,j}/2)$, $j = 1, 2, \dots, n$. The associated Christoffel–Darboux functions or Wronskians

$$W_n(x) = G'_n(x)G_{n-1}(x) - G'_{n-1}(x)G_n(x), \quad n \geq 1,$$

which do not necessarily remain positive throughout $[-1, 1]$, but satisfy at the zeros of G_n

$$W_n(x_{n,j}) > 0, \quad j = 1, 2, \dots, n \quad \text{and} \quad n \geq 1.$$

The last part of the lemma, proved in [8], can be obtained as follows. From (2.9)

$$W_n(x) = \frac{(4z)^{-(n-1)}}{z-1} [2zV_n(z) - R_{n-1}(z)R_n(z)], \quad n \geq 1$$

and, hence,

$$\frac{z_{n,j}^{-(n-2)}}{z_{n,j} - 1} V_n(z_{n,j}) = 2^{2n-3} W_n(x_{n,j}), \quad j = 1, 2, \dots, n, \quad n \geq 1.$$

2.1. Recovering the first positive measure

Now we briefly state the results obtained in [8] under the stronger restriction that $\{d_n\}_{n=1}^\infty$ is a positive chain sequence. By considering the rational functions $\tilde{A}_n(z)/(z-1)R_n(z)$, $n \geq 1$, where $\tilde{A}_n(z) = R_n(z) - Q_n(z)$, there hold

$$\begin{aligned} \tilde{F}_0(z) - \frac{\tilde{A}_n(z)}{(z-1)R_n(z)} &= \frac{\bar{\gamma}_n}{\bar{r}_{n,n}} z^n + O(z^{n+1}) \\ \tilde{F}_\infty(z) - \frac{\tilde{A}_n(z)}{(z-1)R_n(z)} &= -\frac{\gamma_n}{r_{n,n}} \frac{1}{z^{n+2}} + O((1/z)^{n+3}), \end{aligned} \quad n \geq 1,$$

where $\tilde{F}_0(z) = -\sum_{n=1}^\infty \tilde{\mu}_n z^{n-1}$ and $\tilde{F}_\infty(z) = \sum_{n=0}^\infty \tilde{\mu}_{-n} z^{-n-1}$, with $\tilde{\mu}_0 = 1$ and

$$\tilde{\mu}_n = 1 + \sum_{j=1}^n v_j, \quad \tilde{\mu}_{-n} = 1 - \sum_{j=1}^n v_{-j+1}, \quad n \geq 1.$$

Since $v_n = -\bar{v}_{-n+1}$, $n \geq 1$, one finds $\tilde{\mu}_n = \bar{\tilde{\mu}}_{-n}$, $n \geq 1$.

If one defines the moment functional $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{M}}[z^{-n}] = \tilde{\mu}_n$, $n = 0, \pm 1, \pm 2, \dots$, then

$$\tilde{\mathcal{M}}[z^{-n}] = 1 - \mathcal{N}\left[\frac{1 - z^{-n}}{1 - z}\right], \quad n = 0, \pm 1, \pm 2, \dots$$

Conversely, $\mathcal{N}[z^{-n}] = \tilde{\mathcal{M}}[z^{-n}(1 - z)]$, $n = 0, \pm 1, \pm 2, \dots$

For the rational functions $\tilde{A}_n(z)/[(z-1)R_n(z)]$ there also hold the partial fraction decompositions

$$\frac{\tilde{A}_n(z)}{(z-1)R_n(z)} = \frac{\tilde{\lambda}_{n,0}}{z-1} + \sum_{j=1}^n \frac{\tilde{\lambda}_{n,j}}{z-z_{n,j}}, \quad n \geq 1, \tag{2.10}$$

where $z_{n,j}, j = 1, 2, \dots, n$, are the zeros of $R_n(z)$. The numbers $\tilde{\lambda}_{n,j}$ are such that

$$\tilde{\lambda}_{n,j} > 0, \quad j = 0, 1, \dots, n \quad \text{and} \quad \sum_{j=0}^n \tilde{\lambda}_{n,j} = 1,$$

for $n \geq 1$. Precisely, $\tilde{\lambda}_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)}$ is such that

$$1 - d_1 = \tilde{\lambda}_{1,0} > \tilde{\lambda}_{2,0} > \dots > \tilde{\lambda}_{n,0} > \tilde{\lambda}_{n+1,0} > \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \tilde{\lambda}_{n,0} = M_0,$$

where $\{M_n\}_{n=0}^\infty$ is the maximal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^\infty$, and

$$\tilde{\lambda}_{n,j} = \frac{U_n(z_{n,j})}{(1-z_{n,j})V_n(z_{n,j})} = \frac{2^{2n-1}d_1d_2 \dots d_n}{z_{n,j}^{-(n-2)}(z_{n,j}-1)^{-1}V_n(z_{n,j})} \frac{z_{n,j}}{(z_{n,j}-1)(1-z_{n,j})} > 0,$$

for $j = 1, 2, \dots, n$ and $n \geq 1$.

Thus, with the application of the Helly Selection Theorem (see, for example, [24]) to the Riemann–Stieltjes integral representation of (2.10), we obtain that $\mathcal{M}(\zeta^{-n}) = \int_{\mathbb{T}} \zeta^{-n} d\tilde{\mu}(\zeta), n = 0, \pm 1, \pm 2, \dots$

Now by considering also some further results obtained in [8] we can state the following theorem.

Theorem 2.3. *Let the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ be such that $\{d_n\}_{n=1}^\infty$ is also a positive chain sequence. Let $\{m_n\}_{n=0}^\infty$ and $\{M_n\}_{n=0}^\infty$ be the respective minimal and maximal parameter sequences of $\{d_n\}_{n=1}^\infty$.*

Associated with the sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ there exists a nontrivial probability measure $\tilde{\mu}$ on the unit circle, with M_0 as its mass at $z = 1$, such that the corresponding sequence of monic OPUC $\{\tilde{\Phi}_n\}$ is given by

$$\tilde{\Phi}_0(z) = 1 \quad \text{and} \quad \tilde{\Phi}_n(z) \prod_{k=1}^n (1 + ic_k) = R_n(z) - 2(1 - m_n)R_{n-1}(z), \quad n \geq 1.$$

Here, R_n are the polynomials obtained from (2.3) with the use of $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$.

In particular, for $\tilde{\mu}$ the Verblunsky coefficients $\tilde{\alpha}_{n-1} = -\tilde{\Phi}_n(0)$ are given by

$$\tilde{\alpha}_{n-1} = \frac{1}{\tau_n} \frac{1 - 2m_n - ic_n}{1 + ic_n}, \quad n \geq 1,$$

where $\tau_n = \prod_{k=1}^n (1 - ic_k)/(1 + ic_k), n \geq 1$.

Moreover, if \mathcal{N} is the moment functional associated with $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, as described in Lemma 2.1, then for any Laurent polynomial ℓ ,

$$\mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta) d\tilde{\mu}(\zeta).$$

Observe that if the sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ are taken to be the same as $\{c_n^{(2)}\}_{n=1}^\infty$ and $\{d_n^{(2)}\}_{n=1}^\infty$ then the measure $\tilde{\mu}$ given by Theorem 2.3 is the same as the measure $\mu(\delta; \cdot)$ considered in the beginning of the section.

3. Recovering another positive measure

With the sequence of polynomials $\{Q_n\}$ and $\{R_n\}$ given by (2.3) we now show that one can also obtain another nontrivial probability measure, say $\hat{\mu}$, under the conditions that the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ are such that $d_1 \neq 0$ and $\{d_{n+1}\}_{n=1}^\infty = \{d_{1,n}\}_{n=1}^\infty$ is a positive chain sequence.

Let $\{\hat{A}_n\}_{n=0}^\infty$ be the sequence of polynomials given by

$$\hat{A}_{n-1}(z) = \frac{(1 + ic_1)}{2z} \left[R_n(z) + \frac{(1 - ic_1)}{2d_1} (z - 1) Q_n(z) \right], \quad n \geq 1. \tag{3.1}$$

With the observation made in Section 2 that $R_n(0) = \prod_{k=1}^n (1 - ic_k), n \geq 1, Q_1(0) = 2d_1$ and $Q_n(0) = 2d_1 \prod_{k=2}^n (1 - ic_k), n \geq 2$, it is easily verified that \hat{A}_{n-1} is a polynomial of degree $n - 1$, with $\prod_{k=1}^n (1 + ic_k)$ as its leading coefficient.

Since

$$\frac{\hat{A}_n(z)}{R_{n+1}(z)} - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = \frac{1+c_1^2}{4d_1} (1-z^{-1}) \left[\frac{Q_{n+1}(z)}{R_{n+1}(z)} - \frac{Q_n(z)}{R_n(z)} \right], \quad n \geq 1,$$

from (2.7) there hold

$$\frac{\hat{A}_n(z)}{R_{n+1}(z)} - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = \begin{cases} -\frac{(1+c_1^2)\bar{\gamma}_n}{4d_1\bar{r}_{n,n}} z^{n-1} + O(z^n), \\ \frac{(1+c_1^2)\gamma_n}{4d_1r_{n,n}} \frac{1}{z^{n+1}} + O((1/z)^{n+2}), \end{cases} \quad n \geq 1.$$

Hence, there exist series expansions \hat{F}_0 and \hat{F}_∞ such that

$$\hat{F}_0(z) - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = -\frac{(1+c_1^2)\bar{\gamma}_n}{4d_1\bar{r}_{n,n}} z^{n-1} + O(z^n), \quad n \geq 1 \quad (3.2)$$

and

$$\hat{F}_\infty(z) - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = \frac{(1+c_1^2)\gamma_n}{4d_1r_{n,n}} \frac{1}{z^{n+1}} + O((1/z)^{n+2}), \quad n \geq 1. \quad (3.3)$$

Setting

$$\hat{F}_0(z) = -\sum_{n=0}^{\infty} \hat{\mu}_{n+1} z^n \quad \text{and} \quad \hat{F}_\infty(z) = \sum_{n=1}^{\infty} \hat{\mu}_{-n+1} z^{-n},$$

we then have from (3.1), (3.2) and (3.3), together with the results of Lemma 2.1, that the numbers $\hat{\mu}_n$ satisfy

$$\hat{\mu}_n = \frac{1+c_1^2}{4d_1} (v_n - v_{n+1}), \quad \hat{\mu}_{-n} = \frac{1+c_1^2}{4d_1} (v_{-n} - v_{-n+1}), \quad n \geq 1,$$

with $\hat{\mu}_0 = 1$. Since $v_j = -\bar{v}_{-j+1}$, $j \geq 1$, hence there hold

$$\hat{\mu}_n = \bar{\hat{\mu}}_{-n}, \quad n \geq 1.$$

If we define the moment functional $\hat{\mathcal{M}}$ by $\hat{\mathcal{M}}[z^{-n}] = \hat{\mu}_n$, $n = 0, \pm 1, \pm 2, \dots$, then

$$\hat{\mathcal{M}}[z^{-n}] = \frac{1+c_1^2}{4d_1} \mathcal{N}[z^{-n} - z^{-n-1}], \quad n = 0, \pm 1, \pm 2, \dots, \quad (3.4)$$

where \mathcal{N} is the linear functional defined as in Lemma 2.1.

Since \hat{A}_{n-1} is of degree $n-1$, with the results of Lemma 2.2 there holds the partial fraction decomposition

$$\frac{\hat{A}_{n-1}(z)}{R_n(z)} = \sum_{j=1}^n \frac{\hat{\lambda}_{n,j}}{z - z_{n,j}},$$

where $z_{n,j} = e^{i\theta_{n,j}}$, $j = 1, 2, \dots, n$, are the zeros of R_n and

$$\hat{\lambda}_{n,j} = \frac{\hat{A}_{n-1}(z_{n,j})}{R'_n(z_{n,j})} = \frac{(1+c_1^2)(z_{n,j}-1)}{4d_1z_{n,j}} \frac{Q_n(z_{n,j})}{R'_n(z_{n,j})}, \quad j = 1, 2, \dots, n.$$

Since $Q_n(z_{n,j})/R'_n(z_{n,j})$ can be written as $U_n(z_{n,j})/V_n(z_{n,j})$, again from Lemma 2.2,

$$\hat{\lambda}_{n,j} = \frac{2^{2n-3}(1+c_1^2)d_2 \dots d_n}{z_{n,j}^{-(n-2)}(z_{n,j}-1)^{-1}V_n(z_{n,j})} > 0, \quad j = 1, 2, \dots, n.$$

In addition to the positiveness of the elements $\hat{\lambda}_{n,j}$, $j = 1, 2, \dots, n$, by considering the limit of $z\hat{A}_{n-1}(z)/R_n(z)$, as $z \rightarrow \infty$, we also have

$$\sum_{j=1}^n \hat{\lambda}_{n,j} = 1.$$

Hence, if the step functions $\hat{\psi}_n(e^{i\theta})$, $n \geq 1$, are defined on $[0, 2\pi]$ by

$$\hat{\psi}_n(e^{i\theta}) = \begin{cases} 0, & 0 < \theta \leq \theta_{n,1}, \\ \sum_{j=1}^k \hat{\lambda}_{n,j}, & \theta_{n,k} < \theta \leq \theta_{n,k+1}, \quad k = 1, 2, \dots, n-1, \\ 1, & \theta_{n,n} < \theta \leq 2\pi. \end{cases}$$

Then from the definition of the Riemann–Stieltjes integrals

$$\frac{\hat{A}_{n-1}(z)}{R_n(z)} = \int_{\mathbb{T}} \frac{1}{z - \zeta} d\hat{\psi}_n(\zeta), \quad n \geq 1.$$

Hence, by the application of the Helly Selection Theorem (see [24]) there exists a subsequence $\{n_j\}$ such that $\hat{\psi}_{n_j}(e^{i\theta})$ converges to a bonded non-decreasing function, say $\hat{\mu}(e^{i\theta})$, in $[0, 2\pi]$.

From (3.2) and (3.3), since

$$\int_{\mathbb{T}} d\hat{\psi}_n(\zeta) = 1 \quad \text{and} \quad \int_{\mathbb{T}} \zeta^k d\hat{\psi}_n(\zeta) = \hat{\mu}_{-k}, \quad k = \pm 1, \pm 2, \dots, \pm n - 2,$$

we also have that

$$\int_{\mathbb{T}} d\hat{\mu}(\zeta) = 1 = \hat{\mathcal{M}}[1] \quad \text{and} \quad \int_{\mathbb{T}} \zeta^n d\hat{\mu}(\zeta) = \hat{\mu}_{-n} = \hat{\mathcal{M}}[z^n], \quad n = \pm 1, \pm 2, \dots \tag{3.5}$$

Now we can state the following theorem.

Theorem 3.1. *Given the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$, where $\{d_{n+1}\}_{n=1}^\infty = \{d_{1,n}\}_{n=1}^\infty$ is also a positive chain sequence then, associated with these, there exists a nontrivial probability measure $\hat{\mu}$ on the unit circle such that for any Laurent polynomial ℓ ,*

$$\int_{\mathbb{T}} \ell(\zeta) d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} \mathcal{N}[\ell(z)(1 - z^{-1})]. \tag{3.6}$$

Here \mathcal{N} is the moment functional associated with $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, as described in Lemma 2.1, where the additional term d_1 is chosen such that $d_1 \neq 0$. The sequence of monic OPUC $\{\hat{\Phi}_n\}$ with respect to $\hat{\mu}$ are

$$\hat{\Phi}_n(z) = \frac{R_{n+1}(z) - 2(1 - m_n)R_n(z)}{(z - 1) \prod_{k=1}^{n+1} (1 + ic_k)}, \quad n \geq 0, \tag{3.7}$$

where R_n are the polynomials obtained from (2.3) with the use of the two sequences $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$. In particular, the Verblunsky coefficients $\hat{\alpha}_{n-1} = -\overline{\hat{\Phi}_n(0)}$ with respect to $\hat{\mu}$ are

$$\hat{\alpha}_{n-1} = -\frac{1}{\tau_n} \frac{1 - 2m_n - ic_{n+1}}{1 - ic_{n+1}}, \quad n \geq 1.$$

Here, $\{m_n\}_{n=0}^\infty$, where $m_n = 1 - \frac{R_{n+1}(1)}{2R_n(1)}$, $n \geq 0$, is the minimal parameter sequence of $\{d_{1,n}\}_{n=1}^\infty$ and $\tau_n = \prod_{k=1}^n (1 - ic_k)/(1 + ic_k)$, $n \geq 1$.

Proof. The required probability measure $\hat{\mu}$ is the one established in (3.5). We then have from (3.4) the moment functional relations in (3.6). From these, clearly,

$$\frac{1 + c_1^2}{4d_1} \mathcal{N}[z^{-k}(1 - z^{-1})p(z)] = \int_{\mathbb{T}} \zeta^{-k} p(\zeta) d\hat{\mu}(\zeta), \quad k = 0, \pm 1, \pm 2, \dots, \tag{3.8}$$

for any polynomial p .

Now, from Lemma 2.1, since

$$\mathcal{N}[z^{-n+k}R_n(z)] = \begin{cases} 0, & k = 0, 1, \dots, n-1, \\ \gamma_n, & k = n, \end{cases} \tag{3.9}$$

where

$$\gamma_0 = \frac{2d_1}{1 + ic_1} \quad \text{and} \quad \gamma_n = \frac{4d_{n+1}}{(1 + ic_{n+1})} \gamma_{n-1}, \quad n \geq 1,$$

by considering the monic polynomial of degree n

$$\hat{\Phi}_n(z) = \frac{1}{\prod_{k=1}^{n+1} (1 + ic_k)} \frac{1}{z-1} \left[R_{n+1}(z) - \frac{R_{n+1}(1)}{R_n(1)} R_n(z) \right],$$

we obtain from (3.8) that

$$\int_{\mathbb{T}} \zeta^{-k} \hat{\Phi}_n(\zeta) d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} \mathcal{N} [z^{-k} (1 - z^{-1}) \hat{\Phi}_n(z)] = \begin{cases} 0, & k = 0, 1, \dots, n - 1, \\ \hat{\kappa}_n^{-2}, & k = n, \end{cases}$$

where $\hat{\kappa}_n^{-2} = (1 - m_n) \prod_{k=0}^n \frac{4d_{k+1}}{1 + c_{k+1}^2}$. Hence, with the use of (2.8), we obtain the results for the OPUC and, consequently, also about the Verblunsky coefficients. \square

Since $R_n^*(z) = R_n(z)$, $n \geq 0$, we also have from Theorem 3.1 that

$$\hat{\Phi}_n^*(z) = \frac{R_{n+1}(z) - 2(1 - m_n)zR_n(z)}{(1 - z) \prod_{k=1}^{n+1} (1 - ic_k)}, \quad n \geq 0. \tag{3.10}$$

Now combining the results obtained in Theorem 2.3 and the above theorem we can state the following.

Theorem 3.2. *If $\{d_{1, n}\}_{n=1}^\infty$ is not a single parameter positive chain sequence then the measure $\hat{\mu}$, derived as in Theorem 3.1, is such that the integral*

$$J = \int_{\mathbb{T}} |\zeta - 1|^{-2} d\hat{\mu}(\zeta) = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} (1 - \zeta)^{-1} d\hat{\mu}(\zeta)$$

exists.

Proof. From Theorem 3.1 the measure $\hat{\mu}$ is such that

$$\int_{\mathbb{T}} \ell(\zeta) d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} \mathcal{N} [\ell(z)(1 - z^{-1})],$$

for any Laurent polynomial ℓ . Here, \mathcal{N} is the moment functional associated with $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, as described in Lemma 2.1, where the additional term d_1 is chosen such that $d_1 \neq 0$.

Since the chain sequence $\{d_{1, n}\}_{n=1}^\infty$ does not determine its parameter uniquely we have $M_1 > 0$, where $\{M_{n+1}\}_{n=0}^\infty = \{M_{1, n}\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_{1, n}\}_{n=1}^\infty$. Hence, if we chose d_1 such that $0 < d_1 \leq M_1$ then $\{d_n\}_{n=1}^\infty$ is also a positive chain sequence and from Theorem 2.3 there exists a probability measure $\tilde{\mu}$ on the unit circle such that

$$\mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta) d\tilde{\mu}(\zeta).$$

Hence,

$$\int_{\mathbb{T}} \ell(\zeta) d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta^{-1})(1 - \zeta) d\tilde{\mu}(\zeta),$$

from which we conclude the required result of the theorem. \square

If the integral $J = \int_{\mathbb{T}} \zeta(1 - \zeta)^{-1} (\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ exists then so does the integral $\int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$. Clearly, the inverse is not always true.

For the next theorem we assume that the measure $\hat{\mu}$ is such that the integral $\int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ exists or, better still, that the Cauchy principal value integral $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ exists.

For the definition of Cauchy principal value integrals see, for example, [29, Eq. (1.4.24)]. For the purpose of the present work, what we specifically mean by $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ is that

$$I = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{2\pi - \epsilon} \frac{e^{i\theta}}{e^{i\theta} - 1} d\hat{\mu}(e^{i\theta}).$$

Clearly, if $\int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ exists then so is the principal value integral $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$. Since $\hat{\mu}$ is a probability measure it is easily verified that $I + \bar{I} = 2\mathcal{R}e(I) = 1$. Moreover, if $\hat{\mu}$ also satisfies the symmetry $d\hat{\mu}(e^{i(2\pi-\theta)}) = -d\hat{\mu}(e^{i\theta})$ then $I = 1/2$.

As an example, we consider the probability measure $\hat{\mu}$ given by

$$\int_{\mathbb{T}} \ell(\zeta) d\hat{\mu}(\zeta) = (1 - \delta) \int_{\mathbb{T}} \ell(\zeta) \frac{1}{2\pi i \zeta} d\zeta + \delta \ell(i),$$

where $0 \leq \delta < 1$. Then

$$I = \int_{\mathbb{T}} \frac{\zeta}{\zeta - 1} d\hat{\mu}(\zeta) = (1 - \delta) \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{0+\epsilon}^{2\pi-\epsilon} \frac{e^{i\theta}}{e^{i\theta} - 1} d\theta + \delta \frac{i}{i - 1}.$$

This can be written as

$$I - \delta \frac{i}{i - 1} = (1 - \delta) \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \left[\int_{0+\epsilon}^{\pi} \frac{e^{i\theta}}{e^{i\theta} - 1} d\theta + \int_{0+\epsilon}^{\pi} \frac{e^{-i\theta}}{e^{-i\theta} - 1} d\theta \right] = (1 - \delta) \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{0+\epsilon}^{\pi} d\theta,$$

from which $I = 1/2 - i\delta/2$.

Theorem 3.3. Let the associated nontrivial probability measure $\hat{\mu}$ on the unit circle as given by Theorem 3.1 is such that the principal value integral $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta)$ exists. If \mathcal{N} is the moment functional associated with $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$, as described in Lemma 2.1, where the additional term d_1 is chosen such that $d_1 \neq 0$, then for any Laurent polynomial ℓ ,

$$\frac{1 + c_1^2}{4d_1} \mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{\zeta}{\zeta - 1} d\hat{\mu}(\zeta) + it \ell(1),$$

where $t = -\mathcal{I}m[(1 + ic_1)I]$. In particular, if $\{d_{1,n}\}_{n=1}^{\infty}$ is not a single parameter positive chain sequence then $t = 0$.

Proof. From (3.6) and the symmetry property (2.5), clearly \mathcal{N} is such that

$$\frac{1 + c_1^2}{4d_1} \mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{\zeta}{\zeta - 1} d\hat{\mu}(\zeta) + it \ell(1),$$

for all Laurent polynomials ℓ , where $-\infty < t < \infty$. Thus, from (3.9), also

$$\begin{aligned} \frac{1 + c_1^2}{4d_1} \mathcal{N}[z^{-n+k} R_n(z)] &= \int_{\mathbb{T}} \zeta^{-n+k} R_n(\zeta) \frac{\zeta}{\zeta - 1} d\hat{\mu}(\zeta) + it R_n(1) \\ &= \begin{cases} 0, & k = 0, 1, \dots, n - 1, \\ \frac{1 + c_1^2}{4d_1} \gamma_n, & k = n. \end{cases} \end{aligned}$$

We can find the value of t from $\mathcal{N}[z^{-1} R_1(z)] = 0$. Since $\int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\hat{\mu}(\zeta) = -\bar{I}$, we then have

$$\mathcal{N}[z^{-1} R_1(z)] = (1 + ic_1)I - (1 - ic_1)\bar{I} + 2it = 0,$$

which gives the required value for t .

Now if $\{d_{1,n}\}_{n=1}^{\infty}$ is not a single parameter positive chain sequence, by letting $d_1 \leq M_1$, then from $\int_{\mathbb{T}} \ell(\zeta) d\hat{\mu}(\zeta) = \frac{1+c_1^2}{4d_1} \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta^{-1})(1 - \zeta) d\tilde{\mu}(\zeta)$ we have

$$I = \frac{1 + c_1^2}{4d_1} \int_{\mathbb{T}} (1 - \zeta) d\tilde{\mu}(\zeta),$$

which gives $\mathcal{I}m(I) = \frac{1+c_1^2}{4d_1} \mathcal{I}m(\tilde{\mu}_1)$, where $\tilde{\mu}_1 = \int_{\mathbb{T}} \zeta^{-1} d\tilde{\mu}(\zeta) = \bar{\mu}_{-1}$. On the other hand from $\mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta) d\tilde{\mu}(\zeta)$ and $\mathcal{N}[z^{-1} R_1(\zeta)] = 0$ we have $c_1 = -\frac{1+c_1^2}{2d_1} \mathcal{I}m(\tilde{\mu}_1)$. Thus, recalling also that $\mathcal{R}e(I) = 1/2$ we have $t = -\mathcal{I}m[(1 + ic_1)I] = 0$. \square

From (3.7) and (3.10), by elimination of R_{n+1} ,

$$\frac{1}{\prod_{k=1}^n (1 + ic_k)} R_n(z) = \frac{1 + ic_{n+1}}{2(1 - m_n)} \left[\hat{\Phi}_n(z) + \tau_{n+1} \hat{\Phi}_n^*(z) \right], \quad n \geq 0. \tag{3.11}$$

Similarly, by elimination of R_n ,

$$\frac{1}{\prod_{k=1}^{n+1} (1 + ic_k)} R_{n+1}(z) = z \hat{\Phi}_n(z) + \tau_{n+1} \hat{\Phi}_n^*(z), \quad n \geq 0. \tag{3.12}$$

Observe that the polynomials appearing on both sides of (3.11) and (3.12) are monic polynomials. Hence, we can also write (3.11) in the form

$$\frac{1}{\prod_{k=1}^n (1 + ic_k)} R_n(z) = \frac{1}{1 - \tau_{n+1} \hat{\alpha}_{n-1}} \left[\hat{\Phi}_n(z) + \tau_{n+1} \hat{\Phi}_n^*(z) \right], \quad n \geq 0.$$

Moreover, using the recurrence formula (1.1) for the OPUC $\hat{\Phi}_n$ in the above expression

$$\frac{1}{\prod_{k=1}^n (1 + ic_k)} R_n(z) = \left[z \hat{\Phi}_{n-1}(z) + \frac{\tau_{n+1} - \bar{\alpha}_{n-1}}{1 - \tau_{n+1} \hat{\alpha}_{n-1}} \hat{\Phi}_{n-1}^*(z) \right], \quad n \geq 1.$$

Comparing this with (3.12) we have $\tau_n = \frac{\tau_{n+1} - \bar{\alpha}_{n-1}}{1 - \tau_{n+1} \hat{\alpha}_{n-1}}$, $n \geq 1$, or equivalently,

$$\tau_{n+1} = \frac{\tau_n + \bar{\alpha}_{n-1}}{1 + \tau_n \hat{\alpha}_{n-1}}, \quad n \geq 1.$$

The above recurrence formula for τ_n is useful if we can find an expression for τ_1 in terms of the measure $\hat{\mu}$.

Assuming $l = \int_{\mathbb{T}} (\zeta - 1)^{-1} \zeta d\hat{\mu}(\zeta)$ exists, from $\mathcal{N}[z^{-1}R_1(z)] = 0$ and $\tau_1 = (1 - ic_1)/(1 + ic_1)$, the required expression for τ_1 is

$$\tau_1 = \frac{l + it}{\bar{l} - it}.$$

From $\tau_n = \prod_{k=1}^n (1 - ic_k)/(1 + ic_k)$, $n \geq 1$, and the recurrence formula for $\{\tau_n\}$,

$$c_1 = i \frac{\tau_1 - 1}{\tau_1 + 1}, \quad c_{n+1} = \frac{\mathcal{I}m(\tau_{n+1} \hat{\alpha}_{n-1})}{1 - \mathcal{R}e(\tau_{n+1} \hat{\alpha}_{n-1})} = \frac{\mathcal{I}m(\tau_n \hat{\alpha}_{n-1})}{1 + \mathcal{R}e(\tau_n \hat{\alpha}_{n-1})}, \quad n \geq 1.$$

From the expression for the Verblunsky coefficients in Theorem 3.1,

$$1 + \tau_n \hat{\alpha}_{n-1} = \frac{2m_n}{1 - ic_{n+1}} \quad \text{and} \quad 1 - \tau_{n+1} \hat{\alpha}_{n-1} = \frac{2(1 - m_n)}{1 + ic_{n+1}} \quad n \geq 1.$$

Hence,

$$m_n = \frac{1}{2} \frac{|1 + \tau_n \hat{\alpha}_{n-1}|^2}{1 + \mathcal{R}e(\tau_n \hat{\alpha}_{n-1})} = \frac{1}{2} \frac{1 - |\tau_{n+1} \hat{\alpha}_{n-1}|^2}{1 - \mathcal{R}e(\tau_{n+1} \hat{\alpha}_{n-1})}, \quad n \geq 1,$$

$$1 - m_n = \frac{1}{2} \frac{1 - |\tau_n \hat{\alpha}_{n-1}|^2}{1 + \mathcal{R}e(\tau_n \hat{\alpha}_{n-1})} = \frac{1}{2} \frac{|1 - \tau_{n+1} \hat{\alpha}_{n-1}|^2}{1 - \mathcal{R}e(\tau_{n+1} \hat{\alpha}_{n-1})}, \quad n \geq 1,$$

and $d_{n+1} = (1 - m_{n-1})m_n$, $n \geq 1$, with $m_0 = 0$.

4. Sequences of para-orthogonal polynomials that satisfy three term recurrence formulas

From what was presented in the beginning of Section 2, given the sequence $\{\Phi_n\}$ of OPUC associated with some non-trivial probability measure μ on the unit circle then the sequence $\{z\Phi_n(z) - \rho_n^{(2)} \Phi_n^*(z)\}$ of para-orthogonal polynomials, where

$$\rho_0^{(2)} = 1 \quad \text{and} \quad \rho_n^{(2)} = \frac{\Phi_n(1)}{\Phi_n^*(1)} = \frac{\rho_{n-1}^{(2)} - \bar{\alpha}_{n-1}}{1 - \rho_{n-1}^{(2)} \alpha_{n-1}}, \quad n \geq 1,$$

satisfy the three term recurrence formula (2.2).

By observing the results obtained in Section 3 we are now able to state the following theorem which gives a general result about how to generate other sequences of para-orthogonal polynomials that satisfy three term recurrence formulas.

Theorem 4.1. Let $\{\alpha_n\}_{n=0}^\infty$ be such that $|\alpha_n| < 1, n \geq 0$, and let μ and $\{\Phi_n\}$ be the respective nontrivial probability measure and sequence of OPUC for which α_n are the Verblunsky coefficients. With ρ_0 chosen such that $|\rho_0| = 1, \rho_0 \neq 1$, let the sequence of numbers $\{\rho_n\}_{n=0}^\infty$ be given by

$$\rho_n = \frac{\rho_{n-1} - \bar{\alpha}_{n-1}}{1 - \rho_{n-1}\alpha_{n-1}}, \quad n \geq 1, \tag{4.1}$$

where $\alpha_{n-1} = -\bar{\Phi}_n(0), n \geq 1$, are the Verblunsky coefficients. Then the sequence $\{R_n\}$ of POPUC given by

$$R_n(z) \prod_{k=0}^{n-1} \frac{1 + \mathcal{R}e(\rho_k \alpha_{k-1})}{1 + \bar{\rho}_k \bar{\alpha}_{k-1}} = z\Phi_{n-1}(z) - \rho_{n-1}\Phi_{n-1}^*(z), \quad n \geq 1,$$

satisfy the three term recurrence formula

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \tag{4.2}$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$. Here, the real sequence $\{c_n\}$ and the positive chain sequence $\{d_{n+1}\}_{n=1}^\infty = \{d_{1,n}\}_{n=1}^\infty$ are such that

$$c_1 = i\frac{\rho_0 + 1}{\rho_0 - 1}, \quad c_{n+1} = \frac{-\mathcal{I}m(\rho_n \alpha_{n-1})}{1 + \mathcal{R}e(\rho_n \alpha_{n-1})} \quad \text{and} \quad d_{1,n} = (1 - m_{n-1})m_n, \quad n \geq 1,$$

where $\{m_n\}_{n=0}^\infty$ is the minimal parameter sequence of $\{d_{1,n}\}_{n=1}^\infty$ given by

$$m_n = \frac{1}{2} \frac{1 - |\rho_n \alpha_{n-1}|^2}{1 + \mathcal{R}e(\rho_n \alpha_{n-1})}, \quad n \geq 0.$$

The measure μ is such that the integral $J = \int_{\mathbb{T}} |\zeta - 1|^{-2} d\mu(\zeta) = \int_{\mathbb{T}} \zeta(1 - \zeta)^{-1}(\zeta - 1)^{-1} d\mu(\zeta)$ exists if and only if there exists a ρ_0 ($|\rho_0| = 1, \rho_0 \neq 1$) such that the corresponding positive chain sequence $\{d_{1,n}\}_{n=1}^\infty$ is not a single parameter positive chain sequence.

Proof. We consider the monic polynomials $P_0(z) = 1$ and $P_n(z) = z\Phi_{n-1}(z) - \rho_{n-1}\Phi_{n-1}^*(z), n \geq 1$, and show that, as in [10, Thm. 2.1],

$$P_{n+1}(z) = (z + \frac{\rho_n}{\rho_{n-1}})P_n(z) - (1 - \bar{\rho}_{n-1}\bar{\alpha}_{n-1})(1 + \rho_{n-1}\alpha_{n-2})zP_{n-1}(z), \quad n \geq 1. \tag{4.3}$$

With the use of (1.1) and (4.1), first observe that

$$P_n(z) = \frac{\Phi_n(z) - \rho_n \Phi_n^*(z)}{1 + \rho_n \alpha_{n-1}}, \quad n \geq 0.$$

Hence for the polynomials

$$p_{n+1}(z) = P_{n+1}(z) + (1 - \bar{\rho}_{n-1}\bar{\alpha}_{n-1})(1 + \rho_{n-1}\alpha_{n-2})zP_{n-1}(z), \quad n \geq 1,$$

we have

$$p_{n+1}(z) = [z\Phi_n(z) - \rho_n \Phi_n^*(z)] + (1 - \bar{\rho}_{n-1}\bar{\alpha}_{n-1})z[\Phi_{n-1}(z) - \rho_{n-1}\Phi_{n-1}^*(z)], \quad n \geq 1.$$

Once again using (1.1) and then (4.1) this reduces to

$$p_{n+1}(z) = z[z\Phi_{n-1}(z) - \rho_{n-1}\Phi_{n-1}^*(z)] + \frac{\rho_n}{\rho_{n-1}}[z\Phi_{n-1}(z) - \rho_{n-1}\Phi_{n-1}^*(z)], \quad n \geq 1,$$

thus proving (4.3). From (4.3), with the observation

$$(1 + \rho_n \alpha_{n-1})(1 - \rho_{n-1} \alpha_{n-1}) = (1 + \bar{\rho}_n \bar{\alpha}_{n-1})(1 - \bar{\rho}_{n-1} \bar{\alpha}_{n-1}) = 1 - |\alpha_{n-1}|^2, \quad n \geq 0,$$

one can easily derive (4.2).

Since, $m_0 = 0, m_n > 0, n \geq 1$, and

$$1 - m_n = \frac{1}{2} \frac{|1 + \rho_n \alpha_{n-1}|^2}{1 + \mathcal{R}e(\rho_n \alpha_{n-1})} > 0, \quad n \geq 1,$$

one can also conclude that $\{d_{1,n}\}_{n=1}^\infty$ is a positive chain sequence and that $\{m_n\}_{n=0}^\infty$ is its minimal parameter sequence.

Before proving the last part of the theorem, observe that

$$\rho_n = - \prod_{k=1}^{n+1} (1 - ic_k)/(1 + ic_k) = -\tau_{n+1}, \quad n \geq 0,$$

where τ_n are the same as in Theorem 3.1. Hence, starting from the three term recurrence formula (4.2) we find that the measure $\hat{\mu}$ given by Theorem 3.1 is the same as the measure μ . Therefore, if $\{d_{1,n}\}_{n=1}^\infty$ is not a single parameter positive chain sequence then, as in Theorem 3.2, the measure μ is such that the integral J exists.

On the other hand, if μ is such that J exists we can consider the probability measure $\tilde{\mu}$ given by

$$d\tilde{\mu}(\zeta) = J^{-1} \frac{\zeta}{(\zeta - 1)(1 - \zeta)} d\mu(\zeta).$$

Hence, for the para-orthogonal polynomials $(z - 1)R_n^{(2)}(z) = (z - 1)R_n^{(2)}(\tilde{\mu}, z)$ we obtain from the results presented in Section 2 that

$$R_{n+1}^{(2)}(z) = [(1 + ic_{n+1}^{(2)})z + (1 - ic_{n+1}^{(2)})]R_n^{(2)}(z) - 4d_{n+1}^{(2)}zR_{n-1}^{(2)}(z),$$

with $R_0^{(2)}(z) = 1$ and $R_1^{(2)}(z) = (1 + ic_1^{(2)})z + (1 - ic_1^{(2)})$, where $\{c_n^{(2)}\}_{n=1}^\infty$ is a real sequence and $\{d_{n+1}^{(2)}\}_{n=1}^\infty$ is a positive chain sequence. Clearly, $\{d_{n+1}^{(2)}\}_{n=1}^\infty$ is also not a single parameter positive chain sequence. Consequently, using results obtained in Theorems 3.1 and 3.2, we conclude that with the choice ρ_0 such that $c_1^{(2)} = i(\rho_0 + 1)/(\rho_0 - 1)$ the resulting positive chain sequence $\{d_{1,n}\}_{n=1}^\infty = \{d_{n+1}^{(2)}\}_{n=1}^\infty$ is not a single parameter positive chain sequence. \square

Note that every choice of ρ_0 generates a different sequence of polynomials $\{R_n\}$ which we can consider as a different extension of the DG1POP. By observing also the results establish in Theorem 3.3, we can state the following Theorem which gives the orthogonality property associated with each of these the extension to the DG1POP.

Theorem 4.2. *Let μ be a nontrivial probability measure on the unit circle such that the principal value integral $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\mu(\zeta)$ exists and let $\{\Phi_n\}$ be the sequence of monic OPUC with respect to μ . Let t be such that $-\infty < t < \infty$ and let the sequence $\{\rho_n(t)\}_{n=0}^\infty$, with the property $|\rho_n(t)| = 1$, be given by*

$$\rho_0(t) = -\frac{I + it}{\bar{I} - it} \quad \text{and} \quad \rho_n(t) = \frac{\rho_{n-1}(t) - \bar{\alpha}_{n-1}}{1 - \rho_{n-1}(t)\alpha_{n-1}}, \quad n \geq 1,$$

where $\alpha_{n-1} = -\overline{\Phi_n(0)}$, $n \geq 0$. Then for the sequence $\{R_n(t; z)\}$ of POPUC, in z , given by

$$R_n(t; z) = \prod_{k=0}^{n-1} \frac{1 + \text{Re}(\rho_k(t)\alpha_{k-1})}{1 + \rho_k(t)\alpha_{k-1}} = z\Phi_{n-1}(z) - \rho_{n-1}(t)\Phi_{n-1}^*(z), \quad n \geq 1,$$

the following three term recurrence formula holds.

$$R_{n+1}(t; z) = [(1 + ic_{n+1}(t))z + (1 - ic_{n+1}(t))]R_n(t; z) - 4d_{n+1}(t)zR_{n-1}(t; z),$$

with $R_0(t; z) = 1$ and $R_1(t; z) = (1 + ic_1(t))z + (1 - ic_1(t))$, where the real sequences $\{c_n(t)\}$ and $\{d_{n+1}(t)\}$ are such that

$$c_1(t) = i \frac{\rho_0(t) + 1}{\rho_0(t) - 1} = -2(t + \mathcal{I}m(I)) \quad \text{and}$$

$$c_{n+1}(t) = \frac{-\mathcal{I}m(\rho_n(t)\alpha_{n-1})}{1 + \text{Re}(\rho_n(t)\alpha_{n-1})}, \quad d_{n+1}(t) = (1 - m_{n-1}(t))m_n(t), \quad n \geq 1,$$

where $\{m_n(t)\}_{n=0}^\infty$ is the minimal parameter sequence of the positive chain sequence $\{d_{n+1}(t)\}_{n=1}^\infty$, which is given by

$$m_n(t) = \frac{1}{2} \frac{1 - |\rho_n(t)\alpha_{n-1}|^2}{1 + \text{Re}(\rho_n(t)\alpha_{n-1})}, \quad n \geq 0.$$

The sequence $\{d_{n+1}(t)\}_{n=1}^\infty$ is always a single parameter positive chain sequence if $t \neq 0$. However, $\{d_{n+1}(0)\}_{n=1}^\infty$ is not a single parameter positive chain sequence if and only if the integral $J = \int_{\mathbb{T}} |\zeta - 1|^{-2} d\mu(\zeta)$ exists.

With respect to the measure μ , the polynomials $R_n(t; \cdot)$ satisfy the L-orthogonality $\mathcal{N}^{(t)}[z^{-n+k}R_n(t; z)] = 0$, $k = 0, 1, \dots, n - 1$, with respect to the moment functional $\mathcal{N}^{(t)}$ given by

$$\frac{1 + 4(t + \mathcal{I}m(I))^2}{4d_1} \mathcal{N}^{(t)}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{\zeta}{\zeta - 1} d\mu(\zeta) + it \ell(1). \tag{4.4}$$

Observe that I in the above theorem is such that $I + \bar{I} = \mu_0 = 1$. Hence, $\text{Re}(I) = 1/2$ and $\rho_0(t) \neq 1$ if $-\infty < t < \infty$.

A further observation is that the L-orthogonality $\mathcal{N}^{(t)}[z^{-n+k}R_n(t; z)] = 0$, $k = 0, 1, \dots, n - 1$, holds even without the constant multiple $(1 + 4(t + \mathcal{I}m(I))^2)/(4d_1)$ in the definition (4.4) of $\mathcal{N}^{(t)}$. However, if we maintain this constant multiple then $\mathcal{N}^{(t)}[1] = 2d_1/(1 + ic_1(t))$ and the modified moments $\nu_n = \mathcal{N}^{(t)}[z^{-n}]$, $n = 0, \pm 1, \pm 2, \dots$, will be exactly the same as those that follow from Lemma 2.1.

5. Some applications

It is known that OPUC are completely characterized in terms of the Verblunsky coefficients as given by the following theorem (see, for example, [33, Thm. 1.7.11]).

Theorem. *Given an arbitrary sequence of complex numbers $\{\alpha_n\}_{n=0}^\infty$, where $|\alpha_n| < 1, n \geq 0$, then associated with this sequence there exists a unique nontrivial probability measure μ on the unit circle such that the monic polynomials $\{\Phi_n\}$ generated by (1.1) are the respective monic OPUC.*

A very nice and short constructive proof of the above theorem can be found in Erdélyi, Nevai, Zhang and Geronimo [16]. The knowledge that lead to this proof can be traced back to results found in [27,28] and references therein.

Many of the recent research on OPUC are about finding the properties of the measure μ from the associated Verblunsky coefficients.

We have seen from Theorems 4.1 and 4.2 how one can detect using the Verblunsky coefficients if the associated measure is such that the integral $J = \int_{\mathbb{T}} |\zeta - 1|^{-2} d\mu(\zeta)$ exists or not. The following theorem generalizes this to the existence of the integral $J(w) = \int_{\mathbb{T}} |\zeta - w|^{-2} d\mu(\zeta)$, where $|w| = 1$. Following the nomenclature adopted in the literature (see, for example, [19,41]) $J(w) = \int_{\mathbb{T}} |\zeta - w|^{-2} d\mu(\zeta)$ is known as the Geronimus transformation of the measure μ .

Theorem 5.1. *Given the sequence of complex numbers $\{\alpha_n\}_{n=0}^\infty$ such that $|\alpha_n| < 1, n \geq 0$, let μ be the positive measure on the unit circle for which α_n are its Verblunsky coefficients. Then for w satisfying $|w| = 1$, the measure μ is such that the integral $J(w) = \int_{\mathbb{T}} |\zeta - w|^{-2} d\mu(\zeta)$ exists if and only if there exists a unique ρ_0 ($|\rho_0| = 1, \rho_0 \neq w$) such that the sequence $\{d_{1,n}\}_{n=1}^\infty$ given by*

$$d_{1,n} = (1 - m_{n-1})m_n, \quad n \geq 1,$$

where

$$m_0 = 0 \quad \text{and} \quad m_n = \frac{1}{2} \frac{1 - |\alpha_{n-1}|^2}{1 + \operatorname{Re}(\overline{w}\rho_n\alpha_{n-1})}, \quad n \geq 1,$$

is not a single parameter positive chain sequence. Here,

$$\rho_n = w \frac{\rho_{n-1} - \overline{\alpha}_{n-1}}{1 - \rho_{n-1}\alpha_{n-1}}, \quad n \geq 1.$$

Proof. Observe that, we have

$$J(w) = \int_{\mathbb{T}} \frac{w\zeta}{(w - \zeta)(\zeta - w)} d\mu(\zeta) = \int_{\mathbb{T}} \frac{1}{|\zeta - 1|^2} d\mu(w\zeta).$$

It is known that the Verblunsky coefficients associated with the rotated measure $\mu(w\zeta)$ are $\{w^n\alpha_{n-1}\}_{n=0}^\infty$. Thus, from results given in Theorem 4.1, we need to look at the positive chain sequence $\{d_{1,n}\}_{n=1}^\infty$ given by

$$d_{1,n} = (1 - m_{n-1})m_n, \quad n \geq 1,$$

where $m_0 = 0$ and

$$m_n = \frac{1}{2} \frac{1 - |\rho_n w^n \alpha_{n-1}|^2}{1 + \operatorname{Re}(\rho_n w^n \alpha_{n-1})}, \quad \rho_n = \frac{\rho_{n-1} - w^{-n} \overline{\alpha}_{n-1}}{1 - \rho_{n-1} w^n \alpha_{n-1}}, \quad n \geq 1,$$

where ρ_0 is such that $|\rho_0| = 1, \rho_0 \neq 1$. Thus, the results of the theorem are established if we replace $\rho_n w^{n+1}$ by ρ_n . □

Observe that by Wall’s criteria for maximal parameter sequences (see the formula prior to Lemma 2.2), the statement “not a single parameter positive chain sequence” in the above theorem is equivalent to the convergence of the infinite series

$$\sum_{n=1}^\infty \prod_{k=1}^n \frac{m_k}{1 - m_k} = \sum_{n=1}^\infty \prod_{k=1}^n \frac{1 - |\alpha_{k-1}|^2}{|1 + \overline{w}\rho_k\alpha_{k-1}|^2}.$$

When the measure μ satisfy the symmetric $d\mu(e^{i\theta}) = -d\mu(e^{i(2\pi-\theta)})$ then the Verblunsky coefficients α_{n-1} are all real and the principal value integral $I = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1} d\mu(\zeta)$ always exists and takes the value 1/2. Consequently from Theorem 4.2, $\rho_n(0) = -1, n \geq 1$, and we can state the following.

Corollary 5.1.1. Given the sequence of real numbers $\{\alpha_n\}_{n=0}^\infty$, such that $|\alpha_n| < 1, n \geq 0$, let μ be the positive measure on the unit circle for which α_n are its Verblunsky coefficients. Then μ is such that integral $J = \int_{\mathbb{T}} \zeta(\zeta - 1)^{-1}(1 - \zeta)^{-1}d\mu(\zeta) = \int_{\mathbb{T}} |\zeta - 1|^{-2}d\mu(\zeta)$ exists if and only if the infinite series

$$\sum_{n=1}^\infty \prod_{k=1}^n \frac{1 + \alpha_{k-1}}{1 - \alpha_{k-1}},$$

is convergent.

We now see how one can obtain in a recursive manner information about the Verblunsky coefficients of the OPUC associated with the positive measure $\hat{\mu}$, given by $d\hat{\mu}(z) = z^{-1}(1 - z)(z - 1)d\mu(z)$, if one knows the Verblunsky coefficients of the OPUC associated with the measure μ .

Theorem 5.2. Let $\{\Phi_n\}$ be the sequence of monic OPUC with respect to the nontrivial measure μ . Let

$$c_n = \frac{-\mathcal{I}m(\rho_{n-1}^{(2)}\alpha_{n-1})}{1 - \mathcal{R}e(\rho_{n-1}^{(2)}\alpha_{n-1})} \quad \text{and} \quad g_{1,n-1} = \frac{1}{2} \frac{|1 - \rho_{n-1}^{(2)}\alpha_{n-1}|^2}{[1 - \mathcal{R}e(\rho_{n-1}^{(2)}\alpha_{n-1})]}, \quad n \geq 1, \tag{5.1}$$

where $\rho_0^{(2)} = 1$,

$$\alpha_{n-1} = -\overline{\Phi_n(0)} \quad \text{and} \quad \rho_n^{(2)} = \frac{\rho_{n-1}^{(2)} - \overline{\alpha_{n-1}}}{1 - \rho_{n-1}^{(2)}\alpha_{n-1}}, \quad n \geq 1.$$

Then the Verblunsky coefficients $\hat{\alpha}_{n-1} = -\overline{\hat{\Phi}_n(0)}$ associated with the measure $\hat{\mu}$, where

$$d\hat{\mu}(\zeta) = |\zeta - 1|^2d\mu(\zeta),$$

are given by

$$\hat{\alpha}_{n-1} = -\frac{1}{\rho_n^{(2)}} \frac{1 - 2m_n - ic_{n+1}}{1 - ic_{n+1}}, \quad n \geq 1, \tag{5.2}$$

where $m_0 = 0$ and $m_n = \frac{(1-g_{1,n-1})}{(1-m_{n-1})} g_{1,n}, n \geq 1$.

Proof. With the $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, where $d_1 = g_{1,0}$ and $d_{n+1} = (1 - g_{1,n-1})g_{1,n}, n \geq 1$, given by (5.1), let \mathcal{N} be the moment functional given by Lemma 2.1. Then from results given in Section 2 that

$$\mathcal{N}[\ell(z)] = a \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta)d\mu(\zeta),$$

for any Laurent polynomial ℓ . Here, a is some nonzero constant. Observe that in Section 2, \mathcal{N} is normalized such that if μ is a probability measure than $a = 1$.

On the other hand from results given in Section 3 we have the following. Associated with $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$ there exists a nontrivial probability measure $\hat{\mu}$ on the unit circle such that

$$\int_{\mathbb{T}} \ell(\zeta)d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} \mathcal{N}[\ell(z)(1 - z^{-1})],$$

for any Laurent polynomial ℓ . Moreover, the Verblunsky coefficients associated with the measure $\hat{\mu}$ are exactly as given in (5.2).

From the two representations for \mathcal{N} we have

$$\int_{\mathbb{T}} \ell(\zeta)d\hat{\mu}(\zeta) = a \frac{1 + c_1^2}{4d_1} \int_{\mathbb{T}} \ell(\zeta)(1 - \zeta^{-1})(1 - \zeta)d\mu(\zeta).$$

Thus, establishing the proof of the theorem. \square

To obtain information regarding the Verblunsky coefficients of the measure $|\zeta - w|^2d\mu(\zeta)$, where $|w| = 1$, we only need to replace $\alpha_{n-1}, n \geq 0$ in Theorem 5.2 by $w^n\alpha_{n-1}, n \geq 0$.

The above theorem is obtained by first using the results obtained in Section 2 and then followed by using the results given in Section 3. Now, if we start by using the results given in Sections 3 and 4 and then use the results given in Section 2 we obtain the following theorem.

Theorem 5.3. Let μ be a positive measure on the unit circle such that the integral $J = \int_{\mathbb{T}} \zeta(1-\zeta)^{-1}(\zeta-1)^{-1}d\mu(\zeta) = \int_{\mathbb{T}} |1-\zeta|^{-2}d\mu(\zeta)$ exists and let $\{\Phi_n\}$ be the associated sequence of monic OPUC. Let the sequence of numbers $\{\rho_n\}_{n=0}^\infty$ be such that

$$\rho_0 = -\frac{I}{I} \quad \text{and} \quad \rho_n = \frac{\rho_{n-1} - \overline{\alpha_{n-1}}}{1 - \rho_{n-1}\alpha_{n-1}}, \quad n \geq 1,$$

where $I = \int_{\mathbb{T}} \zeta(\zeta-1)^{-1}d\mu(\zeta)$ and $\alpha_{n-1} = -\overline{\Phi_n(0)}$, $n \geq 0$. Let the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_{1,n}\}_{n=1}^\infty$ be such that

$$c_1 = i\frac{\rho_0 + 1}{\rho_0 - 1} \quad \text{and} \quad c_{n+1} = \frac{-\text{Im}(\rho_n\alpha_{n-1})}{1 + \text{Re}(\rho_n\alpha_{n-1})}, \quad d_{1,n} = (1 - m_{n-1})m_n, \quad n \geq 1,$$

where $\{m_n\}_{n=0}^\infty$ is the minimal parameter sequence of the chain sequence $\{d_{n+1}\}_{n=1}^\infty$ given by

$$m_n = \frac{1}{2} \frac{1 - |\rho_n\alpha_{n-1}|^2}{1 + \text{Re}(\rho_n\alpha_{n-1})}, \quad n \geq 0.$$

Let $\{M_n\}_{n=1}^\infty$ be the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^\infty$. The existence of the integral J assures that $M_1 > 0$.

With $0 \leq M_0 < 1$, let

$$m_1 = (1 - M_0)M_1 \quad \text{and} \quad m_{n+1} = d_{n+1}/(1 - m_n), \quad n \geq 1.$$

Then

$$\tilde{\alpha}_{n-1} = -\frac{1}{\rho_{n-1}} \frac{1 - 2m_n - ic_n}{1 + ic_n}, \quad n \geq 1,$$

are the Verblunsky coefficients associated with the probability measure $\tilde{\mu}$ given by

$$\int_{\mathbb{T}} f(\zeta)d\tilde{\mu}(\zeta) = M_0f(1) + \frac{1 - M_0}{J} \int_{\mathbb{T}} f(\zeta) \frac{1}{|1 - \zeta|^2} d\mu(\zeta).$$

To obtain information about the Verblunsky coefficients associated with the probability measure $\tilde{\mu}(w; z)$ given by

$$\int_{\mathbb{T}} f(\zeta)d\tilde{\mu}(w; \zeta) = M_0f(w) + \frac{1 - M_0}{J(w)} \int_{\mathbb{T}} f(\zeta) \frac{1}{|w - \zeta|^2} d\mu(\zeta),$$

where $|w| = 1$, we only need to replace α_{n-1} , $n \geq 0$ in **Theorem 5.3** by $w^n\alpha_{n-1}$, $n \geq 0$.

In [19], some aspects of the orthogonal polynomials associated with the measure $|\zeta - w|^{-2}d\mu(\zeta)$ when $|w| > 1$ are considered.

6. Examples

Example 1. As a first example we consider the real sequence $\{c_n\}_{n=1}^\infty$ and the positive chain sequence $\{d_{n+1}\}_{n=1}^\infty$, where

$$c_1 = c, \quad c_{n+1} = 0, \quad n \geq 1, \quad d_2 = \frac{1}{2}, \quad d_{n+2} = \frac{1}{4}, \quad n \geq 1, \tag{6.1}$$

with c real. From (2.3) one can easily obtain that

$$R_n(z) = (1 + ic)z^n + (1 - ic) \quad \text{and} \quad Q_n(z) = 2d_1 \frac{z^n - 1}{z - 1}, \quad n \geq 1,$$

where $d_1 \neq 0$ is arbitrary. Thus, we can verify that

$$\begin{aligned} -\sum_{k=0}^\infty \frac{-2d_1}{1 - ic} z^k - \frac{Q_n(z)}{R_n(z)} &= \frac{\bar{\gamma}_n z^n}{\bar{r}_{n,n}} + O(z^{n+1}), \\ \sum_{k=1}^\infty \frac{2d_1}{1 + ic} z^{-k} - \frac{Q_n(z)}{R_n(z)} &= \frac{\gamma_n}{r_{n,n} z^{n+1}} + O((1/z)^{n+2}), \end{aligned} \quad n \geq 0,$$

with $\gamma_0 = \frac{2d_1}{1 + ic}$, $\gamma_n = \frac{4d_1}{1 + ic}$, $n \geq 1$, $r_{0,0} = 1$ and $r_{n,n} = 1 + ic$, $n \geq 1$.

It follows from **Lemma 2.1** that the moment functional \mathcal{N} associated with the sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ is given by

$$\mathcal{N}[z^{-n}] = \nu_n = \frac{-2d_1}{1 - ic}, \quad \mathcal{N}[z^{n-1}] = \nu_{-n+1} = \frac{2d_1}{1 + ic}, \quad n \geq 1.$$

Now for the polynomials \hat{A}_n defined in Section 3 we have

$$\hat{A}_{n-1}(z) = \frac{(1 + ic_1)}{2z} \left[R_n(z) + \frac{(1 - ic_1)}{2d_1} (z - 1) Q_n(z) \right] = (1 + ic)z^{n-1}, \quad n \geq 1.$$

Consequently, with the series expansions $\hat{F}_0(z) = -\sum_{n=0}^{\infty} \hat{\mu}_{n+1} z^n$ and $\hat{F}_{\infty}(z) = \sum_{n=1}^{\infty} \hat{\mu}_{-n+1} z^{-n}$, where $\hat{\mu}_0 = 1$ and $\hat{\mu}_n = 0$, $n \geq 1$, we can verify that

$$\hat{F}_0(z) - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = -\frac{(1 + c^2)\overline{\gamma}_n}{4d_1\overline{r}_{n,n}} z^{n-1} + O(z^n), \quad n \geq 1$$

and

$$\hat{F}_{\infty}(z) - \frac{\hat{A}_{n-1}(z)}{R_n(z)} = \frac{(1 + c^2)\gamma_n}{4d_1r_{n,n}} \frac{1}{z^{n+1}} + O((1/z)^{n+2}), \quad n \geq 1.$$

Since the zeros $z_{n,j}$ of R_n are the n^{th} roots of $(1 - ic)/(1 + ic)$, we also directly obtain that

$$\frac{\hat{A}_{n-1}(z)}{R_n(z)} = \sum_{j=1}^n \frac{1/n}{z - z_{n,j}}, \quad n \geq 1.$$

Observing that the zeros $z_{n,j}$, $j = 1, 2, \dots, n$, are of equal distance from each other in the unit circle, we conclude that the probability measure $\hat{\mu}$ given by Theorem 3.1 is $d\hat{\mu}(e^{i\theta}) = (2\pi)^{-1}d\theta$. Note that we can also write $d\hat{\mu}(\zeta) = (2\pi i\zeta)^{-1}d\zeta$.

The measure $d\hat{\mu}(\zeta) = (2\pi i\zeta)^{-1}d\zeta$ is well known (in Simon [33, p. 71] referred to as the free case) and the associated monic OPUC are $\Phi_n(z) = z^n$, $n \geq 0$.

Since the measure $d\hat{\mu}(\zeta) = (2\pi i\zeta)^{-1}d\zeta$ is also such that $d\hat{\mu}(e^{i(2\pi-\theta)}) = -d\hat{\mu}(e^{i\theta})$, we have $I = \int_{\mathbb{T}} (\zeta - 1)^{-1} \zeta d\hat{\mu}(\zeta) = 1/2$. Hence, from Theorem 4.2

$$\rho_n(t) = -\frac{1 + i2t}{1 - i2t}, \quad n \geq 0$$

and the polynomials $\{R_n(t; \cdot)\}$ given by

$$R_n(t; z) \prod_{k=0}^{n-1} \frac{1 + \mathcal{R}e(\rho_k(t)\alpha_{k-1})}{1 + \rho_k(t)\alpha_{k-1}} = z\Phi_{n-1}(z) - \rho_{n-1}(t)\Phi_{n-1}^*(z), \quad n \geq 1,$$

satisfy the three term recurrence formula

$$R_{n+1}(t; z) = [(1 + ic_{n+1}(t))z + (1 - ic_{n+1}(t))]R_n(t; z) - 4d_{n+1}(t)zR_{n-1}(t; z),$$

with $R_0(t; z) = 1$ and $R_1(t; z) = (1 + ic_1(t))z + (1 - ic_1(t))$, where the real sequences $\{c_n(t)\}$ and $\{d_{n+1}(t)\}$ are such that

$$c_1(t) = -2t, \quad c_{n+1}(t) = 0, \quad n \geq 1, \quad d_2(t) = \frac{1}{2}, \quad d_{n+2}(t) = \frac{1}{4}, \quad n \geq 1.$$

The L-orthogonality property $\tilde{\mathcal{N}}^{(t)}[z^{-n+k}R_n(t; z)] = 0$, $0 \leq k \leq n - 1$, holds with the moment functional $\tilde{\mathcal{N}}^{(t)}$ given by

$$\tilde{\mathcal{N}}^{(t)}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{\zeta}{\zeta - 1} \frac{1}{2\pi i\zeta} d\zeta + it\ell(1),$$

for any Laurent polynomial ℓ .

Hence, the polynomials R_n that follow from (6.1) satisfy the L-orthogonality

$$\tilde{\mathcal{N}}^{(-c/2)}[z^{-n+k}R_n(z)] = 0, \quad 0 \leq k \leq n - 1$$

and

$$v_n = \mathcal{N}[z^{-n}] = \frac{4d_1}{1 + c^2} \tilde{\mathcal{N}}^{(-c/2)}[z^{-n}], \quad n = 0, \pm 1, \pm 2, \dots$$

Since the integral $J = \int_{\mathbb{T}} \zeta(1 - \zeta)^{-1}(\zeta - 1)^{-1}d\hat{\mu}(\zeta)$ does not exist, the sequence $\{d_{n+1}(t)\}_{n=0}^{\infty}$ is a single parameter positive chain sequence for all t , as confirmed by Theorem 4.2.

Example 2. Here we start with the probability measure μ given by

$$\int_{\mathbb{T}} \ell(\zeta) d\mu(\zeta) = \int_{\mathbb{T}} \ell(\zeta) \frac{(1-\zeta)(\zeta-1)}{4\pi i \zeta^2} d\zeta,$$

and derive the polynomials $R_n(t; \cdot)$ and their L-orthogonality property given by [Theorem 4.2](#). Again, many information regarding the OPUC associated with the measure μ are well known.

For example, the associated Verblunsky coefficients are

$$\alpha_{n-1} = -\overline{\Phi_n(0)} = -\frac{1}{n+1}, \quad n \geq 1.$$

Clearly the integral $J = \int_{\mathbb{T}} (1-\zeta)^{-1}(\zeta-1)^{-1} \zeta d\mu(\zeta)$ exists and that it is easy to show that

$$I = \int_{\mathbb{T}} \frac{\zeta}{\zeta-1} d\mu(\zeta) = \int_{\mathbb{T}} \frac{\zeta}{\zeta-1} d\mu(\zeta) = \frac{1}{2}.$$

We also verify by induction that the sequence $\{\rho_n(t)\}$ generated by

$$\rho_0(t) = -\frac{I+it}{I-it} \quad \text{and} \quad \rho_n(t) = \frac{\rho_{n-1}(t) - \overline{\alpha_{n-1}}}{1 - \rho_{n-1}(t)\alpha_{n-1}}, \quad n \geq 1,$$

can be explicitly given as

$$\rho_n(t) = -\frac{1+i(n+1)(n+2)t}{1-i(n+1)(n+2)t}, \quad n \geq 0.$$

With this we obtain

$$c_{n+1}(t) = -\frac{2(n+1)t}{1+n(n+1)^2t^2} \quad \text{and} \quad m_n(t) = \frac{n}{2(n+1)} \frac{1+(n+1)^2(n+2)^2t^2}{1+n(n+1)^2(n+2)t^2}, \quad n \geq 0.$$

For the sequence of polynomials $\{R_n(t; \cdot)\}$ given by

$$R_{n+1}(t; z) = [(1+ic_{n+1}(t))z + (1-ic_{n+1}(t))]R_n(t; z) - 4d_{n+1}(t)zR_{n-1}(t; z),$$

with $R_0(t; z) = 1$ and $R_1(t; z) = (1+ic_1(t))z + (1-ic_1(t))$, the L-orthogonality property $\tilde{\mathcal{N}}^{(t)}[z^{-n+k}R_n(t; z)] = 0$, $k = 0, 1, \dots, n-1$, holds with respect to the moment functional

$$\tilde{\mathcal{N}}^{(t)}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{1-\zeta}{4\pi i \zeta} d\zeta + it \ell(1).$$

Since,

$$1 - m_n(t) = \frac{n+2}{2(n+1)} \frac{1+n^2(n+1)^2t^2}{1+n(n+1)^2(n+2)t^2}, \quad n \geq 1,$$

we have

$$\sum_{n=1}^{\infty} \prod_{k=1}^n \frac{m_k(t)}{1-m_k(t)} = \frac{2}{1+4t^2} \sum_{n=1}^{\infty} \left[\frac{1}{(n+1)(n+2)} + (n+1)(n+2)t^2 \right].$$

The above infinite series converge if $t = 0$ and diverge otherwise. Hence, using Wall's criteria (see the formula prior to [Lemma 2.2](#)), the positive chain sequence $\{d_{n+1}(t)\} = \{d_{1,n}(t)\}$, where

$$d_{1,n}(t) = (1 - m_{n-1}(t))m_n(t), \quad n \geq 1,$$

is not a single parameter positive chain sequence if $t = 0$ and is a single parameter positive chain sequence otherwise. Again, confirming the results in [Theorem 4.2](#).

Example 3. Let the real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_{n+1}\}_{n=1}^{\infty}$ be given by

$$c_n = \frac{\eta}{\lambda+n}, \quad d_{n+1} = \frac{1}{4} \frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)}, \quad n \geq 1,$$

where $\lambda, \eta \in \mathbb{R}$ and $\lambda > -1$.

Observe that $d_{n+1} = d_{1,n} = (1 - m_{n-1})m_n$, $n \geq 1$, where

$$m_n = \frac{n}{2(\lambda + n + 1)}, \quad n \geq 0. \quad (6.2)$$

Hence, with our assumption $\lambda > -1$, the sequence $\{d_{1,n}\}_{n=1}^{\infty}$ is a positive chain sequence with $\{m_n\}_{n=0}^{\infty}$ as its minimal parameter sequence.

The polynomials R_n obtained from the above sequences $\{c_n\}$ and $\{d_{n+1}\}$, together with the recurrence formula (2.3), are

$$R_n(z) = \frac{(2\lambda + 2)_n}{(\lambda + 1)_n} {}_2F_1(-n, b + 1; b + \bar{b} + 2; 1 - z), \quad n \geq 1, \quad (6.3)$$

where $b = \lambda + i\eta$. Here, ${}_2F_1(-n, b; c; z)$ represents a Gaussian hypergeometric polynomial of degree n in z . More about the properties of such polynomials and about general Gaussian hypergeometric functions ${}_2F_1(a, b; c; z)$, we cite [1].

With $d_1 \neq 0$ arbitrary, by considering the Gauss type continued fraction expansion of ${}_2F_1(1, -b; \bar{b} + 2; z)$ (see [37, p. 4264] for a very similar expansion), we have for the rational functions in Lemma 2.1 that correspond to the above $\{c_n\}$ and $\{d_{n+1}\}$,

$$\frac{2d_1}{1 - ic_1} {}_2F_1(1, -b; \bar{b} + 2; z) = \frac{Q_n(z)}{R_n(z)} + O(z^{n+1}).$$

Hence, from the series expansion of ${}_2F_1(1, -b; \bar{b} + 2; z)$

$$v_n = d_1 \frac{b + \bar{b} + 2}{b + 1} \frac{(-b - 1)_n}{(\bar{b} + 1)_n}, \quad n \geq 1. \quad (6.4)$$

Since $v_n = -\bar{v}_{-n+1}$, with the convention $(a)_n = \Gamma(a + n)/\Gamma(a)$ for all integer values of n , the above expression for v_n is also valid for $n \leq 0$.

For the moment functional \mathcal{N} defined by $\mathcal{N}[\zeta^{-n}] = v_n$, $n = 0, \pm 1, \pm 2, \dots$, we have

$$\mathcal{N}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) d\psi(b; \zeta), \quad (6.5)$$

for all Laurent polynomials ℓ , where $d\psi(b; z) = d_1 \frac{i|\Gamma(b + 1)|^2(b + \bar{b} + 2)}{2\pi \Gamma(b + \bar{b} + 2)} (-z)^{-\bar{b}-1} (1 - z)^{b + \bar{b} + 1} dz$.

Here, the branch cuts in $(-z)^{-\bar{b}} = (e^{-i\pi z})^{-\bar{b}}$ and $(1 - z)^{b + \bar{b}} = (e^{-i\pi(z-1)})^{b + \bar{b}}$ are along the positive real axis.

For the proofs of the above results, starting from (6.3), we cite for example [8] and [10]. However, considering what was necessary, it was assumed in [8] and [10] that $\lambda > -1/2$. We remark that when $\lambda > -1/2$ the maximal parameter sequence $\{M_{1,n}\}_{n=0}^{\infty}$ of $\{d_{1,n}\}_{n=1}^{\infty}$ is

$$M_{1,n} = \frac{1}{2} \frac{2\lambda + n + 1}{\lambda + n + 1}, \quad n \geq 0.$$

Now, the nontrivial probability measure $\hat{\mu}$ that follows from Theorem 3.1 is

$$d\hat{\mu}(\zeta) = \frac{|\Gamma(b + 2)|^2}{2\pi \Gamma(b + \bar{b} + 3)} (-z)^{-\bar{b}-2} (1 - z)^{b + \bar{b} + 2} dz. \quad (6.6)$$

Observe that (6.6) can also be written in the equivalent form

$$d\hat{\mu}(e^{i\theta}) = \frac{2^{b + \bar{b} + 2} |\Gamma(b + 2)|^2}{2\pi \Gamma(b + \bar{b} + 3)} e^{(\pi - \theta)\Im m(b + 1)} [\sin^2(\theta/2)]^{\Re e(b + 1)} d\theta.$$

The moments associated with $\hat{\mu}$ are

$$\bar{\mu}_{-n} = \mu_n = \int_{\mathbb{T}} \zeta^{-n} d\hat{\mu}(\zeta) = \frac{1 + c_1^2}{4d_1} [v_n - v_{n+1}] = \frac{(-b - 1)_n}{(\bar{b} + 2)_n}, \quad n = 0, 1, 2, \dots$$

Hence, from results established in [37] the associated monic OPUC and Verblunsky coefficients are

$$\hat{\Phi}_n(z) = \frac{(b + \bar{b} + 3)_n}{(b + 2)_n} {}_2F_1(-n, b + 2; b + \bar{b} + 3; 1 - z),$$

$$\hat{\alpha}_{n-1} = -\frac{(b + 1)_n}{(\bar{b} + 2)_n}, \quad n \geq 1.$$

The above expression for $\hat{\Phi}_n$ can also be easily derived from (3.7) and (6.3).

It is important to clarify that the above hypergeometric polynomials R_n and $\hat{\Phi}_n$ are polynomials belonging to a family of biorthogonal polynomials mentioned in the comments of Prof. Richard Askey on the Gabor Szegő: Collected Papers [2, p. 304]. However, that $\hat{\Phi}_n$ are OPUC with respect to the nontrivial probability measure in (6.6) and also that R_n are para-orthogonal polynomials were first observed in [37]. See also the comments given at the end of [37].

Recall that we have assumed $\lambda > -1$ and for the integral $I = \int_{\mathbb{T}} (\zeta - 1)^{-1} \zeta d\hat{\mu}(\zeta)$ we have from (6.4), (6.5) and (6.6) that

$$I = \frac{\bar{b} + 1}{b + \bar{b} + 2}.$$

Let t be such that $-\infty < t < \infty$ and let the sequence $\{\rho_n(t)\}_{n=0}^\infty$, with the property $|\rho_n(t)| = 1$, be given by

$$\rho_0(t) = -\frac{I + it}{\bar{I} - it} \quad \text{and} \quad \rho_n(t) = \frac{\rho_{n-1}(t) - \bar{\alpha}_{n-1}}{1 - \rho_{n-1}(t)\alpha_{n-1}}, \quad n \geq 1, \tag{6.7}$$

where $\alpha_{n-1} = \hat{\alpha}_{n-1} = -\overline{\hat{\Phi}_n(0)}$, $n \geq 0$. Then from Theorem 4.2 for the sequence $\{R_n(t; \cdot)\}$ of POPUC, in z , given by

$$R_n(t; z) \prod_{k=0}^{n-1} \frac{1 + \mathcal{R}e(\rho_k(t)\alpha_{k-1})}{1 + \rho_k(t)\alpha_{k-1}} = z\hat{\Phi}_{n-1}(z) - \rho_{n-1}(t)\hat{\Phi}_{n-1}^*(z), \quad n \geq 1,$$

the following three term recurrence formula holds.

$$R_{n+1}(t; z) = [(1 + ic_{n+1}(t))z + (1 - ic_{n+1}(t))]R_n(t; z) - 4d_{n+1}(t)zR_{n-1}(t; z),$$

with $R_0(t; z) = 1$ and $R_1(t; z) = (1 + ic_1(t))z + (1 - ic_1(t))$, where the real sequences $\{c_n(t)\}$ and $\{d_{n+1}(t)\}$ are such that

$$c_1(t) = i\frac{\rho_0(t) + 1}{\rho_0(t) - 1} \quad \text{and} \quad c_{n+1}(t) = \frac{\mathcal{I}m(\rho_n(t)\alpha_{n-1})}{1 - \mathcal{R}e(\rho_n(t)\alpha_{n-1})}, \quad d_{n+1}(t) = (1 - m_{n-1}(t))m_n(t), \quad n \geq 1,$$

where $\{m_n(t)\}_{n=0}^\infty$ is the minimal parameter sequence of the chain sequence $\{d_{n+1}(t)\}_{n=1}^\infty$, which is given by

$$m_n(t) = \frac{1}{2} \frac{1 - |\rho_n(t)\alpha_{n-1}|^2}{1 + \mathcal{R}e(\rho_n(t)\alpha_{n-1})}, \quad n \geq 0.$$

With respect to the measure μ , the polynomials $R_n(t; \cdot)$ satisfy the L-orthogonality $\mathcal{N}^{(t)}[z^{-n+k}R_n(t; z)] = 0$, $k = 0, 1, \dots, n - 1$, with respect to the moment functional $\mathcal{N}^{(t)}$ given by

$$\frac{1 + 4(t + \mathcal{I}m(I))^2}{4d_1} \mathcal{N}^{(t)}[\ell(z)] = \int_{\mathbb{T}} \ell(\zeta) \frac{\zeta}{\zeta - 1} d\mu(\zeta) + it\ell(1).$$

Clearly, the sequence of polynomials $\{R_n(0; \cdot)\}$ and the chain sequence $\{d_{n+1}(0)\}_{n=1}^\infty$ obtained above are respectively the same as the sequences $\{R_n\}$ and $\{d_{n+1}\}_{n=1}^\infty$. In fact, from (6.7) we can also easily verify that $\rho_n(0) = -(\bar{b} + 1)_{n+1}/(b + 1)_{n+1}$, $n \geq 0$.

Not that when $\lambda > -1/2$ the maximal parameter sequence $\{M_{1,n}\}_{n=0}^\infty$ of $\{d_{n+1}(0)\}_{n=1}^\infty$ is different from its minimal parameter sequence $\{m_n\}_{n=0}^\infty$. Since the integral $J = \int_{\mathbb{T}} \zeta(1 - \zeta)^{-1}(\zeta - 1)^{-1}d\hat{\mu}(\zeta)$ exists when $\lambda > -1/2$, this is exactly what we expect from Theorem 4.2. However, when $-1/2 \geq \lambda > -1$, the integral $J = \int_{\mathbb{T}} \zeta(1 - \zeta)^{-1}(\zeta - 1)^{-1}d\hat{\mu}(\zeta)$ does not exist and, hence from Theorem 4.2, the minimal parameter sequence of $\{d_{n+1}(0)\}_{n=1}^\infty$ is also its maximal parameter sequence. This we can verify as follows.

From (6.2) we have

$$\sum_{n=1}^\infty \left[\prod_{k=1}^n \frac{m_k(0)}{1 - m_k(0)} \right] = \sum_{n=1}^\infty \left[\prod_{k=1}^n \frac{m_k}{1 - m_k} \right] = \sum_{n=1}^\infty \frac{(1)_n}{(2\lambda + 3)_n}.$$

The infinite series on the right hand side can also be written as ${}_2F_1(1, 1; 2\lambda + 3; 1)$, which is (see [1, p. 62]) convergent for $\lambda > -1/2$ and divergent for $-1/2 \geq \lambda > -1$.

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