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Persistence of periodic orbits from planar piecewise linear Hamiltonian differential systems with two or three zones

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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A minha mãe Sônia;

A minha família e amigos;

A meus mestres;

Dedico.

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"Feliz aquele que transfere o que sabe e aprende o que ensina."

(Cora Coralina, 1983, p.136)

Resumo

Neste trabalho, nosso objetivo é estimar o número de ciclos limites do tipo costura em sistemas diferenciais Hamiltonianos lineares por partes planares com duas ou três zonas separadas por retas de modo que os sistemas lineares que definem o por partes têm pontos singulares isolados, ou seja, centros ou selas. Mais precisamente, começaremos determinando o número de ciclos limites de sistemas diferenciais Hamiltonianos lineares por partes contínuos ou descontínuos com duas ou três zonas. Neste caso, mostraremos que se o sistema for descontínuo com três zonas, então ele tem no máximo um ciclo limite, e forneceremos exemplos com um ciclo limite. Em seguida, estimaremos o número de ciclos limites que podem bifurcar de um anel de órbitas periódicas de um sistema diferencial Hamiltoniano linear descontínuo por partes com três zonas, após perturbações polinomiais de grau n, para $n = 1, 2, 3$. Para estes casos, denotando por $H(n)$ o número de ciclos limites que podem bifurcar do anel de órbitas periódicas do sistema, provaremos que se o sistema diferencial linear definido na região entre as duas retas paralelas (chamado de subsistema central) possui um centro na origem e os demais subsistemas possuem centros ou selas, então $H(1) \geq 3$, $H(2) \geq 4$ e $H(3) \geq 7$. Agora, para o caso particular em que o subsistema central possui um centro e os demais subsistemas possuem apenas selas reais, se o centro for real (não necessariamente na origem) ou se estiver sobre a fronteira da região central, então $H(1) \geq 6$, e se for virtual, então $H(1) \geq 4$. Finalmente, se o subsistema central possui uma sela real e os demais subsistemas possuem centros ou selas, então $H(1) \geq 6$. Para isso, estudaremos o número de zeros de suas funções de Melnikov definidas em duas e três zonas. Além disso, fornecemos métodos analíticos detalhados para estudar o número de zeros das funções de Melnikov.

Palavras-chave: Ciclo limite. Sistema diferencial linear por partes. Sistema Hamiltoniano. Função de Melnikov.

Abstract

In this work, our goal is estimate the number of crossing limit cycles in planar piecewise linear Hamiltonian differential systems with two or three zones separated by straight lines such that the linear systems that define the piecewise one have isolated singular points, that is, centers or saddles. More precisely, we will start with the study of the number of limit cycles for continuous or discontinuous piecewise linear Hamiltonian differential systems with two or three zones. In this case, we show that if the system is discontinuous with three zones then it has at most one limit cycle, and we will provide examples with one limit cycle. Next, we will estimate the number of limit cycles that can bifurcate from a periodic annulus in a discontinuous piecewise linear Hamiltonian differential system with three zones, after polynomials functions perturbations of degree n*, for* $n = 1, 2, 3$*. For theses cases, denoting by* $H(n)$ *the number of limit cycles that can bifurcate from this periodic annulus, we prove that if the linear differential system defined in the region between the two parallel lines (called of central subsystem) has a center at the origin and the others subsystems have centers or saddles then* $H(1) \geq 3$*,* $H(2) \geq 4$ *and* $H(3) \geq 7$*. Now, for the particular case where the central subsystem has a center and the others subsystems have only real saddles, if the central subsystem has a real (not necessarily at the origin) or boundary center then* $H(1) \geq 6$ *and if it has a virtual center then* $H(1) \geq 4$*. Finally, if the central subsystem has a real saddle and the others subsystems have centers or saddles then* $H(1) \geq 6$*. For this, we study the number of zeros of its Melnikov functions defined in two or three zones. Moreover, we provide detailed analytical methods to study the number of zeros from Melnikov functions defined in two or three zones.*

Keywords: Limit cycle. Piecewise linear differential system. Hamiltonian system. Melnikov function.

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Summary

Introduction

The qualitative theory of differential systems is important tools for investigate properties inherent to the solutions of the equations that define the problem, without worrying about the possible expressions that such solutions may have. One of the most famous problems of this theory, formulated for the planar case, consist in to determine the number of limit cycles of polynomial differential systems, which was proposed by Hilbert in 1900 as part of Hilbert's 16th problem, (see [25]). Nowadays, this problem has been considered for piecewise differential systems. The study of piecewise differential systems begins with the pioneering work of Andronov [1]. After Filippov [12] establishes the conventions that regulate the transitions of solutions of systems between different regions, piecewise differential systems have piqued the attention of researchers in qualitative theory of differential equations, mainly because many phenomena, for instance in mechanics, electrical circuits, control theory, neurobiology, etc, can be described by models that involve this kind of differential equations (see the books [10, 23, 48] and the papers [7, 13, 45, 47]).

Piecewise linear differential systems are an interesting class of piecewise differential systems and, unlike the smooth case, have a rich dynamic that is far from being fully understood. In addition to numerous applications in various areas of knowledge. In 1990, Lum and Chua [8] conjectured that a continuous piecewise differential systems in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [15]. On the other hand, when the piecewise linear differential system is discontinuous, that is, the subsystems do not coincide on the switching curve, is known that the maximum number of limit cycles is at least three. Apparently, for it case, determining the exact number of limit cycles is a hard task. However, important partial results about this problem have been obtained. In summary, the results about the number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line are given in Table . The symbol "—" indicates that those cases appear repeated in the table and the empty entries on it correspond to cases not studied in the literature, at least as far as we know.

	F_r	F_v	F_b	S_r	S_r^0	N_v	iN_v	C	C_b
F_r		3	2^*	3	2^*	3	3	2^*	2^*
F_v	—	2	2^*	2	1^*		2	1^*	1^*
F_b	—	—	1^*	2^*	1^*	2^*	2^*	1^*	1^*
S_r	—	—	—	2^*	1^*	2	2	1^*	1^*
S_r^0	—	—	—	—	0^*	1^*	1^*	0^*	0^*
N_v	—	—	—	—	—			1^*	1^*
iN_v	—	—	—	—	—	—	2	1^*	1^*
C	—	—	—	—	—	—	—	0^*	0^*
C_b	—	—	—	—	—	—	—	—	0^*

Table 1: Lower bounds (Upper bounds*) of the maximum number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line. Here F_r, F_v, F_b, S_r, S_r^0, N_v, iN_v, C and C_b denote real focus, virtual focus, boundary focus, real saddle, real saddle with zero trace, virtual node, improper node, center and boundary center, respectively.

We denote the lower bounds in the entrances from Table by the symbols that indicate its position on the table. For example, the lower bound for the case with a real focus F_r and a virtual focus F_v is detonated by F_rF_v, that is, $F_rF_v = 3$. A proof for the lower bound F_rF_v can be found in [32]. A proof for the lower bound F_rS_r can be found in [29]. A proof for the lower bounds F_rN_v and F_riN_v can be found in [17]. A proof for the lower bound F_vF_v can be found in [16]. A proof for the lower bound F_vS_r can be found in [55]. A proof for the lower bound F_viN_v can be found in [56]. A proof for the upper bound S_rS_r can be found in [2]. A proof for the lower bounds S_rN_v and S_riN_v can be found in [33]. A proof for the lower bound iN_viN_v can be found in [27]. The other cases listed in Table can be found in [35]. In the papers [3, 5, 18, 41, 54] we can also find proofs for some lower bounds of Table .

If the curve between two linear zones is not a straight line it is possible to obtain as many cycles as you want. This fact has been conjectured by Braga and Mello in [4] and firstly proved by Novaes and Ponce in [49]. Exact number of limit cycles, for discontinuous piecewise linear systems with two zones separated by a straight line, were obtained in particular cases. Llibre and Teixeira [39] proved that if the linear systems, that define the piecewise one, has no singular point, then it has at most one limit cycle. Medrado and Torregrossa [46] proved that if the straight line has only crossing sewing points and the piecewise linear system has only a monodromic singular point on it, then the system has at most one limit cycle.

Recently, piecewise differential system defined in regions with more than two zones have attracted the attention of researchers, see for instance [11, 26, 36, 53, 60]. Results imposing restrictive hypotheses on the systems, such as symmetry and linearity, have been obtained.

For instance, conditions for nonexistence and existence of one, two or three limit cycles for symmetric continuous piecewise linear differential systems with three zones can be found in [34]. Now, for the nonsymmetric case, examples with two limit cycles surrounding a unique singular point at the origin was found in [38, 42]. When we remove these restrictions, that is, when we consider piecewise discontinuous differential system with three zones, there are few works in the literature. In fact, the works available deal with planar piecewise linear near-Hamiltonian differential systems with three zones, given by

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \epsilon g(x,y), \end{cases} \tag{1}$$

with

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, \ x \leq -1, \\ H^C(x,y) = \dfrac{b_C}{2}y^2 - \dfrac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, \ -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, \ x \geq 1, \end{cases}$$

$$f(x,y) = \begin{cases} f_L(x,y), & x \leq -1, \\ f_C(x,y), & -1 \leq x \leq 1, \\ f_R(x,y), & x \geq 1, \end{cases}$$

$$g(x,y) = \begin{cases} g_L(x,y), & x \leq -1, \\ g_C(x,y), & -1 \leq x \leq 1, \\ g_R(x,y), & x \geq 1, \end{cases}$$

where the functions f_i and g_i are C^∞, for $i = L, C, R$, and $0 \leq \epsilon << 1$. When $\epsilon = 0$ (shortly $(1)|_{\epsilon=0}$) we say that system (1) is a piecewise linear Hamiltonian differential system. We call system (1) of *left subsystem* for $x \leq -1$, *right subsystem* for $x \geq 1$ and *central subsystem* for $-1 \leq x \leq 1$.

In [40], Llibre and Teixeira study the existence of limit cycles for system $(1)|_{\epsilon=0}$ when the subsystems that define the piecewise one have a unique singular point, which is a center. More precisely, in the continuous case, they prove that system $(1)|_{\epsilon=0}$ has no limit cycles. Now, in the discontinuous case, system $(1)|_{\epsilon=0}$ has at most one limit cycle and there are examples with one. Fonseca, Llibre and Mello, in [14], proved that if system $(1)|_{\epsilon=0}$ is discontinuous and without singular points, then the maximum number of limit cycles is also one. Many authors have contributed to estimate the number of limit cycles of system (1) that can bifurcate, after polynomial perturbations, from the periodic annulus of the unperturbed system $(1)|_{\epsilon=0}$. For instance, in [59] the authors showed that at least seven limit cycles can bifurcate from a periodic annulus, after linear perturbations, from system $(1)|_{\epsilon=0}$ with subsystems without singular points and a boundary pseudo–focus. On the other hand, in [61], 5, 7 and 12 limit cycles were obtained by linear, quadratic and cubic polynomials perturbations, respectively. But, in this paper, the periodic annulus of the unperturbed system $(1)|_{\epsilon=0}$ was obtained from a real saddle in the central subsystem and two virtual centers in the others subsystems. For the same type of periodic annulus, in [58], 10 limit cycles were obtained through cubic perturbations.

The search for upper bound for the number of limit cycles that a piecewise linear system with three zones can have is what motivates most of the recently works found in the literature about this topic. However, all cases are interesting in themselves, that is, the search for better quotas for the number of limit cycles can not be used to discourage the study of particular families. So the type of singular points of the subsystems and their positions, that is, whether they are real, virtual or boundary, are important questions in this study and should be considered for all subclasses of piecewise linear systems with three zones. Motivated by the previously mentioned works, in this thesis we contribute along these study lines. Our goal is to estimate the number of crossing limit cycles that can bifurcate from periodic annulus of piecewise linear near-Hamiltonian differential systems with two or three zones. For this purpose, we have the text organized as follows.

In Chapter 1, we introduce some definitions and basic results about piecewise smooth vector fields with two or three zones that will be essential in the development of this work. Moreover, in Section 1.3 we will present the Melnikov function method, that is an important tool to investigate the number of limit cycles that emerge from a periodic annulus of a piecewise Hamiltonian differential system.

In Chapter 2, motivated by the paper [40], we study the existence of crossing limit cycles in continuous and discontinuous planar piecewise linear Hamiltonian differential system with two or three zones separated by straight lines, and parallel if there are more than one, such that the linear systems that define the piecewise one have isolated singular points, that is, centers or saddles. In this case, we show that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones or continuous with three zones, then it has no crossing limit cycles. Now, if the planar piecewise linear Hamiltonian differential system is discontinuous with three zones, that is, if system $(1)|_{\epsilon=0}$ is discontinuous, then it has at most one crossing limit cycle, and there are examples with one limit cycle. More precisely, without taking into account the position of the singular points in the zones, we present examples with the unique crossing limit cycle for all possible combinations of saddles and centers. This chapter is the content of paper I of the list of published and submitted papers derived from this thesis available at the end of this introduction.

In Chapter 3, we study the number of crossing limit cycles that can bifurcated, after linear perturbations, from a periodic annulus formed by periodic orbits of system $(1)|_{\epsilon=0}$ that passes through the three zones. We prove that if the central subsystem has a center at the origin and the others subsystems have centers or saddles, then we have at least three crossing limit cycles visiting the three zones. Our results are obtained by studying the number of zeros of the first order Melnikov function associated to system (1). Moreover, we will present, in Section 1, a normal form for system $(1)|_{\epsilon=0}$ in order to simplify the compute. This normal form will also be used in Chapters 4 and 5. This study resulted in paper II of the list of published and submitted papers derived from this thesis.

In Chapter 4, motivated by the results obtained in Chapter 3, we estimated the lower bounds for the number of crossing limit cycles that can bifurcate from periodic annulus of piecewise system (1), assuming similar hypotheses to system $(1)|_{\epsilon=0}$ from Chapter 3, but with the difference that we perturb system $(1)|_{\epsilon=0}$ by polynomial functions of degree n, for $n = 2, 3$. Denoting by $H(n)$ the number of crossing limit cycles that can bifurcate from this periodic annulus, we prove that $H(2) \geq 4$ and $H(3) \geq 7$. A part of these results are published in the paper III of the list of published and submitted papers derived from this thesis.

In Chapter 5, we study the number of crossing limit cycles that can bifurcated, after linear perturbations, from periodic annulus of system $(1)|_{\epsilon=0}$, assuming that the central subsystem has a center and the other subsystems have only saddles. What differs the results of this chapter from the one of Chapter 3 is that now the center of the central subsystem is not fixed at the origin and therefore, we will have periodic annulus formed by crossing periodic orbits passing through two and three zones. We prove that if the central subsystem from system $(1)|_{\epsilon=0}$ has a real or a boundary center, then the maximum number of limit cycles that bifurcate from the periodic annulus is at least six. Four passing through the three zones and two passing through two zones. Now, if the central subsystem has a virtual center, then we have at least four limit cycles bifurcating from the periodic annulus, three passing through the three zones and one passing through two zones. This study resulted in preprint IV of the list of published and submitted papers derived from this thesis.

Finally, in Chapter 6, we study the number of crossing limit cycles that can bifurcated from periodic annulus of system $(1)|_{\epsilon=0}$, assuming that the central subsystem has a real saddle and the other subsystems have centers or saddles. We prove that the maximum number of limit cycles that can bifurcate from the periodic annulus of this kind of piecewise linear Hamiltonian differential systems, after linear perturbations, is at least five, when the right and left subsystems have saddles, and at least six, when the right subsystem has a saddle or center and the left subsystem has a center (vice-versa). Our study is concentrated in the neighborhood of the double homoclinic loop which separates the periodic annulus of orbits that pass through three zones from the two periodic annulus of orbits that pass through only two zones. Thus, to estimate the zeros of the Melnikov functions we consider its expansions at the point corresponding to this orbit. Moreover, in order to simplifies the computation, we obtain a normal form to system $(1)|_{\epsilon=0}$ when the central subsystem has a real saddle. This study resulted in preprint V of the list of published and submitted papers derived from this thesis.

Published and submitted papers derived from this thesis

I. Pessoa, C., Ribeiro, R.: Limit cycles of planar piecewise linear Hamiltonian differential systems with two or three zones, Electron. J. Qual. Theory Differ. Equ., **27**, 1-19 (2022). doi:10.14232/ejqtde.2022.1.27

II. Pessoa, C., Ribeiro, R.: Limit cycles bifurcating from a periodic annulus in discontinuous planar piecewise linear Hamiltonian differential system with three zones, Int. J. Bifur. Chaos Appl. Sci. Eng., **32**, 2250114, 16 pp (2022). doi:10.1142/S0218127422501140

III. Pessoa, C., Ribeiro, R.: Persistence of periodic solutions from discontinuous planar piecewise linear Hamiltonian differential systems with three zones, São Paulo J. Math. Sci., **16**, 932-956 (2022). doi:10.1007/s40863-022-00313-z

IV. Pessoa, C., Ribeiro, R.: Bifurcation of limit cycles from a periodic annulus formed by a center and two saddles in piecewise linear differential system with three zones, Preprint (2022). Available in doi:10.48550/arXiv.2207.05177

V. Euzébio, R., Gouveia, M., Novaes, D., Pessoa, C., Ribeiro, R.: On cyclicity in discontinuous piecewise linear near-Hamiltonian differential systems with three zones having a saddle in the central one, Preprint (2022). Available in doi:10.48550/arXiv.2212.00828

CHAPTER

1

Basic results

In this chapter, we present the definitions, notations and some basic results about piecewise smooth vector fields with two or three zones that will be used throughout this work. Moreover, we present a method, that is the first order Melnikov function, to investigate the number of limit cycles that can emerging from a period annulus of a piecewise Hamiltonian differential system.

1.1 Piecewise smooth vector fields with two or three zones

Let us consider a codimension one manifolds $\Sigma_i \subset \mathbb{R}^2$, for $i = C, L, R$, given by $\Sigma_i = f_i^{-1}(0)$, where $f_i : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function such that $0 \in \mathbb{R}$ is a regular value, i.e., $\nabla f_i(p) \neq 0$, for all $p \in f_i^{-1}(0)$. We call Σ_i, for $i = C, L, R$, of *switching line* and we assume that Σ_C separates $\mathbb{R}^2$ into two disjoint regions given by

$$R_R = \{q \in \mathbb{R}^2 : f_C(q) > 0\} \quad \text{and} \quad R_L = \{q \in \mathbb{R}^2 : f_C(q) < 0\},$$

that is, $\mathbb{R}^2 = R_L \cup \Sigma_C \cup R_R$, and we also assume that Σ_L and Σ_R separates $\mathbb{R}^2$ into three disjoint regions given by

$$R_R = \{q \in \mathbb{R}^2 : f_R(q) > 0\}, \quad R_C = \{q \in \mathbb{R}^2 : f_R(q) < 0 < f_L(q)\}$$

and

$$R_L = \{q \in \mathbb{R}^2 : f_L(q) < 0\},$$

that is, $\mathbb{R}^2 = R_L \cup \Sigma_L \cup R_C \cup \Sigma_R \cup R_R$.

Denote by $\mathcal{X}(U)$ the space of C^r-vector fields on $U \subset \mathbb{R}^2$ endowed with the C^r-topology, with $r \geq 1$.

Definition 1. *A planar piecewise smooth vector field with two zones can be written as* $Z : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$Z(q) = \begin{cases} X_R(q), & \text{if} \quad f_C(q) \geq 0, \\ X_L(q), & \text{if} \quad f_C(q) \leq 0, \end{cases} \tag{1.1}$$

where $X_R, X_L \in \mathcal{X}(U)$*. For simplify the notation, we denote vector field* (1.1) *by* $Z = (X_L, X_R)$*. We call vector field* (1) *of left vector field when* $f_C(q) \leq 0$ *and right vector field when* $f_C(q) \geq 0$*.*

Definition 2. *A planar piecewise smooth vector field with three zones can be written as* $Z : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$Z(q) = \begin{cases} X_R(q), & \text{if} \quad f_R(q) \geq 0, \\ X_C(q), & \text{if} \quad f_R(q) \leq 0 \leq f_L(q), \\ X_L(q), & \text{if} \quad f_L(q) \leq 0, \end{cases} \tag{1.2}$$

where $X_R, X_C, X_L \in \mathcal{X}(U)$*. For simplify the notation, we denote vector field* (1.2) *by* $Z = (X_L, X_C, X_R)$*. We call vector field* (2) *of left vector field when* $f_L(q) \leq 0$*, right vector field when* $f_R(q) \geq 0$ *and central vector field when* $f_R(q) \leq 0 \leq f_L(q)$*.*

Next we present some basic concepts of piecewise smooth vector field theory, most of which were introduced by Filippov in [12]. We will use the vector field X_L and the switching line Σ_C, from piecewise vector field (1), in the next definitions. However, we can easily adapt the definitions to the vector fields X_C and X_R and the switching lines Σ_L and Σ_R from piecewise smooth vector field (2). The derivative of function f_C in the direction of the vector field X_L, i.e., the expression

$$X_L f_C(p) = \langle X_L(p), \nabla f_C(p) \rangle,$$

also known as the *Lie derivative* of f_C with respect to X_L, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$, characterizes the contact between the vector field X_L and the switching line Σ_C. Analogously, we have

$$X_L^i f_C(p) = \langle X_L(p), \nabla X_L^{i-1} f_C(p) \rangle,$$

for $i \geq 2$. We can distinguish the followings subsets of Σ_C (see Figure 1.1) by

Crossing set:

$$\Sigma_C^c = \{p \in \Sigma_C : X_L f_C(p) \cdot X_R f_C(p) > 0\};$$

Sliding set:

$$\Sigma_C^s = \{p \in \Sigma_C : X_L f_C(p) > 0, X_R f_C(p) < 0\};$$

Escaping set:

$$\Sigma_C^e = \{p \in \Sigma_C : X_L f_C(p) < 0, X_R f_C(p) > 0\}.$$

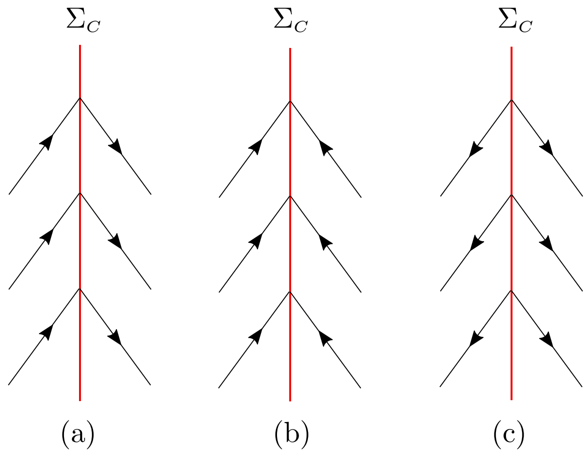


Figure 1.1: (a) Crossing set; (b) Sliding set; (c) Escaping set.

On the sets Σ_C^s and Σ_C^e we define the *Filippov vector field* F_Z associated to $Z = (X_L, X_R)$, as follows: if $p \in \Sigma_C^s \cup \Sigma_C^e$, then $F_Z(p)$ denotes the vector tangent to Σ_C in the cone spanned by $X_L(p)$ and $X_R(p)$. A point $p \in \Sigma_C$ is called a Σ_C-*regular point* of Z if $p \in \Sigma_C^c \cup \Sigma_C^s \cup \Sigma_C^e$, then $F_Z(p) \neq 0$. Otherwise, p is a Σ_C-*singular point* of Z.

When $p \in \Sigma_C$ and $X_L f_C(p) = 0$ we say that p is a *tangent point* of X_L. A tangent point p is a *fold point* of X_L if $X_L f_C(p) = 0$ and $X_L^2 f_C(p) \neq 0$. Moreover, p is a *visible* (resp. *invisible*) fold point of X_L if $X_L f_C(p) = 0$ and $X_L^2 f_C(p) < 0$ (resp. $X_L^2 f_C(p) > 0$). We can classify a singular point p of vector field X_L according to its position in relation to the vector field. More precisely, we say that p is a *real singular point* of X_L if $X_L(p) = 0$ and $p \in R_L$, a *virtual singular point* of X_L if $X_L(p) = 0$ and $p \in (R_L \cup \Sigma_C)^c$, where $(R_L \cup \Sigma_C)^c$ denotes the complement of $R_L \cup \Sigma_C$ in $\mathbb{R}^2$, and a *boundary singular point* of X_L if $X_L(p) = 0$ and $p \in \Sigma_C$.

Definition 3. *Piecewise smooth vector field* (1.1) *is called continuous if*

$$X_L(p) = X_R(p), \quad \forall\, p \in \Sigma_C.$$

Otherwise, it is called discontinuous. Similarly, piecewise smooth vector field (1.2) *is called continuous if*

$$X_L(p) = X_C(p), \quad \forall\, p \in \Sigma_L \quad \textit{and}$$
$$X_C(q) = X_R(q), \quad \forall\, q \in \Sigma_R.$$

Otherwise, it is called discontinuous.

The following definitions describe the concepts about local and global trajectories of a piecewise smooth vector field with two zones, and can be extended to three zones.

Denote by $\phi_{X_L}(t,p)$ the flow of vector field $X_L \in \mathcal{X}(U)$, such that

$$\begin{cases} \dfrac{d}{dt}\phi_{X_L}(t,p) = X_L\big(\phi_{X_L}(t,p)\big), \\ \phi_{X_L}(0,p) = p, \end{cases}$$

where $t \in I \subset \mathbb{R}$ and I depends on the point $p \in U$ and the vector field X_L.

Definition 4. *A local trajectory (or local orbit) $\phi_Z(t,p)$ of vector field* (1)*, through the point p, is obtained by concatenating the flows of the vector fields X_R, X_L and F_Z.*

Definition 5. *The global trajectory (or global orbit) $\Phi_Z(t,p)$ passing through p is a union*

$$\Phi_Z(t,p) = \bigcup_{i\in\mathbb{Z}} \{\sigma_i(t,p_i) : t_i \le t \le t_{i+1}\}$$

of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_i,p_i) = p_i$ and $\sigma_i(t_{i+1},p_i) = p_{i+1}$. An global trajectory is positive (resp. negative) if $i \in \mathbb{N}$ (resp. $-i \in \mathbb{N}$) and $t_0 = 0$.

Definition 6. *Let $\Phi_Z(t,p)$ a global trajectory of the piecewise smooth vector field* (1)*. We say that Φ_Z is periodic orbit if Φ_Z is periodic in the variable t, that is, if exists $T > 0$ such that $\Phi_Z(t+T,p) = \Phi_Z(t,p)$ for all $t \in \mathbb{R}$.*

In a piecewise smooth vector field, a periodic orbit can be of two types: *sliding/escaping periodic orbit* or *crossing periodic orbit*; the first one contains some segment of sliding or escaping sets (see Figure 1.2 (a)), and the second one does not contain any segments of sliding or escaping sets (see Figure 1.2 (b)). A *limit cycle* of vector field (1) is an isolated periodic orbit in the set of all periodic orbits of vector field (1). Therefore, a limit cycle can be of two types: *sliding/escaping limit cycle* or *crossing limit cycle*. In this work, unless mentioned otherwise, when we talk about limit cycles, we are talking about crossing limit cycles.

Consider a point $p \in \Sigma_C$ such that the orbit $\phi_{X_L}(t,p)$ of the vector field X_L passing through p returns to Σ_C after a positive time $t_L(p)$, called *flight time*. We define the *half return map associated with* X_L by

$$\pi_{X_L}(p) = \phi_{X_L}(t_L(p),p) = p_L \in \Sigma_C.$$

Similarly, when the orbit $\phi_{X_R}(t,p_L)$ of the vector field X_R passing through $p_L \in \Sigma_C$ returns to Σ_C after a positive time $t_R(p_L)$, we define the half return map associated with X_R by

$$\pi_{X_R}(p_L) = \phi_{X_R}(t_R(p_L),p_L) = p_R \in \Sigma_C.$$

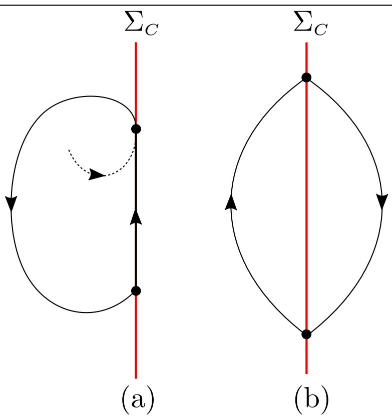


Figure 1.2: (a) Escaping periodic orbit; (b) Crossing periodic orbit.

The *first return map* (or *Poincaré map*) associated with $Z = (X_L, X_R)$ is defined by

$$\pi_Z(p) = \pi_{X_R} \circ \pi_{X_L}(p) = \phi_{X_R}\big(t_R(p_L), \phi_{X_L}(t_L(p), p)\big),$$

or in reverse order, applying ϕ_{X_R} and after ϕ_{X_L}. Moreover, if $p \in \Sigma_C$ is a isolated fixed point to map π_Z, then the global trajectory passing through p is a limit cycle of the piecewise smooth vector field (1).

Instead of studying the fixed point of the first return map, some times it is more convenient to study the zeros of the correspondent *displacement map*, given by

$$d(p) = \pi_Z(p) - p.$$

In the following, we introduce the notions of separatrix, separatrix connections and equivalence between two piecewise smooth vector fields. For more details see [19].

Definition 7. *An unstable separatrix of a saddle point p of $Z = (X_L, X_R)$ is the invariant unstable manifold $W^u(p)$ given by*

$$W^u(p) = \{q \in U : \Phi_Z(t, q) \text{ is defined for } t \in (-\infty, 0) \text{ and } \lim_{t\to-\infty} \Phi_Z(t, q) = p\}.$$

A stable separatrix of a saddle point p of $Z = (X_L, X_R)$ is the invariant stable manifold $W^s(p)$ given by

$$W^s(p) = \{q \in U : \Phi_Z(t, q) \text{ is defined for } t \in (0, \infty) \text{ and } \lim_{t\to\infty} \Phi_Z(t, q) = p\}.$$

If a separatrix is simultaneously stable and unstable it is a separatrix connection.

Here we distinguish two types of separatrix connections:

- *Homoclinic loop* : for this type we have $W^u(p) = W^s(p)$, see Figure 1.3 (a).
- *Heteroclinic orbits* : for this case we have two saddle points $p \neq q$ such that $W^u(p) = W^s(q)$ and $W^s(p) = W^u(q)$, see Figure 1.3 (b).

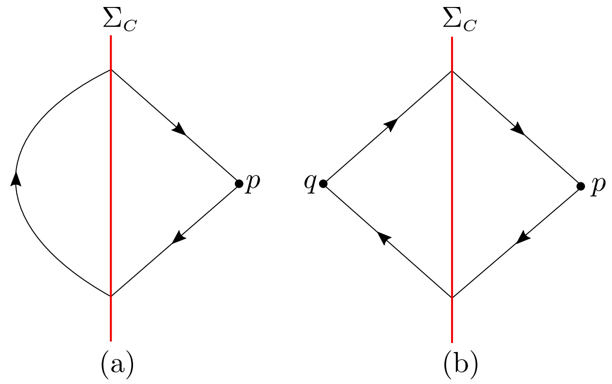


Figure 1.3: (a) Homoclinic loop; (b) Two heteroclinic orbits.

In the follow, we introduce the notion of Σ-equivalence between two piecewise smooth vector fields, which allows comparing phase portraits.

Definition 8. *Two piecewise smooth vector fields $Z_1 : U_1 \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $Z_2 : U_2 \subset \mathbb{R}^2 \to \mathbb{R}^2$ with switching lines $\Sigma_1 \subset U_1$ and $\Sigma_2 \subset U_2$, respectively, are Σ_1-equivalent if there exists an orientation preserving homeomorphism $h : U_1 \to U_2$ which sends Σ_1 to Σ_2 and sends orbits of Z_1 to orbits of Z_2.*

We can see that any Σ-equivalence sends regular orbits to regular orbits and distinguished singular point to distinguished singular point. Moreover, as it sends arrival and departing points to arrival and departing points, Σ^c , Σ^s and Σ^e are preserved, and thus it also sends sliding orbits to sliding orbits and preserves separatrices, separatrix connections, periodic orbits and cycles.

1.2 Piecewise smooth Hamiltonian vector fields with two or three zones

In this section, some basic concepts about piecewise smooth Hamiltonian vector fields are presented.

Definition 9. *Consider $H : U \to \mathbb{R}$ a C^r function, with U an open subset of $\mathbb{R}^2$ and $r > 1$. Fixed a coordinate system $(x, y) \in \mathbb{R}^2$, a smooth Hamiltonian vector field is a vector field given by*

$$X(x, y) = (H_y(x, y), -H_x(x, y)), \tag{1.3}$$

where $H_i(x, y)$ denotes the derivative of the function H with respect the variable $i = x, y$. We say that the function H is the Hamiltonian function of vector field (1.3)*.*

Unless there is a coordinate change (symplectic), every Hamiltonian vector field is like this. Now, let us remember some important properties of Hamiltonian vector fields.

1. H is constant along the solutions of vector field (1.3), that is, H is a first integral of the system
$$\begin{cases} \dot{x} = H_y(x, y), \\ \dot{y} = -H_x(x, y), \end{cases}$$
where the dot denotes the derivative with respect to the independent variable t, here called the time.

2. $X : \mathbb{R}^2 \to \mathbb{R}^2$ is a Hamiltonian vector field if and only if $\operatorname{div}(X) = 0$.

3. The only non-degenerate singular points of a planar Hamiltonian vector field are centers or saddles.

Definition 10. *Consider $H^i : U \subset \mathbb{R}^2 \to \mathbb{R}$, C^r, $r > 1$, and $f_i : \mathbb{R}^2 \to \mathbb{R}$ a smooth function such that $0 \in \mathbb{R}$ is a regular value, for $i = R, C, L$. A piecewise smooth Hamiltonian vector field with two zones is a vector field given by*

$$Z(x, y) = \begin{cases} X_R(x, y) = (H_y^R(x, y), -H_x^R(x, y)), & \text{if} \quad f_C(x, y) \geq 0, \\ X_L(x, y) = (H_y^L(x, y), -H_x^L(x, y)), & \text{if} \quad f_C(x, y) \leq 0. \end{cases}$$

A piecewise smooth Hamiltonian vector field with three zones is a vector field given by

$$Z(x, y) = \begin{cases} X_R(x, y) = (H_y^R(x, y), -H_x^R(x, y)), & \text{if} \quad f_R(x, y) \geq 0, \\ X_C(x, y) = (H_y^C(x, y), -H_x^C(x, y)), & \text{if} \quad f_R(x, y) \leq 0 \leq f_L(x, y), \\ X_L(x, y) = (H_y^L(x, y), -H_x^L(x, y)), & \text{if} \quad f_L(x, y) \leq 0. \end{cases}$$

When the vector fields X_i, $i = R, C, L$ that define the piecewise smooth vector field $Z = (X_R, X_L)$ (or $Z = (X_R, X_C, X_L)$) are Hamiltonian, then the solutions curves of the respective differential equations are contained in the level sets of the Hamiltonian functions. Thus, the first return map can be studied by seeking by points in Σ_C (or Σ_R and Σ_L) that are in the same level curves of these Hamiltonian functions.

1.3 The Melnikov function method

There are two standard methods for studying the number of limit cycles that bifurcate from a periodic annulus of a smooth vector field by small perturbations. The first one is the Averaging method, which has an extension to the piecewise smooth case, see for example [22, 31, 37], and the second one is the Melnikov function method, which also has an extension to piecewise smooth case, see [20, 43]. In [21], the authors proved that these two methods are equivalent to each other. The main goal of both methods is to transform the problem of estimate the number of limit cycles of a piecewise system into a problem of finding zeros of a given function. In the planar case, there are situations in which some limit cycles are not detected by the Melnikov function method when we perturb a periodic annulus of a piecewise Hamiltonian differential system (see [6, 44]). These limit cycles in this context are called alien limit cycles. In this work, we will use the number of zeros of the Melnikov function to estimate the number of limit cycles that can bifurcate from a periodic annulus. Furthermore, in this section, we will present expressions for the Melnikov functions associated with a discontinuous planar piecewise Hamiltonian differential system with two or three zones.

To establish the Melnikov functions, consider $H^i : U \subset \mathbb{R}^2 \to \mathbb{R}$, a C^r function with $r > 1$ and $h_i : \mathbb{R}^2 \to \mathbb{R}$, with $h_i(x, y) = x - x_i$ (note that $0 \in \mathbb{R}$ is a regular value), for $i = R, C, L$. A *discontinuous planar piecewise near–Hamiltonian differential system with n zones*, for $n = 2, 3$, is a piecewise differential system given by

$$\begin{cases} \dot{x} = H_y(x, y) + \epsilon f(x, y), \\ \dot{y} = -H_x(x, y) + \epsilon g(x, y), \end{cases} \tag{1.4}$$

where, for $n = 2$, we have that

$$H(x, y) = \begin{cases} H^L(x, y), & h_C(x, y) \leq 0, \\ H^R(x, y), & h_C(x, y) \geq 0, \end{cases}$$

$$f(x, y) = \begin{cases} f_L(x, y), & h_C(x, y) \leq 0, \\ f_R(x, y), & h_C(x, y) \geq 0, \end{cases}$$

$$g(x,y)=\begin{cases} g_L(x,y), & h_C(x,y)\le 0,\\ g_R(x,y), & h_C(x,y)\ge 0,\end{cases}$$

and, for $n=3$, we have that

$$H(x,y)=\begin{cases} H^L(x,y), & h_L(x,y)\le 0,\\ H^C(x,y), & h_R(x,y)\le 0\le h_L(x,y),\\ H^R(x,y), & h_R(x,y)\ge 0,\end{cases}$$

$$f(x,y)=\begin{cases} f_L(x,y), & h_L(x,y)\le 0,\\ f_C(x,y), & h_R(x,y)\le 0\le h_L(x,y),\\ f_R(x,y), & h_R(x,y)\ge 0,\end{cases}$$

$$g(x,y)=\begin{cases} g_L(x,y), & h_L(x,y)\le 0,\\ g_C(x,y), & h_R(x,y)\le 0\le h_L(x,y),\\ g_R(x,y), & h_R(x,y)\ge 0,\end{cases}$$

where the functions f_i and g_i are C^∞, for $i=L,C,R$, and $0\le\epsilon<<1$. When $\epsilon=0$ system (1.4) is a piecewise Hamiltonian differential system.

Regarding system $(1.4)|_{\epsilon=0}$, we make some assumptions:

(H1) For the case $n=3$, suppose that unperturbed system $(1.4)|_{\epsilon=0}$ has an open interval $J_0=(\alpha_0,\beta_0)$ and a periodic annulus consisting of a family of crossing periodic orbits L_h^0, with $h\in J_0$, such that each orbit of this family crosses the straight lines $x=x_R$ and $x=x_L$ in four points, $A(h)=(x_R,a(h))$, $A_1(h)=(x_R,a_1(h))$, $A_2(h)=(x_L,a_2(h))$ and $A_3(h)=(x_L,a_3(h))$, with $a_1(h)<a(h)$ and $a_2(h)<a_3(h)$, through the three zones with clockwise orientation (see Figure 1.4), satisfying the following equations

$$\begin{aligned} H^R(A(h)) &= H^R(A_1(h)),\\ H^C(A_1(h)) &= H^C(A_2(h)),\\ H^L(A_2(h)) &= H^L(A_3(h)),\\ H^C(A_3(h)) &= H^C(A(h)),\end{aligned}$$

and, for $h \in J_0$,

$$H_y^R(A(h))\, H_y^R(A_1(h))\, H_y^L(A_2(h))\, H_y^L(A_3(h)) \neq 0,$$

$$H_y^C(A(h))\, H_y^C(A_1(h))\, H_y^C(A_2(h))\, H_y^C(A_3(h)) \neq 0.$$

Therefore, for each $h \in J_0$, the crossing periodic orbit L_h^0 from system $(1.4)|_{\epsilon=0}$ is given by $L_h^0 = \widehat{AA_1} \cup \widehat{A_1A_2} \cup \widehat{A_2A_3} \cup \widehat{A_3A}$ (see Figure 1.4).

(H2) For the case $n = 2$, suppose that unperturbed system $(1.4)|_{\epsilon=0}$ has a open interval $J_1 = (\alpha_1, \beta_1)$ and a periodic annulus consisting of a family of crossing periodic orbits L_h^1, $h \in J_1$, such that each orbit of this family crosses the straight lines $x = x_C$ in two points, $B(h) = (x_C, b(h))$ and $B_1(h) = (x_C, b_1(h))$, with $b_1(h) < b(h)$, through the two zones with clockwise orientation (see Figure 1.4), satisfying the following equations

$$H^R(B(h)) = H^R(B_1(h)),$$
$$H^L(B_1(h)) = H^L(B(h)),$$

and, for $h \in J_1$,

$$H_y^R(B(h))\, H_y^R(B_1(h))\, H_y^L(B(h))\, H_y^L(B_1(h)) \neq 0.$$

Therefore, for each $h \in J_1$, the crossing periodic orbit L_h^1 from system $(1.4)|_{\epsilon=0}$ is given by $L_h^1 = \widehat{BB_1} \cup \widehat{B_1B}$ (see Figure 1.4).

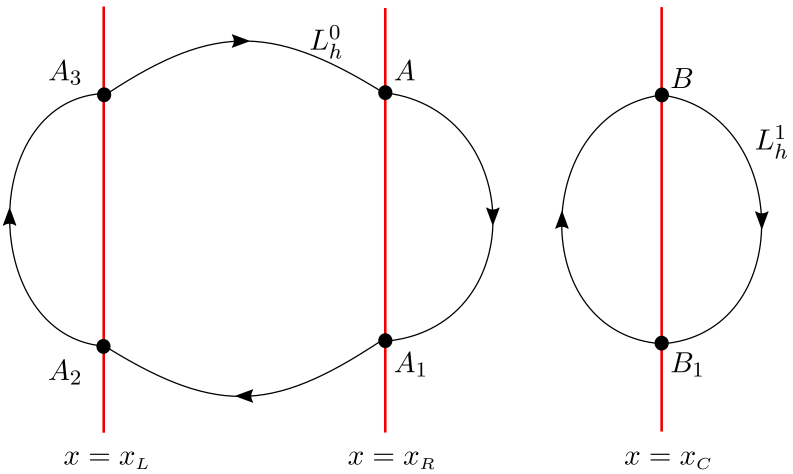


Figure 1.4: The crossing periodic orbits of system $(1.4)|_{\epsilon=0}$ with three and two zones, respectively.

For the perturbed system (1.4) when $n = 3$, consider, for $h \in J_0$, the solution of right subsystem from (1.4) starting from point $A(h)$. Let $A_{1\epsilon}(h) = (x_R, a_{1\epsilon}(h))$ be the first intersection point of this orbit with straight line $x = x_R$. Denote by $A_{2\epsilon}(h) = (x_L, a_{2\epsilon}(h))$ the first intersection point of the orbit from central subsystem from (1.4) starting at $A_{1\epsilon}(h)$ with straight line $x = x_L$,

$A_{3\epsilon}(h) = (x_L, a_{3\epsilon}(h))$ the first intersection point of the orbit from left subsystem from (1.4) starting at $A_{2\epsilon}(h)$ with straight line $x = x_L$ and $A_\epsilon(h) = (x_R, a_\epsilon(h))$ the first intersection point of the orbit from central subsystem from (1.4) starting at $A_{3\epsilon}(h)$ with straight line $x = x_R$ (see Figure 1.5). Similarly, for the perturbed system (1.4) when $n = 2$, consider, for $h \in J_1$, the solution of right subsystem from (1.4) starting from point $B(h)$. Let $B_{1\epsilon}(h) = (x_C, b_{1\epsilon}(h))$ be the first intersection point of this orbit with straight line $x = x_C$ and $B_\epsilon(h) = (x_C, b_\epsilon(h))$ the first intersection point of the orbit from central subsystem from (1.4) starting at $B_{1\epsilon}(h)$ with straight line $x = x_C$ (see Figure 1.5).

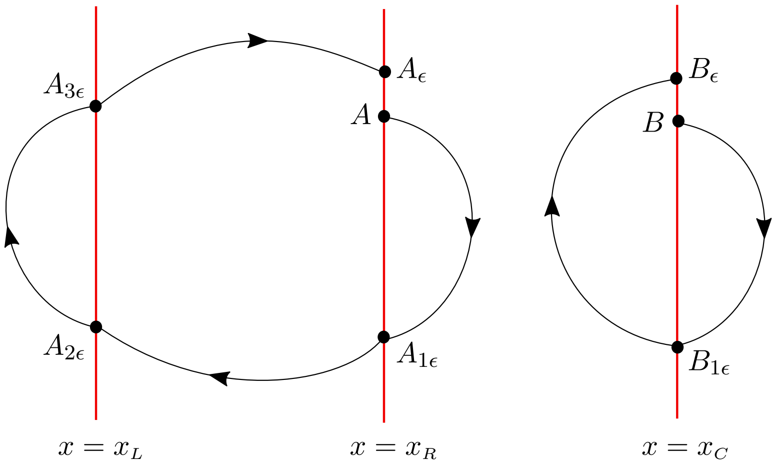


Figure 1.5: Poincaré maps of system (1.4) with three and two zones, respectively.

Thus, we can define the displacement maps of piecewise system (1.4) as

$$\mathcal{B}_0(h,\epsilon) = H^R(A_\epsilon(h)) - H^R(A(h)) = \epsilon\mathcal{F}_0(h,\epsilon) = \epsilon M_0(h) + \mathcal{O}(\epsilon^2), \quad \forall\, h \in J_0, \tag{1.5}$$

$$\mathcal{B}_1(h,\epsilon) = H^R(B_\epsilon(h)) - H^R(B(h)) = \epsilon\mathcal{F}_1(h,\epsilon) = \epsilon M_1(h) + \mathcal{O}(\epsilon^2), \quad \forall\, h \in J_1, \tag{1.6}$$

where M_0 and M_1 are called *the first order Melnikov functions* associated to piecewise system (1.4) with three or two zones, respectively. Note that, by the Mean Value Theorem,

$$\begin{aligned}\mathcal{B}_0(h,\epsilon) &= DH^R\big(A(h) + \mathcal{O}\big(A_\epsilon(h) - A(h)\big)\big)(A_\epsilon(h) - A(h))\\ &= \big(DH^R(A(h)) + \mathcal{O}(\epsilon)\big)(A_\epsilon(h) - A(h))\\ &= \big(H^R_y(A(h)) + \mathcal{O}(\epsilon)\big)(a_\epsilon(h) - a(h)).\end{aligned}$$

As, by hypothesis (H1), $H^R_y(A(h)) \neq 0$, we have that $\mathcal{B}_0(h,\epsilon) = 0$ if and only if $a(h) = a_\epsilon(h)$. Similarly, $\mathcal{B}_1(h,\epsilon) = 0$ if and only if $b(h) = b_\epsilon(h)$. Therefore, for $0 < \epsilon \ll 1$, system (1.4) has a limit cycle in L_h if and only if $\mathcal{F}_i$, $i = 0, 1$, has an isolated zero in h. Thus, by Implicit Function Theorem, the number of simple zeros of the function M_i, $i = 0, 1$, can be used to estimate the number of limit cycles emerging from the periodic annulus from system (1.4).

In what follows, we provide an expression for the first order Melnikov functions M_0 and M_1, that we will use frequently in the text. Then, following the steps of the proof of Theorem 1.1 in [43] for the case with two zones and, doing the obvious adaptations, for the case with three zones, we can prove the following theorem.

Theorem 1. *Consider system* (1.4) *with* $0 \leq \epsilon \ll 1$ *and suppose that the unperturbed system* $(1.4)|_{\epsilon=0}$ *satisfies the hypotheses* (*Hi*), $i = 1, 2$. *Then, the first order Melnikov functions associated to system* (1.4) *with three and two zones can be expressed, respectively, as*

$$\begin{aligned} M_0(h) = & \frac{H_y^R(A)}{H_y^C(A)} I_C^0 + \frac{H_y^R(A) H_y^C(A_3)}{H_y^C(A) H_y^L(A_3)} I_L^0 + \frac{H_y^R(A) H_y^C(A_3) H_y^L(A_2)}{H_y^C(A) H_y^L(A_3) H_y^C(A_2)} \bar{I}_C^0 \\ & + \frac{H_y^R(A) H_y^C(A_3) H_y^L(A_2) H_y^C(A_1)}{H_y^C(A) H_y^L(A_3) H_y^C(A_2) H_y^R(A_1)} I_R^0, \quad h \in J_0, \end{aligned} \tag{1.7}$$

and

$$M_1(h) = \frac{H_y^R(B)}{H_y^L(B)} I_L^1 + \frac{H_y^R(B) H_y^L(B_1)}{H_y^L(B) H_y^R(B_1)} I_R^1, \quad h \in J_1, \tag{1.8}$$

where

$$I_C^0 = \int_{\widehat{A_3 A}} g_C dx - f_C dy, \quad I_L^0 = \int_{\widehat{A_2 A_3}} g_L dx - f_L dy, \quad \bar{I}_C^0 = \int_{\widehat{A_1 A_2}} g_C dx - f_C dy,$$

$$I_R^0 = \int_{\widehat{A A_1}} g_R dx - f_R dy, \quad I_L^1 = \int_{\widehat{B_1 B}} g_L dx - f_L dy \quad \text{and} \quad I_R^1 = \int_{\widehat{B B_1}} g_R dx - f_R dy,$$

Furthermore, if M_i, *for* $i = 0, 1$, *has a simple zero at* h^*, *then for* $0 < \epsilon << 1$, *system* (1.4) *has a unique limit cycle near* L_{h^*}.

Proof. Firstly, let us obtain the expression of the Melnikov function (1.7). From (1.5), the displacement map is given by

$$H^R(A_\epsilon(h)) - H^R(A(h)) = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8, \tag{1.9}$$

where $\delta_1 = H^R(A_\epsilon(h)) - H^C(A_\epsilon(h))$, $\delta_2 = H^C(A_\epsilon(h)) - H^C(A_{3\epsilon}(h))$, $\delta_3 = H^C(A_{3\epsilon}(h)) - H^L(A_{3\epsilon}(h))$, $\delta_4 = H^L(A_{3\epsilon}(h)) - H^L(A_{2\epsilon}(h))$, $\delta_5 = H^L(A_{2\epsilon}(h)) - H^C(A_{2\epsilon}(h))$, $\delta_6 = H^C(A_{2\epsilon}(h)) - H^C(A_{1\epsilon}(h))$, $\delta_7 = H^C(A_{1\epsilon}(h)) - H^R(A_{1\epsilon}(h))$ and $\delta_8 = H^R(A_{1\epsilon}(h)) - H^R(A(h))$.

Note that

$$\delta_2 = H^C(A_\epsilon(h)) - H^C(A_{3\epsilon}(h)) = \int_{\widehat{A_{3\epsilon} A_\epsilon}} dH^C(x, y) = \int_{\widehat{A_{3\epsilon} A_\epsilon}} H_x^C dx + H_y^C dy.$$

From the right side of (1.4), we can substitute dx by $(H_y^C + \epsilon f_C)dt$ and dy by $(-H_x^C + \epsilon g_C)dt$, respectively. Then, δ_2 becomes

$$\begin{aligned}
\delta_2 &= \int_{\widehat{A_{3\epsilon}A_\epsilon}} H_x^C(H_y^C + \epsilon f_C)dt + \int_{\widehat{A_{3\epsilon}A_\epsilon}} H_y^C(-H_x^C + \epsilon g_C)dt \\
&= \epsilon \int_{\widehat{A_{3\epsilon}A_\epsilon}} (H_x^C f_C + H_y^C g_C)dt \\
&= \epsilon \int_{\widehat{A_{3\epsilon}A_\epsilon}} f_C(-dy + \epsilon g_C dt) + \epsilon \int_{\widehat{A_{3\epsilon}A_\epsilon}} g_C(dx - \epsilon f_C dt) \\
&= \epsilon \int_{\widehat{A_3A}} g_C dx - f_C dy + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Thus, we have that

$$\frac{\partial \delta_2}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{A_3A}} g_C dx - f_C dy. \tag{1.10}$$

With the same argument, we obtain

$$\frac{\partial \delta_4}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{A_2A_3}} g_L dx - f_L dy, \tag{1.11}$$

$$\frac{\partial \delta_6}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{A_1A_2}} g_C dx - f_C dy, \tag{1.12}$$

and

$$\frac{\partial \delta_8}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{AA_1}} g_R dx - f_R dy. \tag{1.13}$$

Moreover, since $A = (x_R, a(h))$, $A_{1\epsilon} = (x_R, a_{1\epsilon}(h))$, $A_{2\epsilon} = (x_L, a_{2\epsilon}(h))$, $A_{3\epsilon} = (x_L, a_{3\epsilon}(h))$ and $A_\epsilon = (x_R, a_\epsilon(h))$, we obtain, from $\delta_2 = H^C(A_\epsilon(h)) - H^C(A_{3\epsilon}(h))$, $\delta_4 = H^L(A_{3\epsilon}(h)) - H^L(A_{2\epsilon}(h))$, $\delta_6 = H^C(A_{2\epsilon}(h)) - H^C(A_{1\epsilon}(h))$ and $\delta_8 = H^R(A_\epsilon(h)) - H^R(A(h))$,

$$\frac{\partial \delta_2}{\partial \epsilon}|_{\epsilon=0} = H_y^C(A(h))\frac{\partial a_\epsilon}{\partial \epsilon}|_{\epsilon=0} - H_y^C(A_3(h))\frac{\partial a_{3\epsilon}}{\partial \epsilon}|_{\epsilon=0}, \tag{1.14}$$

$$\frac{\partial \delta_4}{\partial \epsilon}|_{\epsilon=0} = H_y^L(A_3(h))\frac{\partial a_{3\epsilon}}{\partial \epsilon}|_{\epsilon=0} - H_y^L(A_2(h))\frac{\partial a_{2\epsilon}}{\partial \epsilon}|_{\epsilon=0}, \tag{1.15}$$

$$\frac{\partial \delta_6}{\partial \epsilon}|_{\epsilon=0} = H_y^C(A_2(h))\frac{\partial a_{2\epsilon}}{\partial \epsilon}|_{\epsilon=0} - H_y^C(A_1(h))\frac{\partial a_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}, \tag{1.16}$$

and

$$\frac{\partial \delta_8}{\partial \epsilon}|_{\epsilon=0} = H_y^R(A_1(h))\frac{\partial a_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.17}$$

From (1.13) and (1.17), we conclude that

$$\frac{\partial a_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0} = \frac{1}{H_y^R(A_1(h))} \int_{\widehat{AA_1}} g_R dx - f_R dy. \tag{1.18}$$

From (1.18), (1.16) and (1.12), we have that

$$\begin{aligned}\frac{\partial a_{2\epsilon}}{\partial \epsilon}|_{\epsilon=0} =& \frac{1}{H_y^C(A_2(h))}\int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ &+ \frac{H_y^C(A_1(h))}{H_y^C(A_2(h))H_y^R(A_1(h))}\int_{\widehat{AA_1}} g_R dx - f_R dy.\end{aligned} \tag{1.19}$$

From (1.19), (1.15) and (1.11), we have that

$$\begin{aligned}\frac{\partial a_{3\epsilon}}{\partial \epsilon}|_{\epsilon=0} =& \frac{1}{H_y^L(A_3(h))}\int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ &+ \frac{H_y^L(A_2(h))}{H_y^L(A_3(h))}\frac{\partial a_{2\epsilon}}{\partial \epsilon}|_{\epsilon=0}.\end{aligned} \tag{1.20}$$

From (1.20), (1.14) and (1.10), we have that

$$\begin{aligned}\frac{\partial a_{\epsilon}}{\partial \epsilon}|_{\epsilon=0} =& \frac{1}{H_y^C(A(h))}\int_{\widehat{A_3A}} g_C dx - f_C dy \\ &+ \frac{H_y^C(A_3(h))}{H_y^C(A(h))}\frac{\partial a_{3\epsilon}}{\partial \epsilon}|_{\epsilon=0}.\end{aligned}$$

Thus, since $\delta_1 = H^R(A_\epsilon(h)) - H^C(A_\epsilon(h))$, it follows that

$$\frac{\partial \delta_1}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^R(A(h)) - H_y^C(A(h))\Big)\frac{\partial a_\epsilon}{\partial \epsilon}|_{\epsilon=0}. \tag{1.21}$$

Similarly, since $\delta_3 = H^C(A_{3\epsilon}(h)) - H^L(A_{3\epsilon}(h))$, $\delta_5 = H^L(A_{2\epsilon}(h)) - H^C(A_{2\epsilon}(h))$ and $\delta_7 = H^C(A_{1\epsilon}(h)) - H^R(A_{1\epsilon}(h))$, we obtain

$$\frac{\partial \delta_3}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^C(A_3(h)) - H_y^L(A_3(h))\Big)\frac{\partial a_{3\epsilon}}{\partial \epsilon}|_{\epsilon=0}, \tag{1.22}$$

$$\frac{\partial \delta_5}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^L(A_2(h)) - H_y^C(A_2(h))\Big)\frac{\partial a_{2\epsilon}}{\partial \epsilon}|_{\epsilon=0}, \tag{1.23}$$

and

$$\frac{\partial \delta_7}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^C(A_1(h)) - H_y^R(A_1(h))\Big)\frac{\partial a_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.24}$$

Therefore, from (1.5) and (1.9), we have that

$$\begin{aligned}M_0(h) =& \frac{\partial}{\partial \epsilon}\Big(H^R(A_\epsilon(h)) - H^R(A(h))\Big) \\ =& \frac{\partial}{\partial \epsilon}\Big(\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8\Big)|_{\epsilon=0}.\end{aligned} \tag{1.25}$$

Replacing (1.10), (1.11), (1.12), (1.13), (1.21), (1.22), (1.23), (1.24) into (1.25), we obtain the expression (1.7).

Now, let us obtain the expression of the Melnikov function (1.8). From (1.6), the displacement map is given by

$$H^R(B_\epsilon(h)) - H^R(B(h)) = \delta_1 + \delta_2 + \delta_3 + \delta_4, \tag{1.26}$$

where $\delta_1 = H^R(B_\epsilon(h)) - H^L(B_\epsilon(h))$, $\delta_2 = H^L(B_\epsilon(h)) - H^L(B_{1\epsilon}(h))$, $\delta_3 = H^L(B_{1\epsilon}(h)) - H^R(B_{1\epsilon}(h))$ and $\delta_4 = H^R(B_{1\epsilon}(h)) - H^R(B(h))$. Note that

$$\delta_2 = H^L(B_\epsilon(h)) - H^L(B_{1\epsilon}(h)) = \int_{\widehat{B_{1\epsilon}B_\epsilon}} dH^L(x,y) = \int_{\widehat{B_{1\epsilon}B_\epsilon}} H_x^L dx + H_y^L dy.$$

From the right side of (1.4), we can substitute dx by $(H_y^L + \epsilon f_L)dt$ and dy by $(-H_x^L + \epsilon g_L)dt$, respectively. Then, δ_2 becomes

$$\begin{aligned}
\delta_2 &= \int_{\widehat{B_{1\epsilon}B_\epsilon}} H_x^L(H_y^L + \epsilon f_L)dt + \int_{\widehat{B_{1\epsilon}B_\epsilon}} H_y^L(-H_x^L + \epsilon g_L)dt \\
&= \epsilon \int_{\widehat{B_{1\epsilon}B_\epsilon}} (H_x^L f_L + H_y^L g_L)dt \\
&= \epsilon \int_{\widehat{B_{1\epsilon}B_\epsilon}} f_L(-dy + \epsilon g_L dt) + \epsilon \int_{\widehat{B_{1\epsilon}B_\epsilon}} g_L(dx - \epsilon f_L dt) \\
&= \epsilon \int_{\widehat{B_1B}} g_L dx - f_L dy + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Thus, we have that

$$\frac{\partial \delta_2}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{B_1B}} g_L dx - f_L dy. \tag{1.27}$$

With the same argument, we obtain

$$\frac{\partial \delta_4}{\partial \epsilon}|_{\epsilon=0} = \int_{\widehat{BB_1}} g_R dx - f_R dy, \tag{1.28}$$

On the other hand, since $B_\epsilon = (x_C, b_\epsilon(h))$ and $B_{1\epsilon} = (x_C, b_{1\epsilon}(h))$, we obtain, from $\delta_2 = H^L(B_\epsilon(h)) - H^L(B_{1\epsilon}(h))$,

$$\frac{\partial \delta_2}{\partial \epsilon}|_{\epsilon=0} = H_y^L(B(h))\frac{\partial b_\epsilon}{\partial \epsilon}|_{\epsilon=0} - H_y^L(B_1(h))\frac{\partial b_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.29}$$

Moreover, since $B = (x_C, b(h))$ and $\delta_4 = H^R(B_{1\epsilon}(h)) - H^R(B(h))$, we obtain

$$\frac{\partial \delta_4}{\partial \epsilon}|_{\epsilon=0} = H_y^R(B_1(h))\frac{\partial b_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.30}$$

From (1.28) and (1.30), we have that

$$\frac{\partial b_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0} = \frac{1}{H_y^R(B_1(h))}\int_{\widehat{BB_1}} g_R dx - f_R dy. \tag{1.31}$$

From (1.27), (1.29) and (1.31), we have that

$$\begin{aligned}\frac{\partial b_\epsilon}{\partial \epsilon}|_{\epsilon=0} =& \frac{1}{H_y^L(B(h))}\int_{\widehat{B_1B}} g_L dx - f_L dy\\ &+ \frac{H_y^L(B_1(h))}{H_y^L(B(h))H_y^R(B_1(h))}\int_{\widehat{BB_1}} g_R dx - f_R dy.\end{aligned}$$

Moreover, we have that

$$\frac{\partial \delta_1}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^R(B(h)) - H_y^L(B(h))\Big)\frac{\partial b_\epsilon}{\partial \epsilon}|_{\epsilon=0}, \tag{1.32}$$

$$\frac{\partial \delta_3}{\partial \epsilon}|_{\epsilon=0} = \Big(H_y^L(B_1(h)) - H_y^R(B_1(h))\Big)\frac{\partial b_{1\epsilon}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.33}$$

Therefore, from (1.6) and (1.26), we have that

$$\begin{aligned}M_1(h) =& \frac{\partial}{\partial \epsilon}\Big(H^R(B_\epsilon(h)) - H^R(B(h))\Big)\\ =& \frac{\partial}{\partial \epsilon}\Big(\delta_1+\delta_2+\delta_3+\delta_4\Big)|_{\epsilon=0}.\end{aligned} \tag{1.34}$$

Replacing (1.27), (1.28), (1.32), (1.33) into (1.34), we obtain the expression (1.8). The rest of the statement from theorem is a straightforward consequence of Implicit Function Theorem. □

Chapter 2

Limit cycles of planar piecewise linear Hamiltonian differential system with two or three zones

In [40], Llibre and Teixeira studied the existence and the number of crossing limit cycles for continuous and discontinuous planar piecewise linear differential systems with two or three zones such that the linear systems involved have a unique singular point which are centers. More precisely, they prove that if the planar piecewise linear differential system is continuous or discontinuous with two zones or continuous with three zones, then it has no limit cycles. Now, if the planar piecewise linear differential system is discontinuous with three zones, then it has at most one limit cycle, and they provided an example with one limit cycle. Motivated by these results, the main purpose of this chapter is to show that if the singular points of the subsystems that define the discontinuous planar piecewise linear differential system are saddles or centers, then the number of crossing limit cycles is also at most one. Moreover, without taking into account the position of the singular points in the zones, we present examples in Section 2.3 of discontinuous planar piecewise linear Hamiltonian differential system with tree zones with this unique limit cycle for all possible combinations of saddles and centers.

2.1 Preliminaries and main results

As we are interested in studying the existence and the number of limit cycles of piecewise linear Hamiltonian differential systems with two or three zones in the plane, we assume the following hypotheses:

(H1) The switching curves are straight lines, and parallel if there are more than one.

(H2) The vector fields which define the piecewise one are linear.

(H3) The vector fields which define the piecewise one are Hamiltonian.

(H4) The vector fields which define the piecewise one have isolated singular points, i.e., centers or saddles.

Moreover, we can classify the systems that satisfy the above hypotheses according to the configuration of their singular points. Thus, denoting the centers by the capital letter C and by S the saddles, in the case of two zones we have systems of the type CC, SC and SS. Thus, CC indicates that the singular points of the linear systems that define the piecewise linear differential system are centers and so on. Following this idea, for three zones, we have the following six classes of piecewise linear Hamiltonian differential systems: CCC, SCC, SCS, CSC, SSS and SSC.

We already know that the case with two zones has been studied in the literature, i.e., the next theorem is already proved.

Theorem 2. *A continuous or discontinuous planar piecewise linear Hamiltonian differential system with two zones separated by a straight line and such that the linear systems that define it have isolated singular points, i.e., centers or saddles, has no limit cycles.*

A proof of Theorem 2 is contained in the proofs of Theorem 2 and 4 from [35]. Alternative proofs can also be found in other papers. See the proof of Theorem 1 from [41] for the cases CC and CS, and see the proof of Theorem 3.4 from [28] for the case SS. We include in Section 2.2 a proof of Theorem 2 just for the sake of completeness.

Assuming hypotheses (H1)–(H4), the main results of this chapter are the following.

Theorem 3. *A continuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e., centers or saddles, has no limit cycles.*

Theorem 4. *A discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e., centers or saddles, has at most one limit cycle.*

Theorems 3 and 4 have been proved for the particular case CCC, see [40]. Theorem 3 has also been proved for the particular case SCS, see the proof of Lemma 11 from [51]. For the other possibilities, as far as we know, the results of Theorems 3 and 4 are new.

In order to prove Theorems 3 and 4, we will make some considerations about the hypotheses (H1), (H2) and (H3). Let $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i = C, L, R$, be the functions $h_C(x, y) = x$, $h_L(x, y) = x+1$ and $h_R(x, y) = x-1$. By the hypothesis (H1), we can assume, by means of rotations, translations and homotheties, that the switching line Σ_C of a piecewise linear system with two zones in the plane is defined as

$$\Sigma_C = h_C^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = 0\}.$$

These straight line decomposes the plane into two regions

$$R_L = \{(x, y) \in \mathbb{R}^2 : x < 0\} \quad \text{and} \quad R_R = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with two zones is given by

$$\begin{aligned} X_L(x, y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), \quad x \le 0, \\ X_R(x, y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), \quad x \ge 0. \end{aligned} \tag{2.1}$$

Note that the Hamiltonian functions that determine the vector field (2.1) are

$$\begin{aligned} H_L(x, y) &= \frac{b_L}{2} y^2 - \frac{c_L}{2} x^2 + a_L xy + \alpha_L y - \beta_L x, \quad x \le 0, \\ H_R(x, y) &= \frac{b_R}{2} y^2 - \frac{c_R}{2} x^2 + a_R xy + \alpha_R y - \beta_R x, \quad x \ge 0. \end{aligned} \tag{2.2}$$

For the case with three zones, the switching lines Σ_L and Σ_R are given by

$$\Sigma_L = h_L^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = -1\},$$

and

$$\Sigma_R = h_R^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = 1\}.$$

This straight lines decomposes the plane into three regions

$$R_L = \{(x, y) \in \mathbb{R}^2 : x < -1\}, \quad R_C = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\},$$

and

$$R_R = \{(x, y) \in \mathbb{R}^2 : x > 1\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with three zones is given by

$$\begin{aligned}
X_L(x, y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), \quad x \leq -1,\\
X_C(x, y) &= (a_C x + b_C y + \alpha_C, c_C x - a_C y + \beta_C), \quad -1 \leq x \leq 1,\\
X_R(x, y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), \quad x \geq 1.
\end{aligned} \tag{2.3}$$

The Hamiltonian functions that determine the vector field (2.3) are

$$\begin{aligned}
H_L(x, y) &= \frac{b_L}{2} y^2 - \frac{c_L}{2} x^2 + a_L xy + \alpha_L y - \beta_L x, \quad x \leq -1,\\
H_C(x, y) &= \frac{b_C}{2} y^2 - \frac{c_C}{2} x^2 + a_C xy + \alpha_C y - \beta_C x, \quad -1 \leq x \leq 1,\\
H_R(x, y) &= \frac{b_R}{2} y^2 - \frac{c_R}{2} x^2 + a_R xy + \alpha_R y - \beta_R x, \quad x \geq 1.
\end{aligned} \tag{2.4}$$

2.2 Proof of Theorems 2-4

This section is devoted to present the proofs of the main results.

Proof of Theorem 2. Consider a discontinuous piecewise linear Hamiltonian vector field with two zones separated by a straight line, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise linear vector field (2.1), with $a_i^2 + b_i c_i \neq 0$, for $i = L, R$. If the piecewise linear vector field has a periodic orbit, then it intersect the straight line $x = 0$ at two points, $(0, y_0)$ and $(0, y_1)$, with $y_1 < y_0$, satisfying

$$\begin{aligned}
H_R(0, y_1) &= H_R(0, y_0),\\
H_L(0, y_0) &= H_L(0, y_1),
\end{aligned}$$

where H_L and H_R are given by (2.2). More precisely, we have the equations

$$\begin{aligned}
-\frac{1}{2}(y_0 - y_1)(b_R(y_0 + y_1) + 2\alpha_R) &= 0,\\
\frac{1}{2}(y_0 - y_1)(b_L(y_0 + y_1) + 2\alpha_L) &= 0.
\end{aligned}$$

As $y_1 < y_0$, if $b_R = 0$ and $\alpha_R \neq 0$ or $b_L = 0$ and $\alpha_L \neq 0$ the above system has no solutions. If $b_R = \alpha_R = 0$ and $b_L \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_L y_1 + 2\alpha_L)/b_L$, with arbitrary y_1. If $b_L = \alpha_L = 0$ and $b_R \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_R y_1 + 2\alpha_R)/b_R$, with arbitrary y_1. If $b_L b_R \neq 0$, then the above system has a solution (y_0, y_1) with $y_1 < y_0$ only when $b_L = b_R = b$ and $\alpha_L = \alpha_R = \alpha$. Moreover, $y_0 = (-by_1 + 2\alpha)/b$ with arbitrary y_1. If $b_R = b_L = \alpha_R = \alpha_L = 0$, then the system has infinitely many solutions. Therefore, the piecewise linear vector field (2.1) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle.

Note that the continuous case is a constraint of the discontinuous one. In fact, the continuous condition is given by

$$X_R(0, y) = X_L(0, y), \quad \forall\, y \in \mathbb{R},$$

which implies

$$a_R = a_L = a, \quad b_R = b_L = b, \quad \alpha_R = \alpha_L = \alpha \quad \text{and} \quad \beta_R = \beta_L = \beta.$$

□

Proof of Theorem 3. Consider a continuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields that define it, have isolated singular points. That is, we have piecewise linear vector field (2.3), with $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, and due to continuity

$$X_R(1, y) = X_C(1, y) \quad \text{and} \quad X_C(-1, y) = X_L(-1, y), \quad \forall\, y \in \mathbb{R}.$$

These equalities imply that

$$a_R = a_C = a_L = a, \quad b_R = b_C = b_L = b, \quad \alpha_R = \alpha_C = \alpha_L = \alpha$$

and

$$\beta_R - \beta_C - c_C + c_R = \beta_L - \beta_C - c_L + c_C = 0.$$

By Theorem 2, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit, then it intersect the straight lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$\begin{aligned}
&H_R(1, y_1) = H_R(1, y_0),\\
&H_C(1, y_0) = H_C(-1, y_3),\\
&H_L(-1, y_3) = H_L(-1, y_2),\\
&H_C(-1, y_2) = H_C(1, y_1),
\end{aligned} \tag{2.5}$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$-\frac{1}{2}(y_0 - y_1)(b(y_0 + y_1) + 2(a + \alpha)) = 0, \tag{2.6}$$

$$a(y_0 + y_3) + \frac{1}{2}(y_0 - y_3)(b(y_0 + y_3) + 2\alpha) - 2\beta_C = 0, \tag{2.7}$$

$$-\frac{1}{2}(y_2 - y_3)(b(y_2 + y_3) - 2(a - \alpha)) = 0, \tag{2.8}$$

$$-a(y_1 + y_2) - \frac{1}{2}(y_1 - y_2)(b(y_1 + y_2) + 2\alpha) + 2\beta_C = 0. \tag{2.9}$$

As $y_1 < y_0$, $y_2 < y_3$ and $a^2 + bc_i \neq 0$, for $i = L, C, R$, if either $b = 0$ and $a + \alpha \neq 0$ or $b = 0$ and $a + \alpha = 0$ the above system has no solutions. If $b \neq 0$, as $y_1 < y_0$ and $y_2 < y_3$, from equation (2.6), we can obtain y_0 as a function of y_1, i.e.,

$$y_0 = \frac{-by_1 - 2(a + \alpha)}{b}. \tag{2.10}$$

Now, from equation (2.8), we can obtain y_2 as a function of y_3, i.e.,

$$y_2 = \frac{-by_3 - 2(\alpha - a)}{b}. \tag{2.11}$$

Substituting (2.10) and (2.11) into equations (2.7) and (2.9), respectively, we obtain a solution (y_0, y_1, y_2, y_3) of system (2.5) satisfying $y_1 < y_0$ and $y_2 < y_3$, given by $(\varphi_1(y_1), y_1, \varphi_2(y_1), \varphi_3(y_1))$, where

$$\begin{aligned}
\varphi_1(y_1) &= \frac{-by_1 - 2(a + \alpha)}{b}, \\
\varphi_2(y_1) &= \frac{a - \alpha + \sqrt{b^2y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b}, \\
\varphi_3(y_1) &= \frac{a - \alpha - \sqrt{b^2y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b},
\end{aligned}$$

with arbitrary y_1. Note that the inequality $b^2y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C \leq 0$ for all $y_1 \in \mathbb{R}$ is not possible. Therefore, piecewise linear vector field (2.3) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle. □

Proof of Theorem 4. Consider a discontinuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields that define it, have isolated singular points. That is, we have piecewise linear vector field (2.3), with $-a_i^2 - b_ic_i \neq 0$, for $i = L, C, R$. By Theorem 2, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit, then it intersect the straight lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$\begin{aligned} &H_R(1, y_1) = H_R(1, y_0), \\ &H_C(1, y_0) = H_C(-1, y_3), \\ &H_L(-1, y_3) = H_L(-1, y_2), \\ &H_C(-1, y_2) = H_C(1, y_1), \end{aligned} \tag{2.12}$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$\frac{1}{2}(y_1 - y_0)(b_R(y_0 + y_1) + 2(a_R + \alpha_R)) = 0, \tag{2.13}$$
$$\frac{1}{2}(y_0 - y_3)(b_C(y_0 + y_3) + 2\alpha_C) - 2\beta_C + a_C(y_0 + y_3) = 0, \tag{2.14}$$
$$\frac{1}{2}(y_3 - y_2)(b_L(y_2 + y_3) - 2(a_L - \alpha_L)) = 0, \tag{2.15}$$
$$\frac{1}{2}(y_2 - y_1)(b_C(y_1 + y_2) + 2\alpha_C) + 2\beta_C - a_C(y_1 + y_2) = 0. \tag{2.16}$$

To determine all the solutions of the above system, restricted to the conditions $y_1 < y_0$, $y_2 < y_3$ and $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, we distinguish two cases. In the first case we assume that $b_R b_L b_C = 0$. For this cases, system (2.13)–(2.16) has no solutions when

- $b_R = 0$ and $a_R + \alpha_R \neq 0$;
- $b_L = 0$ and $a_L - \alpha_L \neq 0$;
- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$;
- $b_R = a_R + \alpha_R = b_C = \alpha_C - a_C = 0$ and $b_L \neq 0$;
- $b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$ and $b_R \neq 0$;
- $b_C = 0$, $b_R b_L \neq 0$ and $b_R\alpha_C(a_L - \alpha_L) + a_C b_R(\alpha_L - a_L) + b_L(a_R + \alpha_R)(a_C + \alpha_C) + 2b_L b_R \beta_C \neq 0$;

and it has infinitely many solutions when

- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = 0$ and $\alpha_C - a_C \neq 0$;
- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = 0$ and $b_C \neq 0$;
- $b_R = a_R + \alpha_R = b_C = 0$, $\alpha_C - a_C \neq 0$ and $b_L \neq 0$;
- $b_R = a_R + \alpha_R = 0$ and $b_L b_C \neq 0$;
- $b_L = a_L - \alpha_L = b_C = 0$, $b_R \neq 0$ and $\alpha_C - a_C \neq 0$;
- $b_L = a_L - \alpha_L = 0$ and $b_R b_C \neq 0$;

- $b_C = 0$, $b_R b_L \neq 0$ and $b_R\alpha_C(a_L-\alpha_L)+a_C b_R(\alpha_L-a_L)+b_L(a_R+\alpha_R)(a_C+\alpha_C)+2b_L b_R\beta_C = 0$.

In the second case, we assume that $b_L b_C b_R \neq 0$. From equation (2.13), we can obtain y_0 as a function of y_1, i.e.,

$$y_0 = \frac{-b_R y_1 - 2(a_R+\alpha_R)}{b_R}. \tag{2.17}$$

Now, from equation (2.15), we can obtain y_2 as a function of y_3, i.e.,

$$y_2 = \frac{-b_L y_3 - 2(\alpha_L - a_L)}{b_L}. \tag{2.18}$$

Substituting (2.17) and (2.18) into equations (2.14) and (2.16), respectively, we obtain the equations of two hyperbolas in the plane y_1y_3, given by

$$\begin{aligned} \frac{(y_1-A)^2}{K} - \frac{(y_3-B)^2}{K} - C = 0,\\ \frac{(y_1-D)^2}{K} - \frac{(y_3-E)^2}{K} - C = 0, \end{aligned} \tag{2.19}$$

with

$$K = \frac{2}{b_C}, \quad A = \frac{b_R(a_C+\alpha_C) - 2b_C(a_R+\alpha_R)}{b_C b_R}, \quad B = \frac{a_C-\alpha_C}{b_C}, \quad C = \frac{2(a_C\alpha_C + b_C\beta_C)}{b_C},$$

$$D = -\frac{(a_C+\alpha_C)}{b_C} \quad \text{and} \quad E = \frac{b_L(\alpha_C-a_C) - 2b_C(\alpha_L-a_L)}{b_C b_L}.$$

Note that system (2.19) is equivalent to the system

$$\begin{aligned} &y_1^2 - 2Ay_1 + A^2 - y_3^2 + 2By_3 - B^2 - KC = 0,\\ &2(A-D)y_1 + 2(E-B)y_3 + D^2 - E^2 + B^2 - A^2 = 0. \end{aligned} \tag{2.20}$$

The above system could have infinitely many solutions (y_1, y_3), for instance when $A = D$ and $B = E$. In this case, the piecewise linear vector field (2.3) has a continuum of periodic orbits, and consequently, it has no limit cycle. Suppose that system (2.20) has finitely many solutions. According to Bezout's Theorem (see [52, p. 144]), if a system of polynomial equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials, that for system (2.20) is two. Therefore, the two above hyperbolas intersect at most two points. Note that, by (2.13)–(2.16), if (y_0, y_1, y_2, y_3) is solution of system (2.12) then (y_1, y_0, y_3, y_2) is also a solution. However, for $y_1 < y_0$ and $y_2 < y_3$ we have at most a single solution. Therefore, the piecewise linear vector field (2.3) can have at most one limit cycle. $\square$

2.3 Examples

In the proof of Theorem 4 it was not necessary to consider the position of the singular points from subsystems that define the piecewise one, i.e., if the singular points are real, virtual or boundary. Therefore, a natural question is if there are examples of discontinuous planar piecewise linear Hamiltonian differential systems with three zones separated by two parallel straight lines with at most one limit cycle, for each position of the singular points. In this section, we answer this question. More precisely, we give examples of piecewise linear Hamiltonian systems of the type CCC, SCC, SCS, CSC, SSS and SSC with exactly one limit cycle, for different positions of the singular points. Denoting a real singular point by the letter R, a boundary singular point by the letter B and a virtual singular point by the letter V, we say that a system is of the type $\mathrm{C_R S_B C_V}$ if the singular point of the right subsystem is a virtual center, of the left subsystem is a real center and of the central subsystem is a boundary saddle.

Example 1. *(Case $C_V C_R C_V$) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with* $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ *and* $\alpha_R = \beta_R = -4$. *We can see that the central vector field from* (2.3) *has a center at the point* $(1/3, -1/3)$, *the right vector field has a center at the point* $(-6, 14)$ *and the left vector field has a center at the point* $(7, -59/16)$. *Therefore, a candidate to limit cycle of vector field* (2.3), *in this case, corresponds to the solution of system* (2.12), *i.e.,*

$$
\begin{aligned}
&(y_1 - y_0)(y_1 + y_0) = 0,\\
&\frac{1}{3}(y_0(2+3y_0) - y_3(2+3y_3) - 4) = 0,\\
&\frac{1}{2}(y_3 - y_2)(8(y_2+y_3) - 5) = 0,\\
&\frac{1}{3}(4 - y_1(2+3y_1) + y_2(2+3y_2)) = 0.
\end{aligned}
$$

After some computation, the unique solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$, *is given by*

$$\left(\frac{31}{48}\sqrt{\frac{1259}{235}}, -\frac{31}{48}\sqrt{\frac{1259}{235}}, \frac{5}{16} - \frac{1}{3}\sqrt{\frac{1259}{235}}, \frac{5}{16} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ *and* $(1, y_0), (1, y_1) \in \Sigma_R$ *are crossing points because*

$$\langle X_L(-1, y_2), (1, 0)\rangle \cdot \langle X_C(-1, y_2), (1, 0)\rangle \approx 1.5518 > 0,$$
$$\langle X_L(-1, y_3), (1, 0)\rangle \cdot \langle X_C(-1, y_3), (1, 0)\rangle \approx 17.4969 > 0,$$
$$\langle X_C(1, y_0), (1, 0)\rangle \cdot \langle X_R(1, y_0), (1, 0)\rangle \approx 10.9315 > 0,$$
$$\langle X_C(1, y_1), (1, 0)\rangle \cdot \langle X_R(1, y_1), (1, 0)\rangle \approx 6.9452 > 0.$$

The orbit $(x_R(t), y_R(t))$ of X_R, such that $(x_R(0), y_R(0)) = (1, y_0)$, is given by

$$x_R(t) = 7\cos(2t) + \frac{31}{48}\sqrt{\frac{1259}{235}}\sin(2t) - 6,$$
$$y_R(t) = \left(\frac{31}{48}\sqrt{\frac{1259}{235}} - 14\right)\cos(2t) - \left(7 + \frac{31}{24}\sqrt{\frac{1259}{235}}\right)\sin(2t) + 14.$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C, such that $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$, is given by

$$x_{C_1}(t) = \frac{2}{3}\cos(2t) + \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}}\right)\sin(2t) + \frac{1}{3},$$
$$y_{C_1}(t) = \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}}\right)\cos(2t) - \frac{1}{3}(1 + 4\cos(t)\sin(t)).$$

The orbit $(x_L(t), y_L(t))$ of X_L, such that $(x_L(0), y_L(0)) = (-1, y_2)$, is given by

$$x_L(t) = -8\cos(2t) - \frac{4}{3}\sqrt{\frac{1259}{235}}\sin(2t) + 7,$$
$$y_L(t) = \left(4 - \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\cos(2t) + \left(4 + \frac{4}{3}\sqrt{\frac{1259}{235}}\right)\cos(t)\sin(t) - \frac{59}{16}.$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, such that $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$, is given by

$$x_{C_2}(t) = -\frac{4}{3}\cos(2t) + \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\sin(2t) + \frac{1}{3},$$
$$y_{C_2}(t) = \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\cos(2t) + \frac{1}{3}(8\cos(t)\sin(t) - 1).$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$, is

$$t_R = \frac{1}{2}\arctan\left(\frac{20832\sqrt{295865}}{25320661}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$, from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$, is

$$t_{C_1} = \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{96(10810 + \sqrt{295865})}{496001}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$*, from* $(-1, y_2) \in \Sigma_L$ *to* $(-1, y_3) \in \Sigma_L$*, is*

$$t_L = \frac{1}{2}\arctan\left(\frac{12\sqrt{295865}}{7201}\right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$*, from* $(-1, y_3) \in \Sigma_L$ *to* $(1, y_0) \in \Sigma_R$*, is*

$$t_{C_2} = -\frac{1}{2}\arctan\left(\frac{96(\sqrt{295865} - 10810)}{496001}\right).$$

The point $(-1, 5/16) \in \Sigma_L$ *is an invisible fold point of* X_L *and the point* $(-1, -1/3) \in \Sigma_L$ *is a visible fold point of* X_C*. They are endpoints of the escaping set in* Σ_L*. The point* $(1, 0) \in \Sigma_R$ *is an invisible fold point of* X_R *and the point* $(1, -1/3) \in \Sigma_R$ *is a visible fold point of* X_C*. They are endpoints of the sliding set in* Σ_R*.*

Using the Mathematica software (see [57]), we can draw the orbits $(x_i(t), y_i(t))$ *for the time* $t \in [0, t_i]$*,* $i = R, L, C_1, C_2$*, i.e., we obtain the limit cycle illustrated in Fig. 2.1 (a). Fig. 2.1 (b) has been made with the help of P5 software (see [24]), and provides the phase portrait of vector field* (2.3) *in this case (the symbol* $\circ$ *indicates a virtual singular point).*

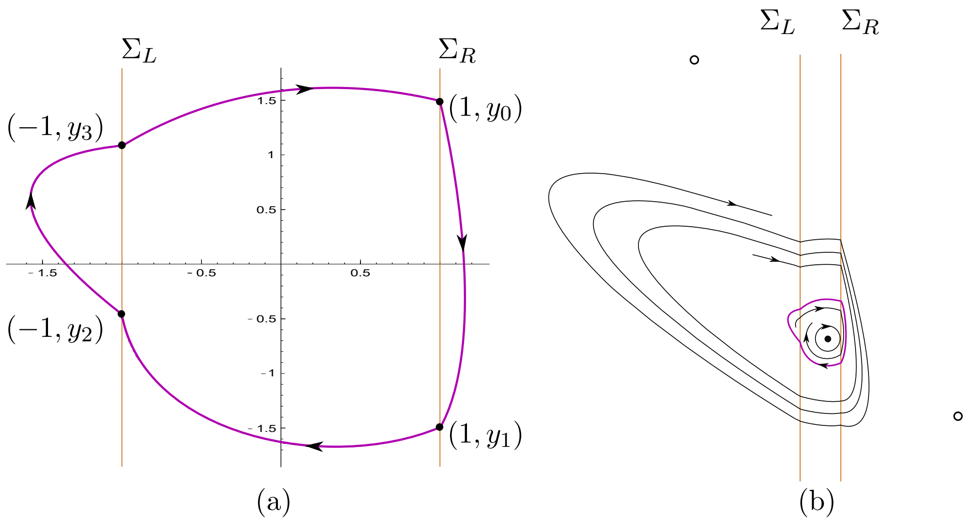


Figure 2.1: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ and $\alpha_R = \beta_R = -4$.

In order to simplify the text, in Table 2.1 we give the parameter values for that the discontinuous planar piecewise linear vector field (2.3) has at most one limit cycle, passing through the points $(1, y_0)$, $(1, y_1)$, $(-1, y_2)$ and $(-1, y_3)$, for the other possible configurations of the type CCC.

Example 2. *(Case* $S_R C_R C_V$*) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with* $a_L = b_L = 1$*,* $\alpha_L = 2/3$*,* $c_L = 35$*,* $\beta_L = 214/3$*,* $a_C = 0$*,* $b_C = 2$*,*

	$C_V C_R C_R$	$C_R C_R C_R$	$C_V C_V C_V$	$C_V C_V C_R$	$C_R C_V C_R$
a_C	0	0	0	0	0
b_C	2	2	2	2	2
c_C	-2	-2	-2	-2	-2
α_C	1	1	1	1	1
β_C	1	1	3	3	3
a_R	8	8	8	8	8
b_R	10	10	10	10	10
c_R	-8	-8	-8	-8	-8
α_R	-8	-8	-8	-8	-8
β_R	10	10	-8	10	10
a_L	16	16	16	16	16
b_L	$\frac{65}{2}$	$\frac{65}{2}$	$\frac{65}{2}$	$\frac{65}{2}$	$\frac{65}{2}$
c_L	-8	-8	-8	-8	-8
α_L	8	8	8	8	8
β_L	0	10	-8	0	$-\frac{9}{2}$
y_0	$\frac{97}{2340}\sqrt{\frac{4873}{2}}$	$\frac{97}{2340}\sqrt{\frac{4873}{2}}$	$\frac{97}{780}\sqrt{\frac{4441}{6}}$	$\frac{97}{780}\sqrt{\frac{4441}{6}}$	$\frac{97}{780}\sqrt{\frac{4441}{6}}$
y_1	$-\frac{97}{2340}\sqrt{\frac{4873}{2}}$	$-\frac{97}{2340}\sqrt{\frac{4873}{2}}$	$-\frac{97}{780}\sqrt{\frac{4441}{6}}$	$-\frac{97}{780}\sqrt{\frac{4441}{6}}$	$-\frac{97}{780}\sqrt{\frac{4441}{6}}$
y_2	$\frac{16}{45}-\frac{1}{36}\sqrt{\frac{4873}{2}}$	$\frac{16}{45}-\frac{1}{36}\sqrt{\frac{4873}{2}}$	$\frac{16}{65}-\frac{1}{12}\sqrt{\frac{4441}{6}}$	$\frac{16}{65}-\frac{1}{12}\sqrt{\frac{4441}{6}}$	$\frac{16}{65}-\frac{1}{12}\sqrt{\frac{4441}{6}}$
y_3	$\frac{16}{45}+\frac{1}{36}\sqrt{\frac{4873}{2}}$	$\frac{16}{45}+\frac{1}{36}\sqrt{\frac{4873}{2}}$	$\frac{16}{65}+\frac{1}{12}\sqrt{\frac{4441}{6}}$	$\frac{16}{65}+\frac{1}{12}\sqrt{\frac{4441}{6}}$	$\frac{16}{65}+\frac{1}{12}\sqrt{\frac{4441}{6}}$

Table 2.1: Parameter values for that vector field (2.3) of the type CCC has one limit cycle.

$\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ *and* $c_R = -10$. *We can see that the central vector field from* (2.3) *has a center at the point* $(1/3, -1/3)$, *the right vector field has a center at the point* $(-6, 14)$ *and the left vector field has a saddle at the point* $(-2, 4/3)$. *Therefore, a candidate to limit cycle of system* (2.3), *in this case, corresponds to the solution of system* (2.12), *i.e.,*

$$
\begin{aligned}
&(y_1 - y_0)(y_1 + y_0) = 0,\\
&\frac{1}{3}(y_0(2+3y_0) - y_3(2+3y_3) - 4) = 0,\\
&\frac{1}{6}(y_3 - y_2)(3(y_2+y_3) - 2) = 0,\\
&\frac{1}{3}(4 - y_1(2+3y_1) + y_2(2+3y_2)) = 0.
\end{aligned}
$$

The only solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$, *is given by*

$$\left(\frac{2\sqrt{5}}{3}, -\frac{2\sqrt{5}}{3}, \frac{1-\sqrt{5}}{3}, \frac{1+\sqrt{5}}{3}\right).$$

The points $(-1,y_2),(-1,y_3)\in\Sigma_L$ *and* $(1,y_0),(1,y_1)\in\Sigma_R$ *are crossing points because*

$$\begin{aligned}
&\langle X_L(-1,y_2),(1,0)\rangle\cdot\langle X_C(-1,y_2),(1,0)\rangle\approx 0.1173>0,\\
&\langle X_L(-1,y_3),(1,0)\rangle\cdot\langle X_C(-1,y_3),(1,0)\rangle\approx 2.1049>0,\\
&\langle X_C(1,y_0),(1,0)\rangle\cdot\langle X_R(1,y_0),(1,0)\rangle\approx 10.8765>0,\\
&\langle X_C(1,y_1),(1,0)\rangle\cdot\langle X_R(1,y_1),(1,0)\rangle\approx 6.9013>0.
\end{aligned}$$

The orbit $(x_R(t),y_R(t))$ *of* X_R*, such that* $(x_R(0),y_R(0))=(1,y_0)$*, is given by*

$$\begin{aligned}
x_R(t)&=7\cos(2t)+\frac{2\sqrt{5}}{3}\sin(2t)-6,\\
y_R(t)&=\frac{2}{3}(\sqrt{5}-21)\cos(2t)-\left(7+\frac{4\sqrt{5}}{3}\right)\sin(2t)+14.
\end{aligned}$$

The orbit $(x_{C_1}(t),y_{C_1}(t))$ *of* X_C*, such that* $(x_{C_1}(0),y_{C_1}(0))=(1,y_1)$*, is given by*

$$\begin{aligned}
x_{C_1}(t)&=\frac{1}{3}\big(2\cos(2t)+\big(1-2\sqrt{5}\big)\sin(2t)+1\big),\\
y_{C_1}(t)&=\frac{1}{3}\big(\big(1-2\sqrt{5}\big)\cos(2t)-2\sin(2t)-1\big).
\end{aligned}$$

The orbit $(x_L(t),y_L(t))$ *of* X_L*, such that* $(x_L(0),y_L(0))=(-1,y_2)$*, is given by*

$$\begin{aligned}
x_L(t)&=\frac{1}{36}\big(18+\sqrt{5}\big)e^{-6t}+\frac{1}{36}\big(18-\sqrt{5}\big)e^{6t}-2,\\
y_L(t)&=-\frac{1}{36}\big(126+7\sqrt{5}\big)e^{-6t}+\frac{1}{36}\big(90-5\sqrt{5}\big)e^{6t}+\frac{4}{3}.
\end{aligned}$$

The orbit $(x_{C_2}(t),y_{C_2}(t))$ *of* X_C*, such that* $(x_{C_2}(0),y_{C_2}(0))=(-1,y_3)$*, is given by*

$$\begin{aligned}
x_{C_2}(t)&=\frac{1}{3}\big(1-4\cos(2t)+\big(2+\sqrt{5}\big)\sin(2t)\big),\\
y_{C_2}(t)&=\frac{1}{3}\big((2+\sqrt{5})\cos(2t)+4\sin(2t)-1\big).
\end{aligned}$$

The flight time of the orbit $(x_R(t),y_R(t))$*, from* $(1,y_0)\in\Sigma_R$ *to* $(1,y_1)\in\Sigma_R$*, is*

$$t_R=\frac{1}{2}\arctan\left(\frac{84\sqrt{5}}{421}\right).$$

The flight time of the orbit $(x_{C_1}(t),y_{C_1}(t))$*, from* $(1,y_1)\in\Sigma_R$ *to* $(-1,y_2)\in\Sigma_L$*, is*

$$t_{C_1}=\frac{\pi}{2}-\frac{1}{2}\arctan(2).$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$, is

$$t_L = \frac{1}{6} \log \left(\frac{329 + 36\sqrt{5}}{319} \right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$, from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$, is

$$t_{C_2} = \frac{1}{2} \arctan(2).$$

The point $(-1, 1/3) \in \Sigma_L$ is an invisible fold point of X_L and the point $(-1, -1/3) \in \Sigma_L$ is a visible fold point of X_C. They are endpoints of the escaping set in Σ_L. The point $(1, 0) \in \Sigma_R$ is an invisible fold point of X_R and the point $(1, -1/3) \in \Sigma_R$ is a visible fold point of X_C. They are endpoints of the sliding set in Σ_R.

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e., we obtain the limit cycle given in Fig. 2.2 (a). Fig. 2.2 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.

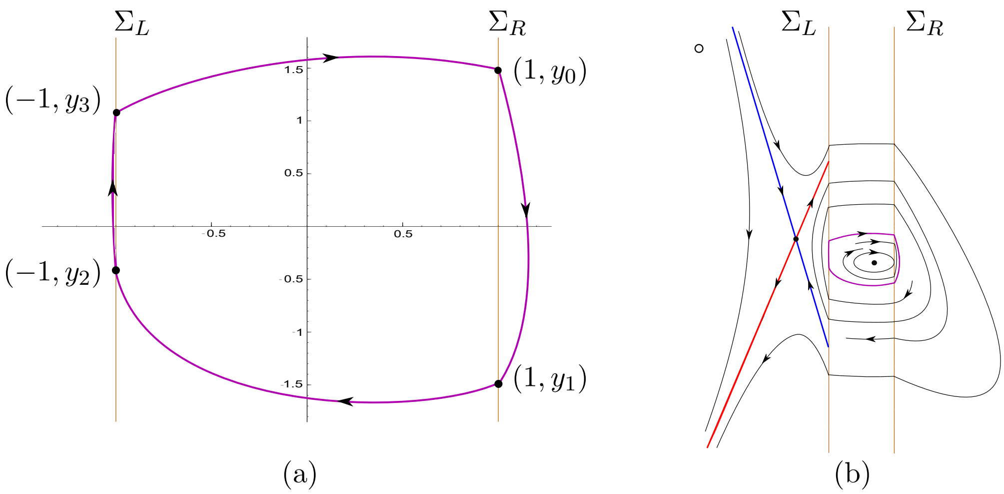


Figure 2.2: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 2/3$, $c_L = 35$, $\beta_L = 214/3$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ and $c_R = -10$.

Example 3. *(Case $S_R C_R S_R$) Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$. We can see that the central vector field from (2.3) has a center at the point $(1/2, -1/2)$, the right vector field has a saddle at the point $(2, -1)$ and the left vector field has a saddle at the point $(-2, 7/5)$. Therefore, a candidate to limit cycle of system (2.3), in this case, corresponds to the solution of* *system* (2.12), *i.e.,*

$$\begin{aligned}
&\frac{1}{2}(y_1+y_0)(y_1-y_0)=0,\\
&y_0^2+y_0-y_3(1+y_3)-2=0,\\
&\frac{1}{10}(y_3-y_2)(5(y_2+y_3)-4)=0,\\
&y_2^2+y_2-y_1(1+y_1)+2=0.
\end{aligned}$$

The only solution (y_0,y_1,y_2,y_3) *of the above system, satisfying the condition* $y_1<y_0$ *and* $y_2<y_3$, *is given by*

$$\left(\frac{18}{5}\sqrt{\frac{2}{7}},-\frac{18}{5}\sqrt{\frac{2}{7}},\frac{2}{5}-2\sqrt{\frac{2}{7}},\frac{2}{5}+2\sqrt{\frac{2}{7}}\right).$$

The points $(-1,y_2),(-1,y_3)\in\Sigma_L$ *and* $(1,y_0),(1,y_1)\in\Sigma_R$ *are crossing points because*

$$\begin{aligned}
&\langle X_L(-1,y_2),(1,0)\rangle\cdot\langle X_C(-1,y_2),(1,0)\rangle\approx 0.3614>0,\\
&\langle X_L(-1,y_3),(1,0)\rangle\cdot\langle X_C(-1,y_3),(1,0)\rangle\approx 4.21>0,\\
&\langle X_C(1,y_0),(1,0)\rangle\cdot\langle X_R(1,y_0),(1,0)\rangle\approx 9.33>0,\\
&\langle X_C(1,y_1),(1,0)\rangle\cdot\langle X_R(1,y_1),(1,0)\rangle\approx 5.4814>0.
\end{aligned}$$

The orbit $(x_R(t),y_R(t))$ *of* X_R, *such that* $(x_R(0),y_R(0))=(1,y_0)$, *is given by*

$$\begin{aligned}
x_R(t)&=-\frac{\left(70+9\sqrt{14}\right)}{140}e^{-4t}+\frac{\left(9\sqrt{14}-70\right)}{140}e^{4t}+2,\\
y_R(t)&=\frac{\left(350+45\sqrt{14}\right)}{140}e^{-4t}+\frac{\left(27\sqrt{14}-210\right)}{140}e^{4t}-1.
\end{aligned}$$

The orbit $(x_{C_1}(t),y_{C_1}(t))$ *of* X_C, *such that* $(x_{C_1}(0),y_{C_1}(0))=(1,y_1)$, *is given by*

$$\begin{aligned}
x_{C_1}(t)&=\cos(t)\Big(\cos(t)+\sin(t)-\frac{36}{5}\sqrt{\frac{2}{7}}\sin(t)\Big),\\
y_{C_1}(t)&=\frac{\left(35-36\sqrt{14}\right)}{70}\cos(2t)-\cos(t)\sin(t)-\frac{1}{2}.
\end{aligned}$$

The orbit $(x_L(t),y_L(t))$ *of* X_L, *such that* $(x_L(0),y_L(0))=(-1,y_2)$, *is given by*

$$\begin{aligned}
x_L(t)&=\frac{\left(21+\sqrt{14}\right)}{42}e^{-6t}-\frac{\left(\sqrt{14}-21\right)}{42}e^{6t}-2,\\
y_L(t)&=-\frac{\left(735+35\sqrt{14}\right)}{210}e^{-6t}-\frac{\left(25\sqrt{14}-525\right)}{210}e^{6t}+\frac{7}{5}.
\end{aligned}$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ *of* X_C*, such that* $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$*, is given by*

$$x_{C_2}(t) = -\frac{3}{2}\cos(2t) + \frac{9}{5}\cos(t)\sin(t) + 2\sqrt{\frac{2}{7}}\sin(2t) + \frac{1}{2},$$
$$y_{C_2}(t) = \left(\frac{9}{10} + 2\sqrt{\frac{2}{7}}\right)\cos(2t) + 3\cos(t)\sin(t) - \frac{1}{2}.$$

The flight time of the orbit $(x_R(t), y_R(t))$*, from* $(1, y_0) \in \Sigma_R$ *to* $(1, y_1) \in \Sigma_R$*, is*

$$t_R = \frac{1}{4}\log\left(\frac{431 + 90\sqrt{14}}{269}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$*, from* $(1, y_1) \in \Sigma_R$ *to* $(-1, y_2) \in \Sigma_L$*, is*

$$t_{C_1} = \arctan\left(\frac{2(7 + 4\sqrt{14})}{35}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$*, from* $(-1, y_2) \in \Sigma_L$ *to* $(-1, y_3) \in \Sigma_L$*, is*

$$t_L = \frac{1}{6}\log\left(\frac{65 + 6\sqrt{14}}{61}\right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$*, from* $(-1, y_3) \in \Sigma_L$ *to* $(1, y_0) \in \Sigma_R$*, is*

$$t_{C_2} = \arctan\left(\frac{2(-7 + 4\sqrt{14})}{35}\right).$$

The point $(-1, 2/5) \in \Sigma_L$ *is an invisible fold point of* X_L *and the point* $(-1, -1/2) \in \Sigma_L$ *is a visible fold point of* X_C*. They are endpoints of the escaping set in* Σ_L*. The point* $(1, 0) \in \Sigma_R$ *is an invisible fold point of* X_R *and the point* $(1, -1/2) \in \Sigma_R$ *is a visible fold point of* X_C*. They are endpoints of the sliding set in* Σ_R*.*

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ *for the time* $t \in [0, t_i]$*,* $i = R, L, C_1, C_2$*, i.e., we obtain the limit cycle illustrated in Fig. 2.3 (a). Fig. 2.3 (b) has been made with the help of P5 software, and provides the phase portrait of vector field* (2.3) *in this case.*

In Table 2.2 we give the parameter values for that the discontinuous planar piecewise linear vector field (2.3) has at most one limit cycle, passing through the points $(1, y_0)$, $(1, y_1)$, $(-1, y_2)$ and $(-1, y_3)$, for the others possibles configurations of the type SCC and SCS.

Example 4. *(Case* $C_V S_R C_V$*) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with* $a_L = 4$*,* $b_L = 8$*,* $\alpha_L = 2$*,* $c_L = -5/2$*,* $\beta_L = 5/2$*,* $a_C = 2/5$*,* $b_C = 24/5$*,*

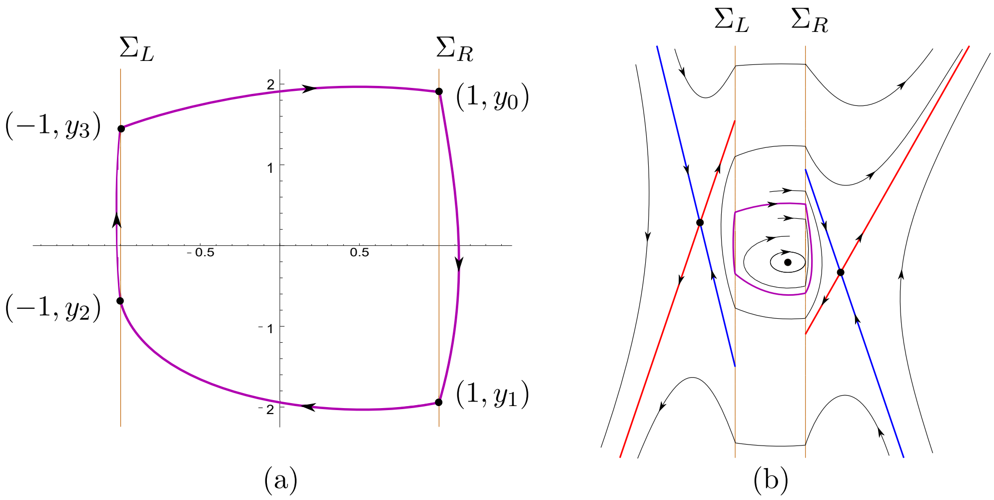


Figure 2.3: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$.

	$S_RC_RC_R$	$S_RC_VC_V$	$S_RC_VC_R$	$S_RC_VS_R$
a_C	0	0	0	0
b_C	2	2	2	2
c_C	-2	-2	-2	-2
α_C	1	1	1	1
β_C	1	3	3	3
a_R	8	8	8	1
b_R	10	10	10	1
c_R	-8	-8	-8	15
α_R	-8	-8	-8	-1
β_R	10	-8	10	-31
a_L	1	1	1	1
b_L	1	1	1	1
c_L	35	35	35	35
α_L	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
β_L	$\frac{357}{5}$	$\frac{357}{5}$	$\frac{357}{5}$	$\frac{357}{5}$
y_0	$\frac{18}{5}\sqrt{\frac{2}{7}}$	$\frac{9}{5}\sqrt{\frac{41}{14}}$	$\frac{9}{5}\sqrt{\frac{41}{14}}$	$\frac{9}{5}\sqrt{\frac{41}{14}}$
y_1	$-\frac{18}{5}\sqrt{\frac{2}{7}}$	$-\frac{9}{5}\sqrt{\frac{41}{14}}$	$-\frac{9}{5}\sqrt{\frac{41}{14}}$	$-\frac{9}{5}\sqrt{\frac{41}{14}}$
y_2	$\frac{2}{5}-2\sqrt{\frac{2}{7}}$	$\frac{2}{5}-\sqrt{\frac{41}{14}}$	$\frac{2}{5}-\sqrt{\frac{41}{14}}$	$\frac{2}{5}-\sqrt{\frac{41}{14}}$
y_3	$\frac{2}{5}+2\sqrt{\frac{2}{7}}$	$\frac{2}{5}+\sqrt{\frac{41}{14}}$	$\frac{2}{5}+\sqrt{\frac{41}{14}}$	$\frac{2}{5}+\sqrt{\frac{41}{14}}$

Table 2.2: Parameter values for that vector field (2.3) of the type SCC and SCS has one limit cycle.

$\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ *and* $\alpha_R = c_R = \beta_R = -8$. *We can see that the central vector field from* (2.3) *has a saddle at the point* $(1/2, 1/3)$, *the right vector field* *has a center at the point* $(-9, 8)$ *and the left vector field has a center at the point* $(-7, 15/4)$. *Therefore, a candidate to limit cycle of system* (2.3), *in this case, corresponds to the solution of system* (2.12), *i.e.,*

$$\begin{aligned}
&5(y_1 - y_0)(y_1 + y_0) = 0,\\
&\frac{1}{15}(3y_0(12y_0 - 7) + 33y_3 - 36y_3^2 + 8) = 0,\\
&2(y_2 - 2y_2^2 + y3(2y_3 - 1)) = 0,\\
&\frac{1}{15}(3y_1(7 - 12y_1) - 33y_2 + 36y_2^2 - 8) = 0.
\end{aligned}$$

The only solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$, *is given by*

$$\left(\frac{5}{12}\sqrt{\frac{7}{3}}, -\frac{5}{12}\sqrt{\frac{7}{3}}, \frac{1}{4} - \frac{7\sqrt{21}}{36}, \frac{1}{4} + \frac{7\sqrt{21}}{36}\right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ *and* $(1, y_0), (1, y_1) \in \Sigma_R$ *are crossing points because*

$$\begin{aligned}
&\langle X_L(-1, y_2), (1, 0)\rangle \cdot \langle X_C(-1, y_2), (1, 0)\rangle \approx 37.617 > 0,\\
&\langle X_L(-1, y_3), (1, 0)\rangle \cdot \langle X_C(-1, y_3), (1, 0)\rangle \approx 23.360 > 0,\\
&\langle X_C(1, y_0), (1, 0)\rangle \cdot \langle X_R(1, y_0), (1, 0)\rangle \approx 10.534 > 0,\\
&\langle X_C(1, y_1), (1, 0)\rangle \cdot \langle X_R(1, y_1), (1, 0)\rangle \approx 28.355 > 0.
\end{aligned}$$

The orbit $(x_R(t), y_R(t))$ *of* X_R, *such that* $(x_R(0), y_R(0)) = (1, y_0)$, *is given by*

$$\begin{aligned}
x_R(t) &= 10\cos(4t) + \frac{25}{24}\sqrt{\frac{7}{3}}\sin(4t) - 9,\\
y_R(t) &= \left(\frac{5}{12}\sqrt{\frac{7}{3}} - 8\right)\cos(4t) - \left(\frac{5}{6}\sqrt{\frac{7}{3}} + 4\right)\sin(4t) + 8.
\end{aligned}$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ *of* X_C, *such that* $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$, *is given by*

$$\begin{aligned}
x_{C_1}(t) &= \frac{\left(18 + 5\sqrt{21}\right)}{30}e^{-2t} - \frac{\left(3 + 5\sqrt{21}\right)}{30}e^{2t} + \frac{1}{2},\\
y_{C_1}(t) &= -\frac{\left(54 + 15\sqrt{21}\right)}{180}e^{-2t} - \frac{\left(6 + 10\sqrt{21}\right)}{180}e^{2t} + \frac{1}{3}.
\end{aligned}$$

The orbit $(x_L(t), y_L(t))$ *of* X_L, *such that* $(x_L(0), y_L(0)) = (-1, y_2)$, *is given by*

$$\begin{aligned}
x_L(t) &= -8\cos(2t) - \frac{7}{3}\sqrt{\frac{7}{3}}\sin(2t) + 7,\\
y_L(t) &= \left(4 - \frac{7}{12}\sqrt{\frac{7}{3}}\right)\cos(2t) + \left(2 + \frac{7}{6}\sqrt{\frac{7}{3}}\right)\sin(2t) - \frac{15}{4}.
\end{aligned}$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ *of* X_C*, such that* $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$*, is given by*

$$
\begin{aligned}
x_{C_2}(t) &= -\frac{\left(15+7\sqrt{21}\right)}{30}e^{-2t} - \frac{\left(30-7\sqrt{21}\right)}{30}e^{2t} + \frac{1}{2},\\
y_{C_2}(t) &= \frac{\left(45+21\sqrt{21}\right)}{180}e^{-2t} - \frac{\left(60-14\sqrt{21}\right)}{180}e^{2t} + \frac{1}{3}.
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$*, from* $(1, y_0) \in \Sigma_R$ *to* $(1, y_1) \in \Sigma_R$*, is*

$$t_R = \frac{1}{4}\arctan\left(\frac{480\sqrt{21}}{6737}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$*, from* $(1, y_1) \in \Sigma_R$ *to* $(-1, y_2) \in \Sigma_L$*, is*

$$t_{C_1} = \frac{1}{2}\log\left(\frac{5+\sqrt{21}}{4}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$*, from* $(-1, y_2) \in \Sigma_L$ *to* $(-1, y_3) \in \Sigma_L$*, is*

$$t_L = \frac{1}{2}\arctan\left(\frac{336\sqrt{21}}{1385}\right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$*, from* $(-1, y_3) \in \Sigma_L$ *to* $(1, y_0) \in \Sigma_R$*, is*

$$t_{C_2} = \frac{1}{2}\log(5+\sqrt{21}).$$

The point $(-1, 1/4) \in \Sigma_L$ *is an invisible fold point of* X_L *and the point* $(-1, 11/24) \in \Sigma_L$ *is a visible fold point of* X_C*. They are endpoints of the sliding set in* Σ_L*. The point* $(1, 0) \in \Sigma_R$ *is an invisible fold point of* X_R *and the point* $(1, 7/24) \in \Sigma_R$ *is a visible fold point of* X_C*. They are endpoints of the escaping set in* Σ_R*.*

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ *for the time* $t \in [0, t_i]$*,* $i = R, L, C_1, C_2$*, i.e., we obtain the limit cycle illustrated in Fig. 2.4 (a). Fig. 2.4 (b) has been made with the help of P5 software, and provides the phase portrait of vector field* (2.3) *in this case.*

In Table 2.3 we give the parameter values for that the discontinuous planar piecewise linear vector field (2.3) has at most one limit cycle, passing through the points $(1, y_0)$, $(1, y_1)$, $(-1, y_2)$ and $(-1, y_3)$, for the others possibles configurations of the type CSC.

Example 5. *(Case* $S_R S_R S_R$*) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with* $a_L = \alpha_L = -2/3$*,* $b_L = 4/3$*,* $c_L = 8/3$*,* $\beta_L = 35/3$*,* $a_C = 2/11$*,* $b_C = 120/11$*,* $\alpha_C = -41/11$*,* $c_C = 4/11$*,* $\beta_C = -4/33$*,* $a_R = -2/11$*,* $b_R = 4/11$*,* $\alpha_R = 1/5$*,* $c_R = 120/11$ *and*

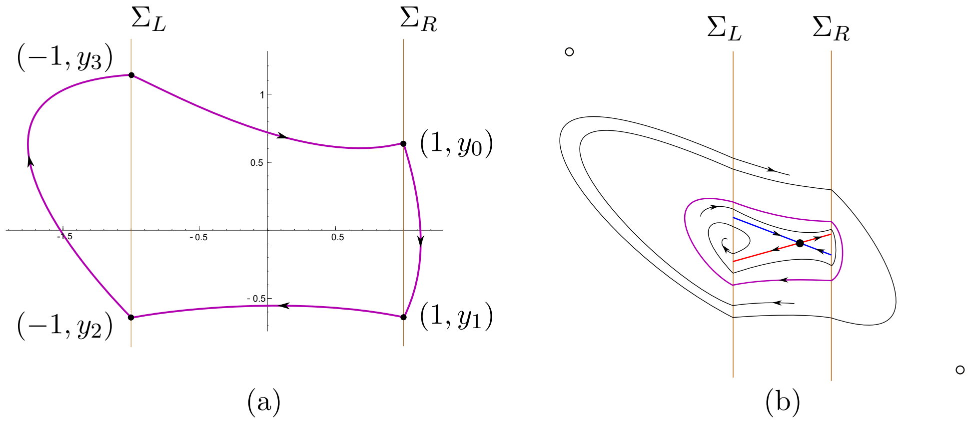


Figure 2.4: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 2$, $c_L = -5/2$, $\beta_L = 5/2$, $a_C = 2/5$, $b_C = 24/5$, $\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ and $\alpha_R = c_R = \beta_R = -8$.

	$C_V S_R C_R$	$C_R S_R C_R$	$C_V S_V C_V$	$C_V S_V C_R$	$C_R S_V C_R$
a_C	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
b_C	$\frac{24}{5}$	$\frac{24}{5}$	$\frac{24}{5}$	$\frac{24}{5}$	$\frac{24}{5}$
c_C	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
α_C	$-\frac{9}{5}$	$-\frac{9}{5}$	$-\frac{9}{5}$	$-\frac{9}{5}$	$-\frac{9}{5}$
β_C	$-\frac{4}{15}$	$-\frac{4}{15}$	-2	-2	-2
a_R	8	8	8	8	8
b_R	10	10	10	10	10
c_R	-8	-8	-8	-8	-8
α_R	-8	-8	-8	-8	-8
β_R	9	9	-8	9	9
a_L	4	8	4	4	4
b_L	8	8	8	8	8
c_L	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{5}{2}$
α_L	2	2	2	2	2
β_L	$\frac{5}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$
y_0	$\frac{5}{12}\sqrt{\frac{7}{3}}$	$\frac{5}{12}\sqrt{\frac{7}{3}}$	$\frac{5}{12}\sqrt{11}$	$\frac{5}{12}\sqrt{11}$	$\frac{5}{12}\sqrt{11}$
y_1	$-\frac{5}{12}\sqrt{\frac{7}{3}}$	$-\frac{5}{12}\sqrt{\frac{7}{3}}$	$-\frac{5}{12}\sqrt{11}$	$-\frac{5}{12}\sqrt{11}$	$\frac{5}{12}\sqrt{11}$
y_2	$\frac{1}{4} - \frac{7}{36}\sqrt{21}$	$\frac{1}{4} - \frac{7}{36}\sqrt{21}$	$\frac{1}{4} - \frac{7}{12}\sqrt{11}$	$\frac{1}{4} - \frac{7}{12}\sqrt{11}$	$\frac{1}{4} - \frac{7}{12}\sqrt{11}$
y_3	$\frac{1}{4} + \frac{7}{36}\sqrt{21}$	$\frac{1}{4} + \frac{7}{36}\sqrt{21}$	$\frac{1}{4} + \frac{7}{12}\sqrt{11}$	$\frac{1}{4} + \frac{7}{12}\sqrt{11}$	$\frac{1}{4} + \frac{7}{12}\sqrt{11}$

Table 2.3: Parameter values for that vector field (2.3) of the type CSC has one limit cycle.

$\beta_R = -749/55$. *We can see that the central vector field from* (2.3) *has a saddle at the point* $(1/2, 1/3)$, *the right vector field has a saddle at the point* $(1509/1210, 89/1210)$ *and the left vector field has a saddle at the point* $(-4, -3/2)$. *Therefore, a candidate to limit cycle of system* (2.3), *in this case, corresponds to the solution of system* (2.12)*, i.e.,*

$$\begin{aligned}
&\frac{1}{55}(10y_1^2 + y_1 - y_0(1 + 10y_0)) = 0,\\
&\frac{1}{33}(9y_0(20y_0 - 13) + 3y_3(43 - 60y_3) + 8) = 0,\\
&\frac{2}{3}(y_3 - y_2)(y_3 + y_2) = 0,\\
&\frac{1}{33}(9y_1(13 - 20y_1) + 3y_2(60y_2 - 43) - 8) = 0.
\end{aligned}$$

The only solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$*, is given by*

$$\left(\frac{43\sqrt{26} - 12}{240}, -\frac{43\sqrt{26} + 12}{240}, -\frac{3}{8}\sqrt{\frac{13}{2}}, \frac{3}{8}\sqrt{\frac{13}{2}}\right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ *and* $(1, y_0), (1, y_1) \in \Sigma_R$ *are crossing points because*

$$\begin{aligned}
&\langle X_L(-1, y_2), (1, 0)\rangle \cdot \langle X_C(-1, y_2), (1, 0)\rangle \approx 18.2786 > 0,\\
&\langle X_L(-1, y_3), (1, 0)\rangle \cdot \langle X_C(-1, y_3), (1, 0)\rangle \approx 8.3123 > 0,\\
&\langle X_C(1, y_0), (1, 0)\rangle \cdot \langle X_R(1, y_0), (1, 0)\rangle \approx 1.9518 > 0,\\
&\langle X_C(1, y_1), (1, 0)\rangle \cdot \langle X_R(1, y_1), (1, 0)\rangle \approx 4.6699 > 0.
\end{aligned}$$

The orbit $(x_R(t), y_R(t))$ *of* X_R*, such that* $(x_R(0), y_R(0)) = (1, y_0)$*, is given by*

$$\begin{aligned}
x_R(t) &= -\frac{\left(3588 + 473\sqrt{26}\right)}{29040}e^{-2t} + \frac{\left(473\sqrt{26} - 3588\right)}{29040}e^{2t} + \frac{1509}{1210},\\
y_R(t) &= \frac{\left(17940 + 2365\sqrt{26}\right)}{29040}e^{-2t} + \frac{\left(2838\sqrt{26} - 21528\right)}{29040}e^{2t} + \frac{89}{1210}.
\end{aligned}$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ *of* X_C*, such that* $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$*, is given by*

$$\begin{aligned}
x_{C_1}(t) &= \frac{\left(112 + 43\sqrt{26}\right)}{88}e^{-2t} - \frac{\left(68 + 43\sqrt{26}\right)}{88}e^{2t} + \frac{1}{2},\\
y_{C_1}(t) &= -\frac{\left(672 + 258\sqrt{26}\right)}{2640}e^{-2t} - \frac{\left(340 + 215\sqrt{26}\right)}{2640}e^{2t} + \frac{1}{3}.
\end{aligned}$$

The orbit $(x_L(t), y_L(t))$ *of* X_L*, such that* $(x_L(0), y_L(0)) = (-1, y_2)$*, is given by*

$$\begin{aligned}
x_L(t) &= \frac{\left(24 + \sqrt{26}\right)}{16}e^{-2t} - \frac{\left(\sqrt{26} - 24\right)}{16}e^{2t} - 4,\\
y_L(t) &= -\frac{\left(24 + \sqrt{26}\right)}{16}e^{-2t} - \frac{\left(\sqrt{26} - 24\right)}{8}e^{2t} - \frac{3}{2}.
\end{aligned}$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, such that $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$, is given by

$$x_{C_2}(t) = \frac{(20 - 45\sqrt{26})}{88}e^{-2t} + \frac{(45\sqrt{26} - 152)}{88}e^{2t} + \frac{1}{2},$$
$$y_{C_2}(t) = \frac{(54\sqrt{26} - 24)}{528}e^{-2t} + \frac{(45\sqrt{26} - 152)}{528}e^{2t} + \frac{1}{3}.$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$, is

$$t_R = \frac{1}{2}\log\left(\frac{718873 + 130548\sqrt{26}}{271415}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$, from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$, is

$$t_{C_1} = \log\left(\frac{\sqrt{23 + 2\sqrt{26}}}{5}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$, is

$$t_L = \frac{1}{2}\log\left(\frac{301 + 24\sqrt{26}}{275}\right).$$

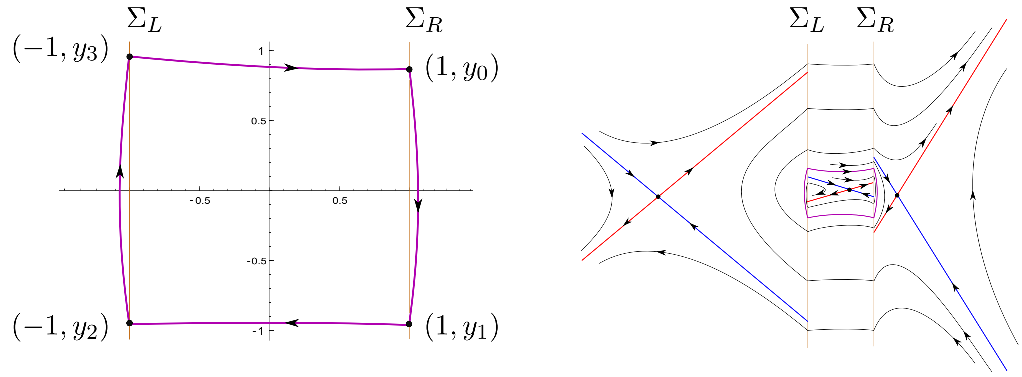
Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$, from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$, is

$$t_{C_2} = \frac{1}{2}\log\left(\frac{23 + 2\sqrt{26}}{17}\right).$$

The point $(-1, 0) \in \Sigma_L$ is an invisible fold point of X_L and the point $(-1, 43/120) \in \Sigma_L$ is a invisible fold point of X_C. They are endpoints of the sliding set in Σ_L. The point $(1, -1/20) \in \Sigma_R$ is an invisible fold point of X_R and the point $(1, 13/40) \in \Sigma_R$ is a visible fold point of X_C. They are endpoints of the escaping set in Σ_R.

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e., we obtain the limit cycle illustrated in Fig. 2.5 (a). Fig. 2.5 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) *in this case.*

Example 6. *(Case $S_RS_RC_V$) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$. We can see that the central vector field from* (2.3) *has a saddle at the point $(1/2, 1/3)$, the right vector field has a center at the point $(-17/2, 15/2)$ and the left vector field has a saddle at the point $(-4, -3/2)$. Therefore, a candidate to limit cycle of system* (2.3)*, in this case, corresponds to*



(a) (b)

Figure 2.5: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = -2/11$, $b_R = 4/11$, $\alpha_R = 1/5$, $c_R = 120/11$ and $\beta_R = -749/55$.

the solution of system (2.12)*, i.e.,*

$$\begin{aligned}
&5y_1^2 + y_1 - y_0(1 + 5y_0) = 0,\\
&\frac{1}{33}(9y_0(20y_0 - 13) + 3y_3(43 - 60y_3) + 8) = 0,\\
&\frac{2}{3}(y_3 - y_2)(y_3 + y_2) = 0,\\
&\frac{1}{33}(9y_1(13 - 20y_1) + 3y_2(60y_2 - 43) - 8) = 0.
\end{aligned}$$

Making a calculation, the only solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$*, is given by*

$$\left(\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{17}{8}\sqrt{\frac{43}{470}}, \frac{17}{8}\sqrt{\frac{43}{470}}\right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ *and* $(1, y_0), (1, y_1) \in \Sigma_R$ *are crossing points because*

$$\begin{aligned}
&\langle X_L(-1, y_2), (1, 0)\rangle \cdot \langle X_C(-1, y_2), (1, 0)\rangle \approx 9.3593 > 0,\\
&\langle X_L(-1, y_3), (1, 0)\rangle \cdot \langle X_C(-1, y_3), (1, 0)\rangle \approx 2.6591 > 0,\\
&\langle X_C(1, y_0), (1, 0)\rangle \cdot \langle X_R(1, y_0), (1, 0)\rangle \approx 6.9128 > 0,\\
&\langle X_C(1, y_1), (1, 0)\rangle \cdot \langle X_R(1, y_1), (1, 0)\rangle \approx 57.1644 > 0.
\end{aligned}$$

The orbit $(x_R(t), y_R(t))$ *of* X_R*, such that* $(x_R(0), y_R(0)) = (1, y_0)$*, is given by*

$$\begin{aligned}
x_R(t) &= \frac{19}{2}\cos(4t) + \frac{43}{48}\sqrt{\frac{215}{94}}\sin(4t) - \frac{17}{2},\\
y_R(t) &= \left(\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{38}{5}\right)\cos(4t) - \left(\frac{19}{5} + \frac{43}{12}\sqrt{\frac{43}{470}}\right)\sin(4t) + \frac{15}{2}.
\end{aligned}$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ *of* X_C*, such that* $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$*, is given by*

$$\begin{aligned}
x_{C_1}(t) &= \frac{\left(5828 + 43\sqrt{20210}\right)}{4136} e^{-2t} - \frac{\left(3760 + 43\sqrt{20210}\right)}{4136} e^{2t} + \frac{1}{2},\\
y_{C_1}(t) &= -\frac{\left(34968 + 258\sqrt{20210}\right)}{124080} e^{-2t} - \frac{\left(18800 + 215\sqrt{20210}\right)}{124080} e^{2t} + \frac{1}{3}.
\end{aligned}$$

The orbit $(x_L(t), y_L(t))$ *of* X_L*, such that* $(x_L(0), y_L(0)) = (-1, y_2)$*, is given by*

$$\begin{aligned}
x_L(t) &= \frac{\left(16920 + 17\sqrt{20210}\right)}{11280} e^{-2t} - \frac{\left(17\sqrt{20210} - 16920\right)}{11280} e^{2t} - 4,\\
y_L(t) &= -\frac{\left(16920 + 17\sqrt{20210}\right)}{11280} e^{-2t} - \frac{\left(34\sqrt{20210} - 33840\right)}{11280} e^{2t} - \frac{3}{2}.
\end{aligned}$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ *of* X_C*, such that* $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$*, is given by*

$$\begin{aligned}
x_{C_2}(t) &= \frac{\left(940 - 51\sqrt{20210}\right)}{4136} e^{-2t} + \frac{\left(51\sqrt{20210} - 7144\right)}{4136} e^{2t} + \frac{1}{2},\\
y_{C_2}(t) &= \frac{\left(306\sqrt{20210} - 5640\right)}{124080} e^{-2t} + \frac{\left(255\sqrt{20210} - 35720\right)}{124080} e^{2t} + \frac{1}{3}.
\end{aligned}$$

The flight time of the orbit $(x_R(t), y_R(t))$*, from* $(1, y_0) \in \Sigma_R$ *to* $(1, y_1) \in \Sigma_R$*, is*

$$t_R = \frac{1}{4} \arctan\left(\frac{39216\sqrt{20210}}{19148449}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$*, from* $(1, y_1) \in \Sigma_R$ *to* $(-1, y_2) \in \Sigma_L$*, is*

$$t_{C_1} = \frac{1}{2} \log\left(\frac{605 + 4\sqrt{20210}}{805}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$*, from* $(-1, y_2) \in \Sigma_L$ *to* $(-1, y_3) \in \Sigma_L$*, is*

$$t_L = \frac{1}{2} \log\left(\frac{621547 + 1224\sqrt{20210}}{596693}\right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$*, from* $(-1, y_3) \in \Sigma_L$ *to* $(1, y_0) \in \Sigma_R$*, is*

$$t_{C_2} = \frac{1}{2} \log\left(\frac{605 + 4\sqrt{20210}}{53}\right).$$

The point $(-1, 0) \in \Sigma_L$ *is an invisible fold point of* X_L *and the point* $(-1, 43/120) \in \Sigma_L$ *is a invisible fold point of* X_C*. They are endpoints of the sliding set in* Σ_L*. The point* $(1, -1/10) \in \Sigma_R$ *is an invisible fold point of* X_R *and the point* $(1, 13/40) \in \Sigma_R$ *is a visible fold point of* X_C*. They are endpoints of the escaping set in* Σ_R*.*

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ *for the time* $t \in [0, t_i]$, $i = R, L, C_1, C_2$, *i.e., we obtain the limit cycle illustrated in Fig. 2.6 (a). Fig. 2.6 (b) has been made with the help of P5 software, and provides the phase portrait of vector field* (2.3) *in this case.*

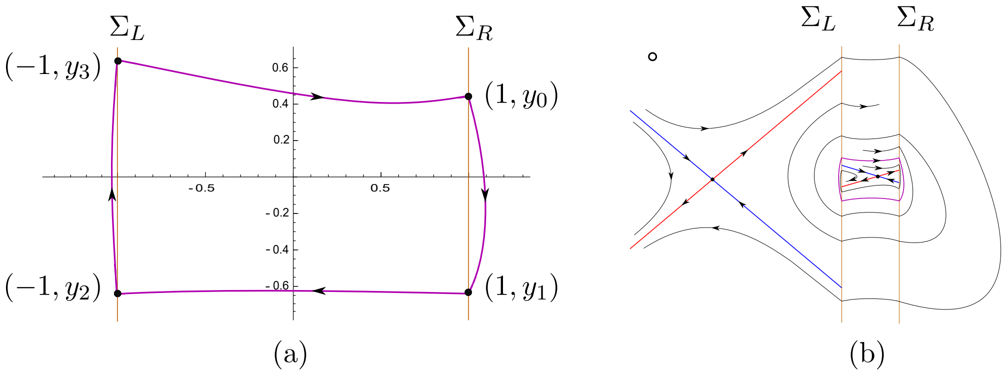


Figure 2.6: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$.

In Table 2.4 we give the parameter values for that the discontinuous planar piecewise linear vector field (2.3) *has at most one limit cycle, passing through the points* $(1, y_0)$, $(1, y_1)$, $(-1, y_2)$ *and* $(-1, y_3)$, *for the others possibles configurations of the type SSC and SSS.*

Example 7. *(Case $C_V S_B C_V$) Consider the discontinuous planar piecewise linear Hamiltonian vector field* (2.3) *with* $a_L = 4$, $\alpha_L = 2$, $b_L = 5/2$, $c_L = -5/2$, $\beta_L = 35/3$, $a_C = \sqrt{2}/11$, $b_C = 60\sqrt{2}/11$, $\alpha_C = -21\sqrt{2}/11$, $c_C = 2\sqrt{2}/11$, $\beta_C = -5\sqrt{2}/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -19/2$ *and* $c_R = \beta_R = -8$. *We can see that the central vector field from* (2.3) *has a saddle at the point* $(1, 1/3)$, *the right vector field has a center at the point* $(-39/4, 35/4)$ *and the left vector field has a center at the point* $(7, -15/4)$. *Therefore, a candidate to limit cycle of system* (2.3), *in this case, corresponds to the solution of system* (2.12), *i.e.,*

$$\begin{aligned}
&-\frac{1}{2}(y_0 - y_1)(10y_0 + 10y_1 - 3) = 0,\\
&\frac{2\sqrt{2}}{33}(5(1-3y_0)^2 + 33y_3 - 45y_3^2) = 0,\\
&2(y_2 - 2y_2^2 + y_3(-1 + 2y_3)) = 0,\\
&-\frac{2\sqrt{2}}{33}(5(1-3y_1)^2 + 33y_2 - 45y_2^2) = 0.
\end{aligned}$$

Making a calculation, the only solution (y_0, y_1, y_2, y_3) *of the above system, satisfying the conditions* $y_1 < y_0$ *and* $y_2 < y_3$, *is given by*

$$\left(\frac{(54 + 7\sqrt{278})}{360}, \frac{(54 - 7\sqrt{278})}{360}, \frac{(90 - 11\sqrt{278})}{360}, \frac{(90 + 11\sqrt{278})}{360}\right).$$

	$\mathrm{S_RS_RC_R}$	$\mathrm{S_RS_VC_V}$	$\mathrm{S_RS_VC_R}$	$\mathrm{S_RS_VS_R}$
a_C	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$
b_C	$\frac{120}{11}$	$\frac{120}{11}$	$\frac{120}{11}$	$\frac{120}{11}$
c_C	$\frac{4}{11}$	$\frac{4}{11}$	$\frac{4}{11}$	$\frac{4}{11}$
α_C	$-\frac{41}{11}$	$-\frac{41}{11}$	$-\frac{41}{11}$	$-\frac{41}{11}$
β_C	$-\frac{4}{33}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$
a_R	8	8	8	$-\frac{2}{11}$
b_R	10	10	10	$\frac{4}{11}$
c_R	-8	-8	-8	$\frac{120}{11}$
α_R	-7	-7	-7	$\frac{1}{5}$
β_R	9	-8	9	$-\frac{749}{55}$
a_L	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
b_L	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
c_L	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{8}{3}$
α_L	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
β_L	$\frac{35}{3}$	$\frac{35}{3}$	$\frac{35}{3}$	$\frac{35}{3}$
y_0	$-\frac{1}{10}+\frac{43}{24}\sqrt{\frac{43}{470}}$	$-\frac{1}{10}+\frac{43}{8}\sqrt{\frac{31}{1410}}$	$-\frac{1}{10}+\frac{43}{8}\sqrt{\frac{31}{1410}}$	$-\frac{1}{20}+\frac{43}{2640}\sqrt{6226}$
y_1	$-\frac{1}{10}-\frac{43}{24}\sqrt{\frac{43}{470}}$	$-\frac{1}{10}-\frac{43}{8}\sqrt{\frac{31}{1410}}$	$-\frac{1}{10}-\frac{43}{8}\sqrt{\frac{31}{1410}}$	$-\frac{1}{20}-\frac{43}{2640}\sqrt{6226}$
y_2	$-\frac{17}{8}\sqrt{\frac{43}{470}}$	$-\frac{17}{8}\sqrt{\frac{93}{470}}$	$-\frac{17}{8}\sqrt{\frac{93}{470}}$	$-\frac{3}{8}\sqrt{\frac{283}{22}}$
y_3	$\frac{17}{8}\sqrt{\frac{43}{470}}$	$\frac{17}{8}\sqrt{\frac{93}{470}}$	$\frac{17}{8}\sqrt{\frac{93}{470}}$	$-\frac{3}{8}\sqrt{\frac{283}{22}}$

Table 2.4: Parameter values for that vector field (2.3) of the type SSC and SSS has one limit cycle.

The points $(-1,y_2),(-1,y_3)\in\Sigma_L$ *and* $(1,y_0),(1,y_1)\in\Sigma_R$ *are crossing points because*

$$\langle X_L(-1,y_2),(1,0)\rangle\cdot\langle X_C(-1,y_2),(1,0)\rangle\approx 19.6852>0,$$

$$\langle X_L(-1,y_3),(1,0)\rangle\cdot\langle X_C(-1,y_3),(1,0)\rangle\approx 12.3493>0,$$

$$\langle X_C(1,y_0),(1,0)\rangle\cdot\langle X_R(1,y_0),(1,0)\rangle\approx 3.52299>0,$$

$$\langle X_C(1,y_1),(1,0)\rangle\cdot\langle X_R(1,y_1),(1,0)\rangle\approx 12.6929>0.$$

The orbit $(x_R(t),y_R(t))$ *of* X_R*, such that* $(x_R(0),y_R(0))=(1,y_0)$*, is given by*

$$x_R(t)=-\frac{39}{4}+\frac{43}{4}\cos(4t)+\frac{7}{72}\sqrt{\frac{139}{2}}\sin(4t),$$
$$y_R(t)=\frac{1}{360}\Big(3150+(7\sqrt{278}-3096)\cos(4t)-2(774+7\sqrt{278})\sin(4t)\Big).$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ *of* X_C*, such that* $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$*, is given by*

$$
\begin{aligned}
x_{C_1}(t) &= \frac{e^{-\sqrt{2}t}}{132}\left(66 + 7\sqrt{278} + 132e^{\sqrt{2}t} - (66 + 7\sqrt{278})e^{2\sqrt{2}t}\right),\\
y_{C_1}(t) &= -\left(\frac{e^{-\sqrt{2}t}}{660} + \frac{e^{\sqrt{2}t}}{792}\right)\left(66 + 7\sqrt{278}\right) + \frac{1}{3}.
\end{aligned}
$$

The orbit $(x_L(t), y_L(t))$ *of* X_L*, such that* $(x_L(0), y_L(0)) = (-1, y_2)$*, is given by*

$$
\begin{aligned}
x_L(t) &= -\cos(4t)\left(\frac{11\sqrt{278}}{45}\sin(t) + 8\right) + 7,\\
y_L(t) &= \frac{1}{360}\left(\big(1440 - 11\sqrt{278}\big)\cos(2t) + \big(720 + 22\sqrt{278}\big)\sin(2t) - 1350\right).
\end{aligned}
$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ *of* X_C*, such that* $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$*, is given by*

$$
\begin{aligned}
x_{C_2}(t) &= \frac{e^{-\sqrt{2}t}}{132}\left(-90 - 11\sqrt{278} + 132e^{\sqrt{2}t} - (174 - 11\sqrt{278})e^{2\sqrt{2}t}\right),\\
y_{C_2}(t) &= \frac{e^{-\sqrt{2}t}}{3960}\left(540 + 66\sqrt{278} + 1320e^{\sqrt{2}t} + (55\sqrt{278} - 870)e^{2\sqrt{2}t}\right).
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$*, from* $(1, y_0) \in \Sigma_R$ *to* $(1, y_1) \in \Sigma_R$*, is*

$$t_R = \frac{1}{4}\arctan\left(\frac{10836\sqrt{278}}{1191341}\right).$$

The flight time of the orbit $(x_{C_1}(t), y_{C_1}(t))$*, from* $(1, y_1) \in \Sigma_R$ *to* $(-1, y_2) \in \Sigma_L$*, is*

$$t_{C_1} = \frac{1}{\sqrt{2}}\log\left(\frac{121 + 6\sqrt{278}}{113}\right).$$

The flight time of the orbit $(x_L(t), y_L(t))$*, from* $(-1, y_2) \in \Sigma_L$ *to* $(-1, y_3) \in \Sigma_L$*, is*

$$t_L = \arctan\left(\frac{11}{360}\sqrt{\frac{139}{2}}\right).$$

Finally, the flight time of the orbit $(x_{C_2}(t), y_{C_2}(t))$*, from* $(-1, y_3) \in \Sigma_L$ *to* $(1, y_0) \in \Sigma_R$*, is*

$$t_{C_2} = \frac{1}{2\sqrt{2}}\log\left(\frac{24649 + 1452\sqrt{278}}{1681}\right).$$

The point $(-1, 1/4) \in \Sigma_L$ *is an invisible fold point of* X_L *and the point* $(-1, 11/30) \in \Sigma_L$ *is a invisible fold point of* X_C*. They are endpoints of the sliding set in* Σ_L*. The point* $(1, 3/20) \in \Sigma_R$ *is an invisible fold point of* X_R *and the point* $(1, 1/3) \in \Sigma_R$ *is a visible fold point of* X_C*. They* *are endpoints of the escaping set in* Σ_R*.*

Hence, using the Mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ *for the time* $t \in [0, t_i]$*,* $i = R, L, C_1, C_2$*, i.e., we obtain the limit cycle illustrated in Fig. 2.7 (a). Fig. 2.7 (b) has been made with the help of P5 software, and provides the phase portrait of vector field* (2.3) *in this case.*

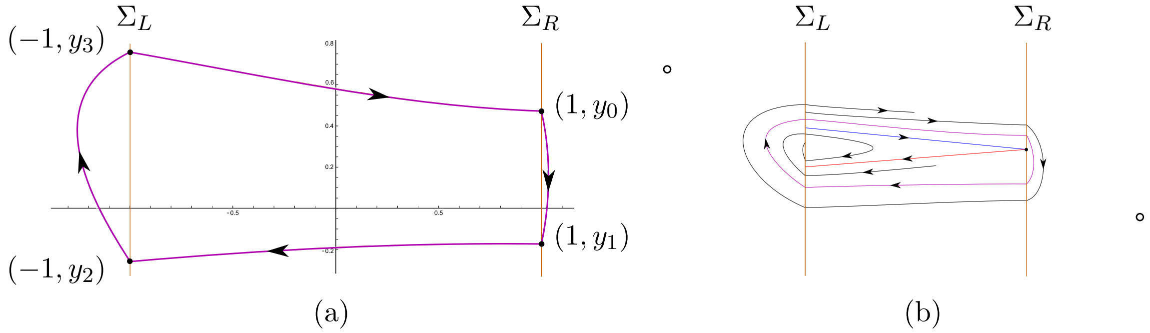


Figure 2.7: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$.

To simplify the text, in Table 2.5 we give the parameter values for that the vector field (2.3) has at most one limit cycle, passing through the points $(1, y_0)$, $(1, y_1)$, $(-1, y_2)$ and $(-1, y_3)$, for some configurations with boundary singular points. In fact, we have 31 possibilities and all are realized by concrete examples, which we do not show for simplicity.

	$\mathrm{C_B C_B C_B}$	$\mathrm{S_R C_B C_B}$	$\mathrm{S_R C_B S_R}$	$\mathrm{S_R S_B C_B}$	$\mathrm{S_R S_B S_R}$
a_C	0	0	0	$\frac{\sqrt{2}}{11}$	$\frac{\sqrt{2}}{11}$
b_C	2	2	2	$\frac{60\sqrt{2}}{11}$	$\frac{60\sqrt{2}}{11}$
c_C	-2	-2	-2	$\frac{2\sqrt{2}}{11}$	$\frac{2\sqrt{2}}{11}$
α_C	1	1	1	$-\frac{21\sqrt{2}}{11}$	$-\frac{21\sqrt{2}}{11}$
β_C	2	2	2	$-\frac{5\sqrt{2}}{33}$	$-\frac{5\sqrt{2}}{33}$
a_R	1	1	1	1	$-\frac{2}{11}$
b_R	1	1	1	1	$\frac{4}{11}$
c_R	-5	-5	15	-5	$\frac{120}{11}$
α_R	$-\frac{9}{10}$	$-\frac{9}{10}$	-1	$-\frac{9}{10}$	$\frac{2}{11}$
β_R	$\frac{49}{10}$	$\frac{49}{10}$	-31	$\frac{49}{10}$	$-\frac{749}{55}$
a_L	4	1	1	$-\frac{2}{3}$	$-\frac{2}{3}$
b_L	8	1	1	$\frac{4}{3}$	$\frac{4}{3}$
c_L	$-\frac{5}{2}$	35	35	$\frac{8}{3}$	$\frac{8}{3}$
α_L	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{3}{5}$	$-\frac{2}{3}$	$-\frac{2}{3}$
β_L	$-\frac{5}{4}$	$\frac{357}{5}$	$\frac{357}{5}$	$\frac{35}{3}$	$\frac{35}{3}$
y_0	$-\frac{1}{10}+\frac{13}{16}\sqrt{\frac{28801}{3201}}$	$\frac{1}{130}\left(9\sqrt{1209}-13\right)$	$\frac{9}{10}\sqrt{\frac{57}{7}}$	$\frac{1}{360}\left(143\sqrt{3}-36\right)$	$\frac{1}{900}\left(506\sqrt{5}-45\right)$
y_1	$-\frac{1}{10}-\frac{13}{16}\sqrt{\frac{28801}{3201}}$	$\frac{1}{130}\left(-9\sqrt{1209}-13\right)$	$-\frac{9}{10}\sqrt{\frac{57}{7}}$	$\frac{1}{360}\left(-143\sqrt{3}-36\right)$	$-\frac{1}{900}\left(506\sqrt{5}+45\right)$
y_2	$\frac{5}{16}-\frac{2}{5}\sqrt{\frac{28801}{3201}}$	$-\frac{2}{65}\left(\sqrt{1209}-13\right)$	$\frac{2}{5}-\frac{1}{2}\sqrt{\frac{57}{7}}$	$-\frac{169}{120\sqrt{3}}$	$-\frac{529}{180\sqrt{5}}$
y_3	$\frac{5}{16}+\frac{2}{5}\sqrt{\frac{28801}{3201}}$	$\frac{2}{65}\left(\sqrt{1209}+13\right)$	$\frac{2}{5}+\frac{1}{2}\sqrt{\frac{57}{7}}$	$\frac{169}{120\sqrt{3}}$	$\frac{529}{180\sqrt{5}}$

Table 2.5: Parameter values for that vector field (2.3) with boundary singular points has one limit cycle.

CHAPTER

3

Persistence of periodic solutions from system $(1)|_{\epsilon=0}$ having a center in the central region

In this chapter, we will study the number of limit cycles that can bifurcate from a periodic annulus of a discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel lines. We will prove that if the central subsystem has a real center at the origin and the right and left subsystems have centers or saddles, then we have at least three limit cycles that appear after linear perturbations of the periodic annulus. For this purpose, in Section 3.2, we will provide a normal form to this class of piecewise linear Hamiltonian system in order to simplify the computations and, in Section 3.4, we will present an expression for the Melnikov function associated with this piecewise system.

3.1 Preliminaries and main result

Our goal is estimate the lower bounds for the number of crossing limit cycles of discontinuous piecewise linear near-Hamiltonian differential systems with three zones given by

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \epsilon g(x,y), \end{cases} \tag{3.1}$$

with

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, & x \le -1, \\ H^C(x,y) = \dfrac{b_C}{2}y^2 - \dfrac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, & -1 \le x \le 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, & x \ge 1, \end{cases} \tag{3.2}$$

$$f(x,y) = \begin{cases} f_L(x,y) = r_{10}x + r_{01}y + r_{00}, & x \le -1, \\ f_C(x,y) = u_{10}x + u_{01}y + u_{00}, & -1 \le x \le 1, \\ f_R(x,y) = p_{10}x + p_{01}y + p_{00}, & x \ge 1, \end{cases} \tag{3.3}$$

$$g(x,y) = \begin{cases} g_L(x,y) = s_{10}x + s_{01}y + s_{00}, & x \le -1, \\ g_C(x,y) = v_{10}x + v_{01}y + v_{00}, & -1 \le x \le 1, \\ g_R(x,y) = q_{10}x + q_{01}y + q_{00}, & x \ge 1. \end{cases} \tag{3.4}$$

Assume that system $(3.1)|_{\epsilon=0}$ satisfies the following hypotheses:

(H1) The unperturbed central subsystem from $(3.1)|_{\epsilon=0}$ has a real center at the origin and the other unperturbed subsystems from $(3.1)|_{\epsilon=0}$ have centers or saddles.

(H2) The unperturbed system from $(3.1)|_{\epsilon=0}$ has only crossing points on the straight lines $x = \pm 1$, except by some tangent points.

(H3) The unperturbed system from $(3.1)|_{\epsilon=0}$ has a periodic annulus consisting of a family of crossing periodic orbits around the origin such that each orbit of this family passes through the three zones with clockwise orientation.

Then, the main result in this chapter is the following.

Theorem 5. *The number of limit cycles of system* (3.1)*, satisfying the hypotheses* (Hi) *for* $i = 1, 2, 3$*, which can bifurcate from the periodic annulus of the unperturbed system* $(3.1)|_{\epsilon=0}$ *is at least three.*

To prove Theorem 5 we study the number of zeros of the first order Melnikov function associated to system (3.1), see Section 1.3 of Chapter 1 for more details about the Melnikov function. Our study is concentrated in the neighborhood of the periodic orbit that bounds the periodic annulus, since to estimate the zeros of the Melnikov function we consider its expansion at the point corresponding to this orbit. What distinguishes our approach apart from others is that we provide a detailed analytical method to study the number of zeros from Melnikov function defined in three zones (see Proposition 3).

3.2 A normal form to system $(3.1)|_{\epsilon=0}$

In order to simplify the computation to prove Theorem 5 it is convenient to perform a linear change of variables which transforms system $(3.1)|_{\epsilon=0}$ into a simple form. This change of variables is a homeomorphism that keeps invariant the straight lines $x = \pm 1$, i.e., this homeomorphism will be a Σ-equivalence between the systems. More precisely, we have the following result.

Proposition 1. *Suppose that system* $(3.1)|_{\epsilon=0}$ *satisfies the hypotheses (Hi),* $i = 1, 2, 3$, *except that the center of the central subsystem need not necessarily be at the origin. Then, after a linear change of variables and a rescaling of the independent variable, we can write it as*

$$\begin{cases} \dot{x} = H_y(x, y), \\ \dot{y} = -H_x(x, y), \end{cases}$$

where

$$H(x, y) = \begin{cases} H^L(x, y) = \dfrac{b_L y^2}{2} - \dfrac{c_L x^2}{2} + a_L xy + a_L y - \beta_L x, & x \leq -1, \\ H^C(x, y) = \dfrac{x^2}{2} + \dfrac{y^2}{2} - \beta_C x, & -1 \leq x \leq 1, \\ H^R(x, y) = \dfrac{b_R y^2}{2} - \dfrac{c_R x^2}{2} + a_R xy - a_R y - \beta_R x, & x \geq 1. \end{cases}$$

Proof. As the central subsystem from $(3.1)|_{\epsilon=0}$ has a center with clockwise orientation of the orbits, then $a_C^2 + b_C c_C < 0$ and $b_C > 0$. Note that $b_i \neq 0$, for $i = L, R$. In fact, if the singular points of the subsystems from $(3.1)|_{\epsilon=0}$ are centers, this is true due to the clockwise orientation of the orbits. Now, if the singular point of the right or left subsystem is a saddle with the correspondent $b_i = 0$ we have a separatrix parallel to straight line $x = 0$. System $(3.1)|_{\epsilon=0}$ has four tangent points given by $P_1 = (1, -(a_C + \alpha_C)/b_C)$, $P_2 = (1, -(a_R + \alpha_R)/b_R)$, $P_3 = (-1, (a_C - \alpha_C)/b_C)$ and $P_4 = (-1, (a_L - \alpha_L)/b_L)$. By hypothesis (H2), we have that system $(3.1)|_{\epsilon=0}$ have only crossing points on the straight lines $x = \pm 1$, except in the tangent points. Hence, for all $y \in \mathbb{R} \setminus \{(\pm a_C - \alpha_C)/b_C, -(a_R + \alpha_R)/b_R), (a_L - \alpha_L)/b_L)\}$, we must have

$$\langle X_L(-1, y), (1, 0)\rangle \langle X_C(-1, y), (1, 0)\rangle > 0 \quad \text{and} \quad \langle X_R(1, y), (1, 0)\rangle \langle X_C(1, y), (1, 0)\rangle > 0,$$

i.e., we have the following inequalities

$$\begin{aligned} &b_C b_L y^2 - (a_L b_C + a_C b_L - b_C \alpha_L + \alpha_C b_L)y + a_C a_L - a_C \alpha_L - a_L \alpha_C + \alpha_C \alpha_L > 0, \\ &b_C b_R y^2 + (a_R b_C + a_C b_R + b_C \alpha_R + \alpha_C b_R)y + a_C a_R + a_C \alpha_R + a_R \alpha_R + \alpha_C \alpha_R > 0. \end{aligned}$$

This implies that $b_L b_C > 0$, $b_R b_C > 0$, $P_1 = P_2$ and $P_3 = P_4$. Therefore, as $b_C > 0$, we have that

$$\alpha_L = \frac{b_L(\alpha_C - a_C) + a_L b_C}{b_C}, \quad b_L > 0, \quad \alpha_R = \frac{b_R(a_C + \alpha_C) - a_R b_C}{b_C} \quad \text{and} \quad b_R > 0. \tag{3.5}$$

Assuming the conditions (3.5), consider the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{a_C}{\omega_C} & \dfrac{b_C}{\omega_C} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{\alpha_C}{\omega_C} \end{pmatrix},$$

with $\omega_C = \sqrt{-a_C^2 - b_C c_C}$. Applying this change of variables and rescaling the time by $\tilde{t} = \omega_C\, t$, we obtain the results after rewriting the parameters. □

Remark 1. *Consider system $(3.1)|_{\epsilon=0}$ in its normal form, i.e., with $\alpha_L = a_L$, $\alpha_R = -a_R$, $b_C = c_C = 1$ and $a_C = \alpha_C = 0$. Note that when $\beta_C = 0$, we have that the singular point of the central subsystem from $(3.1)|_{\epsilon=0}$ is at the origin. In this case, assuming the hypotheses (Hi), $i = 1, 2, 3$, the periodic annulus of system $(3.1)|_{\epsilon=0}$ has all its periodic orbits passing through the three zones bounded by the orbit L_0 of the central subsystem tangent to straight lines $x = \pm 1$ in the points $P_R = (1, 0)$ and $P_L = (-1, 0)$ (see Figure 3.1 (a)).*

If $\beta_C \neq 0$, by a reflection around the straight line $x = 0$ (if necessary), we can assume without loss of generality that $\beta_C > 0$. In this case, the periodic annulus of system $(3.1)|_{\epsilon=0}$ has all its periodic orbits passing through the three zones bounded by the orbit L_0^1 that is obtained by the orbit of the central subsystem tangent to straight lines $x = -1$ in the point $P_L = (-1, 0)$. Note that, L_0^1 intercept crosswise the straight line $x = 1$ at two distinct points. Moreover, the periodic annulus has periodic orbits passing by two zones, which are bounded by only L_0^1 when $\beta_C \geq 1$, or bounded by L_0^1 and L_0^2 when $0 < \beta_C < 1$, where L_0^2 is the orbit obtained by the orbit of the central subsystem tangent to straight lines $x = 1$ in the point $P_R = (1, 0)$ contained in the region bounded by L_0^1 (see Figure 3.1 (b)–(c)).

In this chapter, we study only the case $\beta_C = 0$ and showed that at least three limit cycles can bifurcate from a periodic annulus when perturbed by linear polynomials. In Chapter 4, we also study the case $\beta_C = 0$ and showed that at least four limit cycles can bifurcate from a periodic annulus when perturbed by quadratic polynomials and at least seven limit cycles can bifurcate from a periodic annulus when perturbed by cubic polynomials. Now, in Chapter 5, we study the case $\beta_C > 0$, where system $(3.1)|_{\epsilon=0}$ is of the type SCS. In this case, we show that either at least six limit cycles can bifurcate from periodic annulus, perturbed by piecewise linear polynomials, near the periodic orbit L_0^1, when $0 < \beta_C \leq 1$ or at least four limit cycles can bifurcate from *periodic annulus, perturbed by piecewise linear polynomials, near the periodic orbit* L_0^1*, when* $\beta_C > 1$.

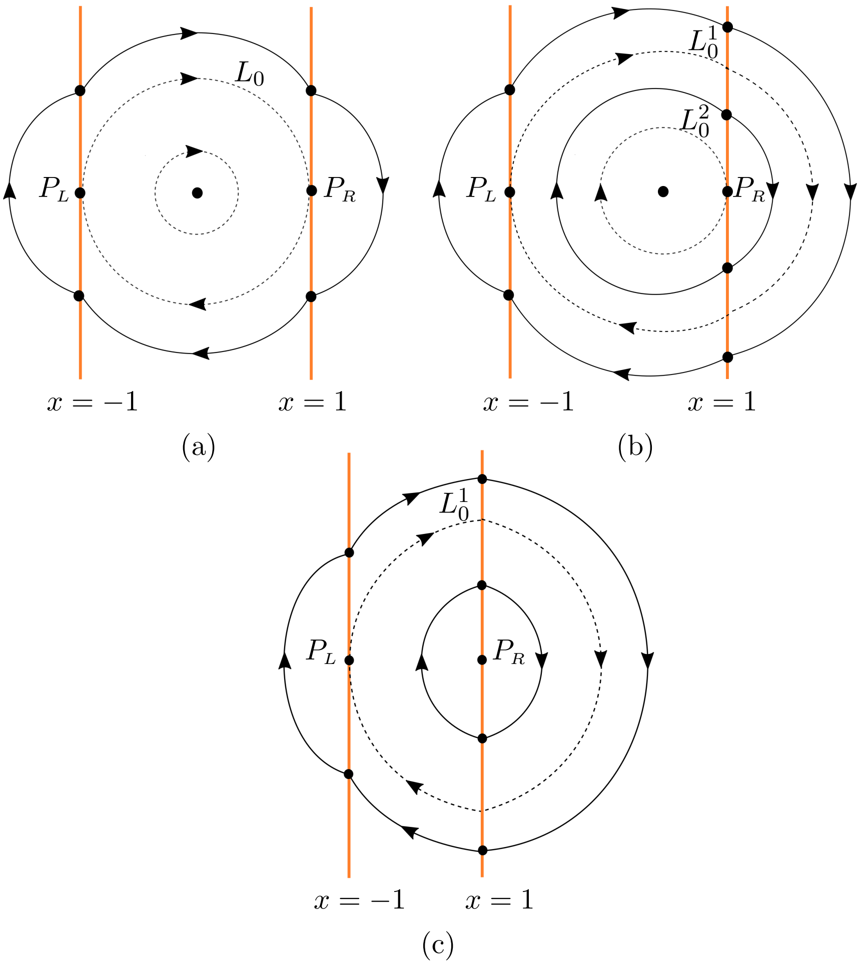


Figure 3.1: Periodic orbits of system $(3.1)|_{\epsilon=0}$ with $\alpha_L = a_L$, $\alpha_R = -a_R$, $b_C = c_C = 1$ and $a_C = \alpha_C = 0$ when (a) $\beta_C = 0$, (b) $0 < \beta_C < 1$ and (c) $\beta_C \geq 1$.

3.3 A classification to system $(3.1)|_{\epsilon=0}$

We can classify the systems that satisfy the hypothesis (H1) according to the configuration of their singular points (without taking into account if the singular points are real or virtual). More precisely, as in Chapter 2, denoting the centers by the capital letter C and by capital letter S the saddles, in the case of three zones, we have the following three classes of piecewise linear Hamiltonian systems: SCS, CCS and CCC.

In order to compute the zeros of the first order Melnikov function M associated to system $(3.1)|_{\epsilon=0}$, that can be expressed as (see Theorem 1)

$$
\begin{aligned}
M(h) = {} & \frac{H_y^R(A)}{H_y^C(A)} I_C + \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} I_L + \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} \bar{I}_C \\
& + \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} I_R, \quad h \in J,
\end{aligned} \tag{3.6}
$$

where

$$
I_C = \int_{\widehat{A_3 A}} g_C dx - f_C dy, \quad I_L = \int_{\widehat{A_2 A_3}} g_L dx - f_L dy,
$$

$$
\bar{I}_C = \int_{\widehat{A_1 A_2}} g_C dx - f_C dy, \quad I_R = \int_{\widehat{A A_1}} g_R dx - f_R dy,
$$

it is necessary to find the open interval J, where it is defined. For this, consider the following proposition.

Proposition 2. *Consider system* (3.1) *in the normal form, i.e., with* $\alpha_L = a_L$, $\alpha_R = -a_R$, $b_C = 1$, $c_C = -1$ *and* $a_C = \alpha_C = \beta_C = 0$, *satisfying the hypotheses (Hi),* $i = 1, 2, 3$.

(a) *If system* $(3.1)|_{\epsilon=0}$ *is of the type SCS or CCS, then* $J = (0, \tau_{RS})$, *where* $\tau_{RS} = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/b_R\omega_{RS}$ *with* $\omega_{RS} = \sqrt{a_R^2 + b_R c_R}$, *and the periodic annuli are equivalent to one of Figure 3.2.*

(b) *If system* $(3.1)|_{\epsilon=0}$ *is of the type CCC, then* $J = (0, \infty)$, *and the periodic annuli are equivalent to one of Figure 3.3.*

Proof. Suppose that system $(3.1)|_{\epsilon=0}$ is of the type SCS or CCS. Note that if the saddles are virtual or if they are on the straight lines $x = \pm 1$, then we have not periodic orbits passing through the three zones. Denote by W_R^u and W_R^s (resp. W_L^u and W_L^s) the unstable and stable separatrices of the saddles of the right (resp. left) subsystems from $(3.1)|_{\epsilon=0}$, respectively. Denote by $P_L^i = W_L^i \cap \{(-1, y) : y \in \mathbb{R}\}$ and $P_R^i = W_R^i \cap \{(1, y) : y \in \mathbb{R}\}$, for $i = u, s$. After some computations, it is possible to show that

$$
P_L^u = (-1, \tau_{LS}), \quad P_L^s = (-1, -\tau_{LS}), \quad P_R^u = (1, -\tau_{RS}), \quad P_R^s = (1, \tau_{RS}),
$$

where $\omega_{LS} = \sqrt{a_L^2 + b_L c_L}$, $\omega_{RS} = \sqrt{a_R^2 + b_R c_R}$, $\tau_{RS} = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/b_R\omega_{RS}$ and $\tau_{LS} = (a_L^2 + b_L\beta_L - \omega_{LS}^2)/b_L\omega_{LS}$. Note that we have a symmetry between the points P_L^u and P_L^s (resp. P_R^u and P_R^s) with respect to x-axis. Define by τ the smallest ordinate value between the points P_R^s and P_L^u, i.e., $\tau = \min\{\tau_{RS}, \tau_{LS}\}$. Then, by a reflection around the y-axis if necessary, we can assume that $\tau = \tau_{RS}$.

As the vector field X_C associated with the central subsystem from $(3.1)|_{\epsilon=0}$ is $X_C(x, y) = (y, -x)$, if system $(3.1)|_{\epsilon=0}$ is of the type SCS and the ordinates of the points P_R^s and P_L^u are distinct, i.e., $\tau_{RS} \neq \tau_{LS}$ (see Figure 3.2 (a)) or if system $(3.1)|_{\epsilon=0}$ is of the type CCS (see Figure 3.2 (c) or (d)), then we have a homoclinic loop passing through the points P_R^s and P_R^u. Otherwise, if system $(3.1)|_{\epsilon=0}$ is of the type SCS and the ordinates of points P_R^s and P_L^u are the same, i.e., $\tau_{RS} = \tau_{LS}$ (see Figure 3.2 (b)) then we have heteroclinic orbits passing through the points P_R^s, P_R^u, P_L^s and P_L^u. Moreover, the central subsystem from $(3.1)|_{\epsilon=0}$ has a periodic orbit tangent to straight lines $x = \pm 1$ in the points $P_R = (1, 0)$ and $P_L = (-1, 0)$. Figure 3.2 illustrates the possible phase portraits of system $(3.1)|_{\epsilon=0}$ of the type SCS and CCS.

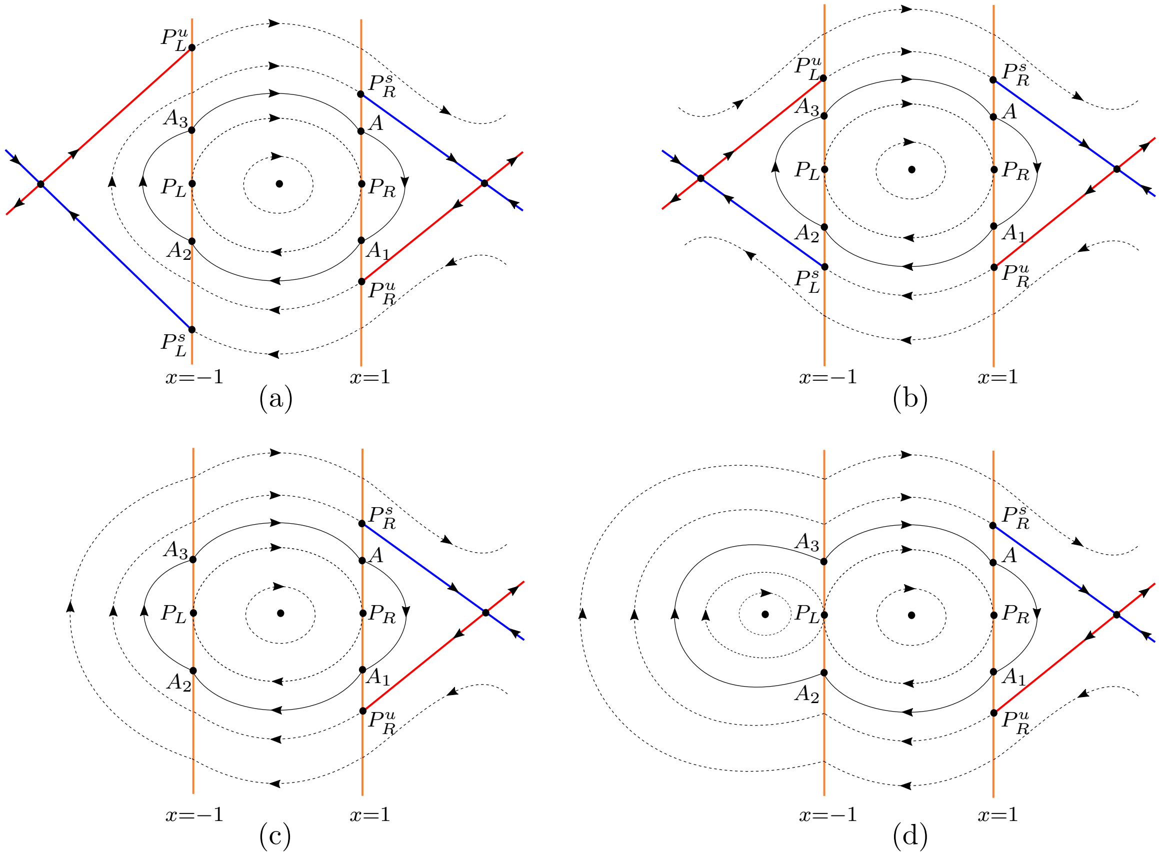


Figure 3.2: Phase portraits of system $(3.1)|_{\epsilon=0}$ of the type : (a) SCS with $\tau_{RS} \neq \tau_{LS}$; (b) SCS with $\tau_{RS} = \tau_{LS}$; (c) CCS when left subsystem has a virtual center; (d) CCS when left subsystem has a real center.

Consider a initial point of form $A(h) = (1, h)$, with $h \in (0, \tau_{RS})$. By the hypothesis (H3), system $(3.1)|_{\epsilon=0}$ has a family of crossing periodic orbits that intersect the straight lines $x = \pm 1$ at four points, $A(h)$, $A_1(h) = (1, a_1(h))$, with $a_1(h) < h$, and $A_2(h) = (-1, a_2(h))$, $A_3(h) = (-1, a_3(h))$, with $a_2(h) < a_3(h)$ satisfying

$$H^R(A(h)) = H^R(A_1(h)),$$
$$H^C(A_1(h)) = H^C(A_2(h)),$$
$$H^L(A_2(h)) = H^L(A_3(h)),$$
$$H^C(A_3(h)) = H^C(A(h)),$$

where H^R, H^C and H^L are given by (3.2). More precisely, we have the equations

$$\frac{b_R}{2}(h - a_1(h))(h + a_1(h)) = 0,$$
$$\frac{1}{2}(a_1(h) - a_2(h))(a_1(h) + a_2(h)) = 0,$$
$$\frac{b_L}{2}(a_2(h) - a_3(h))(a_2(h) + a_3(h)) = 0,$$
$$\frac{1}{2}(a_3(h) - h)(a_3(h) + h) = 0.$$

As $a_1(h) < h$, $a_2(h) < a_3(h)$, $b_R > 0$ and $b_L > 0$, the only solution of the above system is $a_1(h) = -h$, $a_2(h) = -h$ and $a_3(h) = h$, i.e., we have the four points given by $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Moreover, system $(3.1)|_{\epsilon=0}$ has a periodic orbit L_h passing through these points, for all $h \in (0, \tau_{RS})$. If $h \in [\tau_{RS}, \infty)$ then the orbit of system $(3.1)|_{\epsilon=0}$ with initial condition in $A(h)$ does not return to the straight line $x = 1$ for positive times, i.e., system $(3.1)|_{\epsilon=0}$ has no periodic orbit passing through the point $A(h)$. Therefore, if $h \in (0, \tau_{RS})$ system $(3.1)|_{\epsilon=0}$ has a periodic annulus, formed by the periodic orbits L_h, bounded by one (see Figure 3.2 (a)–(c)) or two (see Figure 3.2 (d)) periodic orbits tangent to the straight lines $x = \pm 1$, when $h = 0$, and a homoclinic loop (see Figure 3.2 (a), (c) and (d)) or two heteroclinic orbits (see Figure 3.2 (b)), when $h = \tau_{RS}$. Therefore, item (a) is proved.

To prove item (b), suppose that system $(3.1)|_{\epsilon=0}$ is of the type CCC. The central subsystem from $(3.1)|_{\epsilon=0}$ has a periodic orbit tangent to straight lines $x = \pm 1$ at the points $P_R = (1, 0)$ and $P_L = (-1, 0)$. Moreover, as in the previous case, for each $h \in (0, \infty)$ we have a periodic orbit L_h passing through the points $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Therefore, system $(3.1)|_{\epsilon=0}$ has a continuum of periodic orbits formed by the periodic orbits L_h, with $h \in (0, \infty)$, and bounded by one (see Figure 3.3 (a)), two (see Figure 3.3 (a)) or three (see Figure 3.3 (c)) periodic orbits tangent to the straight lines $x = \pm 1$, when $h = 0$. Figure 3.3 shows the possible phase portraits of system $(3.1)|_{\epsilon=0}$ of the type CCC. $\square$

As $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$, the coefficients that multiply the integrals of first order Melnikov function (3.6) associated with system (3.1) can be easily calculated. More precisely, we have the immediate corollary.

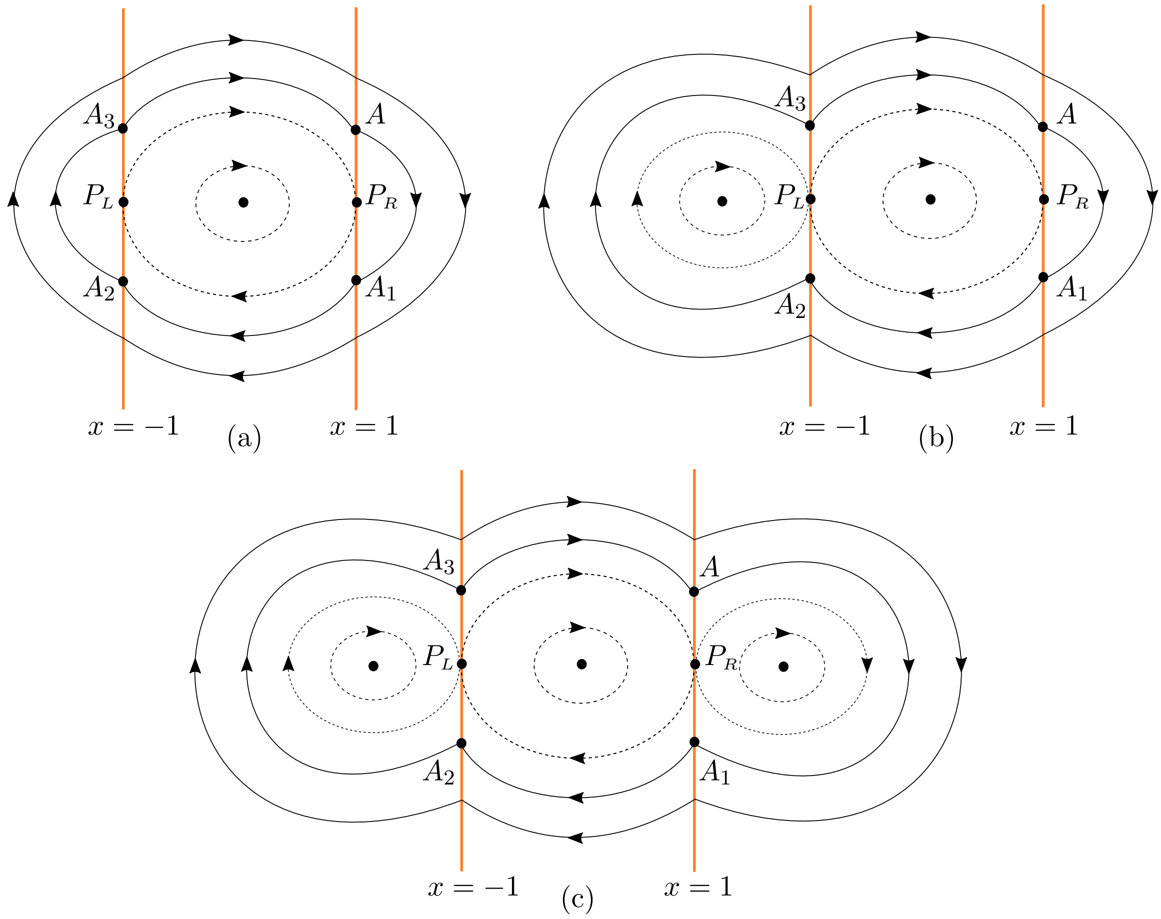


Figure 3.3: Phase portraits of system $(3.1)|_{\epsilon=0}$ of the type CCC when: (a) the left and right subsystems have virtual centers; (b) the left subsystem has a real center and right subsystem has a virtual center; (c) the left and right subsystems have real centers.

Corollary 1. *Let J be the interval of definition of Melnikov function* (3.6)*. For $h \in J$,*

$$\frac{H_y^R(A)}{H_y^C(A)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} = b_R, \quad \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} = \frac{b_R}{b_L}$$

and

$$\frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} = 1.$$

Then, the first order Melnikov function associated to system (3.1) *can be written as*

$$\begin{aligned} M(h) = {} & b_R \int_{\widehat{A_3A}} g_C dx - f_C dy + \frac{b_R}{b_L} \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ & + b_R \int_{\widehat{A_1A_2}} g_C dx - f_C dy + \int_{\widehat{AA_1}} g_R dx - f_R dy. \end{aligned}$$

3.4 Melnikov functions associated with system (3.1)

In what follows, we determinate the first order Melnikov function associated to system (3.1) when system $(3.1)|_{\epsilon=0}$ is of the type SCS, CCS and CCC. For this, we define the functions:

$$\begin{aligned}
f_0(h) =& \ h, \quad h \in (0,\infty),\\
f_C^C(h) =& \ (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right), \quad h \in (0,\infty),\\
f_R^C(h) =& \ (h^2+\tau_{RC}^2)\left(2\pi\mu_1+(-1)^{\mu_1}\arccos\left(\frac{\tau_{RC}^2-h^2}{\tau_{RC}^2+h^2}\right)\right), \quad h \in (0,\infty),\\
f_L^C(h) =& \ (h^2+\tau_{LC}^2)\left(2\pi\mu_2+(-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2-h^2}{\tau_{LC}^2+h^2}\right)\right), \quad h \in (0,\infty),\\
f_R^S(h) =& \ (h^2-\tau_{RS}^2)\log\left(\frac{\tau_{RS}+h}{\tau_{RS}-h}\right), \quad h \in (0,\tau_{RS}),\\
f_L^S(h) =& \ (h^2-\tau_{LS}^2)\log\left(\frac{\tau_{LS}+h}{\tau_{LS}-h}\right), \quad h \in (0,\tau_{RS}),
\end{aligned} \tag{3.7}$$

with $\mu_i = 0,1$, for $i = 1,2$, $\tau_{RC} = (a_R^2 - b_R\beta_R + \omega_R^2)/(b_R\omega_R)$ and $\tau_{LC} = (a_L^2 + b_L\beta_L + \omega_L^2)/(b_L\omega_L)$.

Theorem 6. *Suppose that system $(3.1)|_{\epsilon=0}$ is of the type SCS. Then the first order Melnikov function M associated with system (3.1) can be expressed, when $\tau_{LS} \neq \tau_{RS}$ (i.e., we have a homoclinic loop), as*

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^S f_L^S(h), \tag{3.8}$$

and, when $\tau_{LS} = \tau_{RS}$ (i.e., we have two heteroclinic orbits), as

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h), \tag{3.9}$$

with $h \in (0,\tau)$, where the functions f_0, f_C^C, f_R^S, f_L^S are the ones defined in (3.7). Here the coefficients k_0 and k_i^j, for $i = L,C,R$ and $j = C,S$, depend on the parameters of system (3.1).

Proof. Firstly, let us obtain the expression of Melnikov function (3.8). The orbit $(x_R(t), y_R(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1,h)$, is given by

$$\begin{aligned}
x_R(t) =& -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(b_R h - b_R e^{2t\omega_{RS}}h - 2e^{t\omega_{RS}}\omega_{RS} + b_R\tau_{RS} - 2b_R e^{t\omega_{RS}}\tau_{RS} + b_R e^{2t\omega_{RS}}\tau_{RS}\Big),\\
y_R(t) =& -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(-a_R h + a_R e^{2t\omega_{RS}}h - \omega_{RS}h - e^{2t\omega_{RS}}\omega_{RS}h - a_R\tau_{RS} + 2a_R e^{t\omega_{RS}}\tau_{RS}\\
&- a_R e^{2t\omega_{RS}}\tau_{RS} - \omega_{RS}\tau_{RS} + e^{2t\omega_{RS}}\omega_{RS}\tau_{RS}\Big).
\end{aligned}$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$t_R = \frac{1}{\omega_{RS}} \log\left(\frac{h + \tau_{RS}}{\tau_{RS} - h}\right).$$

Now, for f_R and g_R defined in (3.3) and (3.4), respectively, we have

$$\begin{aligned} I_R &= \int_{\widehat{AA_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \big(g_R(x_R(t), y_R(t))x'_R(t) - f_R(x_R(t), y_R(t))y'_R(t)\big)dt \\ &= \int_0^{t_R} \left(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x'_R(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y'_R(t)\right)dt \\ &= \alpha_1 f_0(h) + \alpha_2 f_R^S(h), \end{aligned} \tag{3.10}$$

with

$$\alpha_1 = \frac{1}{\omega_{RS}}\Big(2(p_{00} + p_{10})\omega_{RS} + b_R(p_{10} + q_{01})\tau_{RS}\Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RS}}\Big(p_{10} + q_{01}\Big).$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$\begin{aligned} x_{C1}(t) &= \cos(t) - h\sin(t), \\ y_{C1}(t) &= -h\cos(t) - \sin(t). \end{aligned}$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$t_{C1} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right).$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned} \bar{I}_C &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ &= \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t))x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t))y'_{C1}(t)\big)dt \\ &= \int_0^{t_{C1}} \left(\sum_{i+j=0}^{1} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x'_{C1}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y'_{C1}(t)\right)dt \\ &= -2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^C(h), \end{aligned} \tag{3.11}$$

with

$$\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{1}{2}\Big(u_{10} + v_{01}\Big).$$

The orbit $(x_L(t), y_L(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned} x_L(t) = &\frac{e^{-t\omega_{LS}}}{2\omega_{LS}}\Big(b_L h - b_L e^{2t\omega_{LS}} h - 2e^{t\omega_{LS}}\omega_{LS} + b_L\tau_{LS} - 2b_L e^{t\omega_{LS}}\tau_{LS} \\ &+ b_L e^{2t\omega_{LS}}\tau_{LS}\Big), \\ y_L(t) = &\frac{e^{-t\omega_{LS}}}{2\omega_{LS}}\Big(-a_L h + a_L e^{2t\omega_{LS}} h - \omega_{LS} h - e^{2t\omega_{LS}}\omega_{LS} h - a_L\tau_{LS} \\ &+ 2a_L e^{t\omega_{LS}}\tau_{LS} - a_L e^{2t\omega_{LS}}\tau_{LS} - \omega_{LS}\tau_{LS} + e^{2t\omega_{LS}}\omega_{LS}\tau_{LS}\Big). \end{aligned}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \frac{1}{\omega_{LS}} \log\left(\frac{h + \tau_{LS}}{\tau_{LS} - h}\right).$$

Now, for f_L and g_L defined in (3.3) and (3.4), respectively, we have

$$\begin{aligned} I_L &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ &= \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\big)dt \\ &= \int_0^{t_L} \left(\sum_{i+j=0}^{1} s_{ij}x_L^i(t)y_L^j(t)x_L'(t) - \sum_{i+j=0}^{1} r_{ij}x_L^i(t)y_L^j(t)y_L'(t)\right)dt \\ &= \alpha_5 f_0(h) + \alpha_6 f_L^S(h), \end{aligned} \tag{3.12}$$

with

$$\alpha_5 = \frac{1}{\omega_{LS}}\Big(2(r_{10} - r_{00})\omega_{LS} + b_L(r_{10} + s_{01})\tau_{LS}\Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LS}}\Big(r_{10} + s_{01}\Big).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$\begin{aligned} x_{C2}(t) &= -\cos(t) + h\sin(t), \\ y_{C2}(t) &= h\cos(t) + \sin(t). \end{aligned}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right).$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned} I_C &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t))x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t))y'_{C2}(t)\big)dt \\ &= \int_0^{t_{C2}} \Bigg(\sum_{i+j=0}^{1} v_{ij}x^i_{C2}(t)y^j_{C2}(t)x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij}x^i_{C2}(t)y^j_{C2}(t)y'_{C2}(t)\Bigg)dt \\ &= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f^C_C(h). \end{aligned} \tag{3.13}$$

Hence, by Corollary 1, the first order Melnikov function associated to system (3.1) is given by

$$M(h) = b_R I_C + \frac{b_R}{b_L} I_L + b_R \bar{I}_C + I_R. \tag{3.14}$$

Replacing (3.10), (3.11), (3.12) and (3.13) into (3.14) we obtain (3.8), with

$$k_0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k^C_C = 2b_R\alpha_4, \quad k^S_R = \alpha_2 \quad \text{and} \quad k^S_L = \frac{b_R}{b_L}\alpha_6.$$

Finally, replacing $\tau_{LS} = \tau_{RS}$ on function M in (3.8) we obtain the expression (3.9) with

$$k_0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L\omega_{LS}}\Big(2(r_{10} - r_{00})\omega_{LS} + b_L(r_{10} + s_{01})\tau_{RS}\Big),$$

$$k^C_C = 2b_R\alpha_4 \quad \text{and} \quad k^S_R = \alpha_2 + \frac{b_R}{b_L}\alpha_6.$$

□

Theorem 7. *Suppose that system* $(3.1)|_{\epsilon=0}$ *is of the type CCS. Then the first order Melnikov function* M *associated to system* (3.1) *can be expressed as*

$$M(h) = k_0 f_0(h) + k^C_C f^C_C(h) + k^S_R f^S_R(h) + k^C_L f^C_L(h), \tag{3.15}$$

for $h \in (0, \tau_{RS})$*, where the functions* f_0, f^C_C, f^C_L, f^S_R *are the ones defined in* (3.7)*, with* $\mu_2 = 0$ $(\mu_2 = 1)$ *if the center of the left subsystem from* $(3.1)|_{\epsilon=0}$ *is virtual (real). Here the coefficients* k_0 *and* k^j_i*, for* $i = L, C, R$ *and* $j = C, S$*, depend on the parameters of system* (3.1)*.*

Proof. The orbit $(x_R(t), y_R(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$
\begin{aligned}
x_R(t) = &-\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(b_R h - b_R e^{2t\omega_{RS}} h - 2e^{t\omega_{RS}}\omega_{RS} + b_R\tau_{RS} - 2b_R e^{t\omega_{RS}}\tau_{RS} + b_R e^{2t\omega_{RS}}\tau_{RS}\Big),\\
y_R(t) = &-\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(- a_R h + a_R e^{2t\omega_{RS}} h - \omega_{RS} h - e^{2t\omega_{RS}}\omega_{RS} h - a_R\tau_{RS} + 2a_R e^{t\omega_{RS}}\tau_{RS}\\
&- a_R e^{2t\omega_{RS}}\tau_{RS} - \omega_{RS}\tau_{RS} + e^{2t\omega_{RS}}\omega_{RS}\tau_{RS}\Big).
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$
t_R = \frac{1}{\omega_{RS}}\log\left(\frac{h + \tau_{RS}}{\tau_{RS} - h}\right).
$$

Now, for f_R and g_R defined in (3.3) and (3.4), respectively, we have

$$
\begin{aligned}
I_R &= \int_{\widehat{AA_1}} g_R dx - f_R dy\\
&= \int_0^{t_R} \big(g_R(x_R(t), y_R(t))x'_R(t) - f_R(x_R(t), y_R(t))y'_R(t)\big)dt\\
&= \int_0^{t_R} \Bigg(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x'_R(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y'_R(t)\Bigg)dt\\
&= \alpha_1 f_0(h) + \alpha_2 f_R^S(h),
\end{aligned}
\tag{3.16}
$$

with

$$
\alpha_1 = \frac{1}{\omega_{RS}}\Big(2(p_{00} + p_{10})\omega_{RS} + b_R(p_{10} + q_{01})\tau_{RS}\Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RS}}\Big(p_{10} + q_{01}\Big).
$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \cos(t) - h\sin(t),\\
y_{C1}(t) &= -h\cos(t) - \sin(t).
\end{aligned}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$
t_{C1} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right).
$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned}\bar{I}_C =& \int_{\widehat{A_1A_2}} g_C dx - f_C dy\\ =& \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t))x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t))y'_{C1}(t)\big)dt\\ =& \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{1} v_{ij}x^i_{C1}(t)y^j_{C1}(t)x'_{C1}(t) - \sum_{i+j=0}^{1} u_{ij}x^i_{C1}(t)y^j_{C1}(t)y'_{C1}(t)\Bigg)dt\\ =& -2v_{00} + \alpha_3 f_0(h) + \alpha_4 f^C_C(h),\end{aligned} \tag{3.17}$$

with

$$\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{1}{2}\Big(u_{10} + v_{01}\Big).$$

The orbit $(x_L(t), y_L(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned}x_L(t) =& -\frac{1}{\omega_{LC}}\Big(\omega_{LC} - b_L\tau_{LC} + b_L\tau_{LC}\cos(t\omega_{LC}) + b_L h\sin(t\omega_{LC})\Big),\\ y_L(t) =& \frac{1}{\omega_{LC}}\Big(-a_L\tau_{LC} + (-\omega_{LC}h + a_L\tau_{LC})\cos(t\omega_{LC}) + (a_L h + \omega_{LC}\tau_{LC})\sin(t\omega_{LC})\Big).\end{aligned}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \frac{1}{\omega_{LC}}\left(2\pi\mu_2 + (-1)^{\mu_2}\arccos\left(\frac{\tau^2_{LC} - h^2}{\tau^2_{LC} + h^2}\right)\right),$$

with $\mu_2 = 0$ ($\mu_2 = 1$) if the center of the left subsystem from $(3.1)|_{\epsilon=0}$ is virtual (real). Now, for f_L and g_L defined in (3.3) and (3.4), respectively, we have

$$\begin{aligned}I_L =& \int_{\widehat{A_2A_3}} g_L dx - f_L dy\\ =& \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x'_L(t) - f_L(x_L(t), y_L(t))y'_L(t)\big)dt\\ =& \int_0^{t_L} \Bigg(\sum_{i+j=0}^{1} s_{ij}x^i_L(t)y^j_L(t)x'_L(t) - \sum_{i+j=0}^{1} r_{ij}x^i_L(t)y^j_L(t)y'_L(t)\Bigg)dt\\ =& \alpha_5 f_0(h) + \alpha_6 f^C_L(h),\end{aligned} \tag{3.18}$$

with

$$\alpha_5 = \frac{1}{\omega_{LC}}\Big(2(r_{10} - r_{00})\omega_{LC} - b_L(r_{10} + s_{01})\tau_{LC}\Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LC}}\Big(r_{10} + s_{01}\Big).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$\begin{aligned} x_{C2}(t) &= -\cos(t) + h\sin(t), \\ y_{C2}(t) &= h\cos(t) + \sin(t). \end{aligned}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos\left(\frac{h^2-1}{h^2+1}\right).$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned} I_C &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t))x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t))y'_{C2}(t)\big)dt \\ &= \int_0^{t_{C2}} \Bigg(\sum_{i+j=0}^{1} v_{ij}x^i_{C2}(t)y^j_{C2}(t)x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij}x^i_{C2}(t)y^j_{C2}(t)y'_{C2}(t)\Bigg)dt \\ &= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f^C_C(h). \end{aligned} \tag{3.19}$$

Hence, by Corollary 1, the first order Melnikov function associated to system (3.1) is given by

$$M(h) = b_R I^1_C + \frac{b_R}{b_L} I^1_L + b_R \bar{I}^1_C + I^1_R. \tag{3.20}$$

Replacing (3.16), (3.17), (3.18) and (3.19) into (3.20) we obtain (3.15), with

$$k_0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k^C_C = 2b_R\alpha_4, \quad k^S_R = \alpha_2 \quad \text{and} \quad k^C_L = \frac{b_R}{b_L}\alpha_6.$$

□

Theorem 8. *Suppose that system $(3.1)|_{\epsilon=0}$ is of the type CCC. Then the first order Melnikov function M associated to system (3.1) can be expressed as*

$$M(h) = k_0 f_0(h) + k^C_C f^C_C(h) + k^C_R f^C_R(h) + k^C_L f^C_L(h), \tag{3.21}$$

for $h \in (0, \infty)$, where the functions f_0, f^C_C, f^C_R, f^C_L are the ones defined in (3.7), with $\mu_1 = 0$ ($\mu_1 = 1$) if the center of the right subsystem from $(3.1)|_{\epsilon=0}$ is virtual (real) and $\mu_2 = 0$ ($\mu_2 = 1$) if the center of the left subsystem from $(3.1)|_{\epsilon=0}$ is virtual (real). Here the coefficients k_0 and k^C_i, for $i = L, C, R$, depend on the parameters of system (3.1).

Proof. The orbit $(x_R(t), y_R(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$
\begin{aligned}
x_R(t) &= \frac{1}{\omega_{RC}}\Big(\omega_{RC} - b_R\tau_{RC} + b_R\tau_{RC}\cos(t\omega_{RC}) + b_R h\sin(t\omega_{RC})\Big),\\
y_R(t) &= \frac{1}{\omega_{RC}}\Big(a_R\tau_{RC} + (\omega_{RC}h - a_R\tau_{RC})\cos(t\omega_{RC}) - (a_R h + \omega_{RC}\tau_{RC})\sin(t\omega_{RC})\Big).
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$
t_R = \frac{1}{\omega_{RC}}\left(2\pi\mu_1 + (-1)^{\mu_1}\arccos\left(\frac{\tau_{RC}^2 - h^2}{\tau_{RC}^2 + h^2}\right)\right),
$$

with $\mu_1 = 0$ ($\mu_1 = 1$) if the center of the right subsystem from $(3.1)|_{\epsilon=0}$ is virtual (real). Now, for f_R and g_R defined in (3.3) and (3.4), respectively, we have

$$
\begin{aligned}
I_R &= \int_{\widehat{AA_1}} g_R dx - f_R dy\\
&= \int_0^{t_R}\Big(g_R(x_R(t), y_R(t))x_R'(t) - f_R(x_R(t), y_R(t))y_R'(t)\Big)dt\\
&= \int_0^{t_R}\left(\sum_{i+j=0}^{1} q_{ij}x_R^i(t)y_R^j(t)x_R'(t) - \sum_{i+j=0}^{1} p_{ij}x_R^i(t)y_R^j(t)y_R'(t)\right)dt\\
&= \alpha_1 f_0(h) + \alpha_2 f_R^S(h),
\end{aligned}
\tag{3.22}
$$

with

$$
\alpha_1 = \frac{1}{\omega_{RC}}\Big(2(p_{00} + p_{10})\omega_{RC} - b_R(p_{10} + q_{01})\tau_{RC}\Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RC}}\Big(p_{10} + q_{01}\Big).
$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \cos(t) - h\sin(t),\\
y_{C1}(t) &= -h\cos(t) - \sin(t).
\end{aligned}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$
t_{C1} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right).
$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned}\bar{I}_C =& \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ =& \int_0^{t_{C1}} \Big(g_C(x_{C1}(t), y_{C1}(t))x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t))y'_{C1}(t)\Big)dt \\ =& \int_0^{t_{C1}} \left(\sum_{i+j=0}^{1} v_{ij} x^i_{C1}(t) y^j_{C1}(t) x'_{C1}(t) - \sum_{i+j=0}^{1} u_{ij} x^i_{C1}(t) y^j_{C1}(t) y'_{C1}(t) \right) dt \\ =& -2v_{00} + \alpha_3 f_0(h) + \alpha_4 f^C_C(h),\end{aligned} \tag{3.23}$$

with

$$\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{1}{2}\Big(u_{10} + v_{01}\Big).$$

The orbit $(x_L(t), y_L(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned}x_L(t) =& \frac{1}{\omega_{LC}}\Big(\omega_{LC} - b_L\tau_{LC} + b_L\tau_{LC}\cos(t\omega_{LC}) + b_L h \sin(t\omega_{LC})\Big), \\ y_L(t) =& \frac{1}{\omega_{LC}}\Big(- a_L\tau_{LC} + (-\omega_{LC}h + a_L\tau_{LC})\cos(t\omega_{LC}) + (a_L h + \omega_{LC}\tau_{LC})\sin(t\omega_{LC})\Big).\end{aligned}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \frac{1}{\omega_{LC}}\left(2\pi\mu_2 + (-1)^{\mu_2} \arccos\left(\frac{\tau^2_{LC} - h^2}{\tau^2_{LC} + h^2}\right)\right),$$

with $\mu_2 = 0$ ($\mu_2 = 1$) if the center of the left subsystem from $(3.1)|_{\epsilon=0}$ is virtual (real). Now, for f_L and g_L defined in (3.3) and (3.4), respectively, we have

$$\begin{aligned}I_L =& \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ =& \int_0^{t_L} \Big(g_L(x_L(t), y_L(t))x'_L(t) - f_L(x_L(t), y_L(t))y'_L(t)\Big)dt \\ =& \int_0^{t_L} \left(\sum_{i+j=0}^{1} s_{ij} x^i_L(t) y^j_L(t) x'_L(t) - \sum_{i+j=0}^{1} r_{ij} x^i_L(t) y^j_L(t) y'_L(t) \right) dt \\ =& \alpha_5 f_0(h) + \alpha_6 f^C_L(h),\end{aligned} \tag{3.24}$$

with

$$\alpha_5 = \frac{1}{\omega_{LC}}\Big(2(r_{10} - r_{00})\omega_{LC} - b_L(r_{10} + s_{01})\tau_{LC}\Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LC}}\Big(r_{10} + s_{01}\Big).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(3.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$x_{C2}(t) = -\cos(t) + h\sin(t),$$
$$y_{C2}(t) = h\cos(t) + \sin(t).$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos\left(\frac{h^2-1}{h^2+1}\right).$$

Now, for f_C and g_C defined in (3.3) and (3.4), respectively, we obtain

$$\begin{aligned}
I_C &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\
&= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t))x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t))y'_{C2}(t)\big)dt \\
&= \int_0^{t_{C2}} \left(\sum_{i+j=0}^{1} v_{ij}x^i_{C2}(t)y^j_{C2}(t)x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij}x^i_{C2}(t)y^j_{C2}(t)y'_{C2}(t)\right)dt \\
&= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f^C_C(h).
\end{aligned} \tag{3.25}$$

Hence, by Corollary 1, the first order Melnikov function associated to system (3.1) is given by

$$M(h) = b_R I^1_C + \frac{b_R}{b_L} I^1_L + b_R \bar{I}^1_C + I^1_R. \tag{3.26}$$

Replacing (3.22), (3.23), (3.24) and (3.25) into (3.26) we obtain (3.21), with

$$k_0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k^C_C = 2b_R\alpha_4, \quad k^S_R = \alpha_2 \quad \text{and} \quad k^C_L = \frac{b_R}{b_L}\alpha_6.$$

□

3.5 Proof of Theorem 5

In order to prove Theorem 5, we study the number of zeros of the Melnikov function M associated with system (3.1). Our study focuses on the neighborhood of the periodic orbits that bound the periodic annulus, since to estimate the zeros of the Melnikov functions we consider their expansions at the point corresponding to these orbits. For this purpose, we need the following results.

Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(h) = \sum_{j=0}^{n} C_j(\delta)(h-\tau)^j + \mathcal{O}((h-\tau)^{n+1}), \tag{3.27}$$

with $\tau \geq 0$ fixed and the coefficients $C_j(\delta)$, $j = 0, \ldots, n$, are smooth functions of the parameters $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. Then we have the following proposition.

Proposition 3. *Suppose that there exist an integer $k \geq 1$ and $\tilde{\delta} \in \mathbb{R}^m$ with $m \geq k+1$ such that*

$$C_j(\tilde{\delta}) = 0, \;\; j = 0, \ldots, k, \quad \textit{and} \quad \operatorname{rank} \frac{\partial(C_0, \ldots, C_k)}{\partial(\delta_1, \ldots, \delta_m)}(\tilde{\delta}) = k+1. \tag{3.28}$$

Then function (3.27) *has at least k real simple zeros in a neighborhood of $h = \tau$ for all δ near $\tilde{\delta}$. Moreover, function* (3.27) *has exactly k real simple zeros, in a neighborhood of $h = \tau$ for all δ near $\tilde{\delta}$, if $C_k(\tilde{\delta}) \neq 0$.*

Proof. By the condition (3.28) we can assume that

$$\det \frac{\partial(C_0, \ldots, C_k)}{\partial(\delta_1, \ldots, \delta_{k+1})}(\tilde{\delta}) \neq 0.$$

Then the change of parameters $\tilde{C}_i = C_i(\delta_1, \ldots, \delta_{k+1}, \tilde{\delta}_{k+2}, \ldots, \tilde{\delta}_m)$, $i = 0, \ldots, k$, has inverse $\delta_j(\tilde{C}_0, \ldots, \tilde{C}_k)$, $j = 1, \ldots, k+1$, and we can write (3.27) as

$$F(h) = \tilde{C}_0 + \tilde{C}_1(h-\tau) + \cdots + \tilde{C}_k(h-\tau)^k + \mathcal{O}((h-\tau)^{k+1}), \tag{3.29}$$

with $\tilde{C}_k(\tilde{\delta}) \neq 0$ and $\tilde{C}_j(\tilde{\delta}) = 0$, $j = 0, \ldots, k-1$.

Let $0 < |\tilde{C}_k - C_k(\tilde{\delta})| \ll 1$ and $0 < \tau - h_k \ll 1$ such that

$$\tilde{C}_k(h_k - \tau)^k > 0.$$

Take $\tilde{C}_{k-1}$ such that $|\tilde{C}_{k-1}| \ll |\tilde{C}_k|$, $\tilde{C}_{k-1}\tilde{C}_k > 0$ and

$$\tilde{C}_{k-1}(h_k - \tau)^{k-1} + \tilde{C}_k(h_k - \tau)^k > 0.$$

Now, as $\tilde{C}_{k-1}\tilde{C}_k > 0$, we can choose h_{k-1}, such that $0 < \tau - h_{k-1} \ll \tau - h_k \ll 1$ and

$$\tilde{C}_{k-1}(h_{k-1} - \tau)^{k-1} + \tilde{C}_k(h_{k-1} - \tau)^k < 0.$$

Therefore, the equation

$$\tilde{C}_{k-1}(h-\tau)^{k-1} + \tilde{C}_k(h-\tau)^k = 0$$

has a zero h_k^*, with $h_k < h_k^* < h_{k-1}$. Continuing with this reasoning, there are $\tilde{C}_0, \ldots, \tilde{C}_k$, such that

$$\tilde{C}_0\tilde{C}_1 > 0,\ \tilde{C}_1\tilde{C}_2 > 0, \ldots,\ \tilde{C}_{k-1}\tilde{C}_k > 0,\ |\tilde{C}_0| \ll |\tilde{C}_1| \ll \cdots \ll |\tilde{C}_k| \ll 1,$$

and equation (3.29) has k real positive zeros. Note that we could have done all of the above computations by adding at each step the remainder of function (3.27) (indicated by the Landau Symbols). Moreover, when $\tilde{C}_k(\tilde{\delta}) \neq 0$ and $C_j(\tilde{\delta}) = 0$, $j = 0, \ldots, k-1$, by Rolle's theorem, for all δ near $\tilde{\delta}$ we can choose the $\tilde{C}_j$ and h_j, $j = 0, \ldots, k$, such that equation (3.29) has exactly k real positive simple zeros near $h = \tau$. $\square$

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^{k+1}, $k \geq 1$. Then*

$$f(x)\log(\tau - x) = \mathcal{P}(x)\log(\tau - x) + \mathcal{R}(x),$$

where $\mathcal{P}(x) = \sum_{i=0}^{k} \frac{f^{(i)}(\tau)}{i!}(x-\tau)^i$ is the Taylor's polynomial of f at $x = \tau$ of degree k and $\mathcal{R}(x) = \log(\tau - x)(f(x) - \mathcal{P}(x))$. Moreover, $\lim_{x\to\tau} \mathcal{R}(x) = 0$.

Proof. To prove the lemma, it suffices to show that $\lim_{x\to\tau} \mathcal{R}(x) = 0$. By the Taylor's formula with Lagrange remainder, there is $c \in \mathbb{R}$ such that

$$\mathcal{R}(x) = \frac{f^{(k+1)}(c)}{(k+1)!}\log(\tau - x)(x-\tau)^{k+1}.$$

By the L'Hospital rule

$$\begin{aligned}
\lim_{x\to\tau} \mathcal{R}(x) &= \frac{f^{(k+1)}(c)}{(k+1)!}\lim_{x\to\tau}\log(\tau - x)(x-\tau)^{k+1} \\
&= \frac{f^{(k+1)}(c)}{(k+1)!}\lim_{x\to\tau}\frac{\log(\tau - x)}{(x-\tau)^{-(k+1)}} \\
&= \frac{f^{(k+1)}(c)}{(k+1)!}\lim_{x\to\tau}\frac{(\tau - x)^{-1}}{(k+1)(x-\tau)^{-(k+2)}} \\
&= -\frac{f^{(k+1)}(c)}{(k+1)!(k+1)}\lim_{x\to\tau}(x-\tau)^{(k+1)} = 0.
\end{aligned}$$

$\square$

Finally, we are able to prove the Theorem 5.

Proof of Theorem 5. In order to prove Theorem 5, let us consider the Melnikov functions associated to system (3.1) given by Theorems 6, 7 and 8. Our study will be concentrated in a neighborhood of the orbit L_h, with $h = \tau_{RS}$ for the cases SCS, CCS and $h = \tau$, $\tau \in (0, \infty)$, for the case CCC. We begin the proof with the case where system $(3.1)|_{\epsilon=0}$ is of the type SCS. For this case, we have two subcases. The first one is when we have a homoclinic loop (i.e. $\tau_{LS} \neq \tau_{RS}$) and the second one is when we have two heteroclinic orbits (i.e. $\tau_{LS} = \tau_{RS}$). The second and third cases are when system $(3.1)|_{\epsilon=0}$ is of the type CCS and CCC.

Case SCS. Consider the Melnikov functions given by Theorem 6. By Lemma 1, we can expand these functions at $h = \tau_{RS}$, when $\tau_{LS} \neq \tau_{RS}$, as

$$M(h) = \sum_{j=0}^{3} C_j^1 (h - \tau_{RS})^j + \sum_{j=1}^{2} D_j^1 \log(\tau_{RS} - h)(h - \tau_{RS})^j + \mathcal{O}((h - \tau_{RS})^4), \tag{3.30}$$

and, when $\tau_{LS} = \tau_{RS}$, as

$$M(h) = \sum_{j=0}^{2} C_j^2 (h - \tau_{RS})^j + \sum_{j=1}^{2} D_j^2 \log(\tau_{RS} - h)(h - \tau_{RS})^j + \mathcal{O}((h - \tau_{RS})^3), \tag{3.31}$$

where

$$\begin{aligned}
C_0^1 &= \tau_{RS}\Bigg(\frac{2b_R}{b_L}(r_{10} - r_{00}) + 2(p_{00} + p_{10} + b_R(v_{01} - u_{10})) + \frac{\tau_{LS} b_R}{\omega_{LS}}(r_{10} + s_{01}) + \frac{\tau_{RS} b_R}{\omega_{RS}} \\
&\quad (p_{10} + q_{01})\Bigg) + b_R(u_{10} + v_{01})(1 + \tau_{RS}^2) \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right) + \frac{b_R}{2\omega_{LS}}(r_{10} + s_{01}) \\
&\quad (\tau_{RS}^2 - \tau_{LS}^2) \log\left(\frac{\tau_{LS} + \tau_{RS}}{\tau_{LS} - \tau_{RS}}\right), \\
C_1^1 &= \frac{2b_R}{b_L}(r_{10} - r_{00}) + 2(p_{00} + p_{10} - 2b_R u_{10}) + \frac{\tau_{LS} b_R}{\omega_{LS}}(p_{10} + q_{01}) + b_R \tau_{RS}\Bigg(2(u_{10} + v_{01}) \\
&\quad \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right) + \frac{\log(2\tau_{RS})}{\tau_{RS}}(p_{10} + q_{01}) + \frac{1}{\tau_{LS}}(r_{10} + s_{01}) \log\left(\frac{\tau_{LS} + \tau_{RS}}{\tau_{LS} - \tau_{RS}}\right)\Bigg), \\
C_2^1 &= \frac{b_R}{2}\Bigg(2(u_{10} + v_{01})\Bigg(\arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right) - \frac{2\tau_{RS}}{1 + \tau_{RS}^2}\Bigg) + \frac{1}{\omega_{RS}}(p_{10} + q_{01})(1 + \log(2) \\
&\quad + \log(\tau_{RS})) + \frac{1}{\omega_{LS}}(r_{10} + s_{01})\left(\frac{2\tau_{RS}\tau_{LS}}{\tau_{LS}^2 - \tau_{RS}^2} + \log\left(\frac{\tau_{LS} + \tau_{RS}}{\tau_{LS} - \tau_{RS}}\right)\right)\Bigg), \\
C_3^1 &= \frac{b_R}{8\omega_{RS}\tau_{RS}}(p_{10} + q_{01}) + \frac{2b_R}{3}\left(\frac{\tau_{LS}^3}{\omega_{LS}(\tau_{LS}^2 - \tau_{RS}^2)^2}(r_{10} + s_{01}) - \frac{2}{(1 + \tau_{RS}^2)^2}(u_{10} + v_{01})\right), \\
C_0^2 &= \tau_{RS}\Bigg(\frac{2b_R}{b_L}(r_{10} - r_{00}) + 2(p_{00} + p_{10} + b_R(v_{01} - u_{10})) + \frac{\tau_{RS} b_R}{\omega_{LS}}(r_{10} + s_{01}) + \frac{\tau_{RS} b_R}{\omega_{RS}} \\
&\quad (p_{10} + q_{01})\Bigg) + b_R(u_{10} + v_{01})(1 + \tau_{RS}^2) \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right),
\end{aligned}$$

$$C_1^2 = \frac{2b_R}{b_L}(r_{10} - r_{00}) + 2(p_{00} + p_{10} + b_R(v_{01} - u_{10})) + \frac{\tau_{RS} b_R}{\omega_{LS}}(r_{10} + s_{01}) + \frac{\tau_{RS} b_R}{\omega_{RS}}$$
$$(p_{10} + q_{01}) + 2\tau_{RS} b_R(u_{10} + v_{01})\left(\arccos\left(\frac{\tau_R^2 - 1}{\tau_{RS}^2 + 1}\right) - \frac{\tau_{RS}^2 + 1}{\tau_{RS}}\right) + \frac{\tau_R b_R \log(2\tau_{RS})}{\omega_{RS}\omega_{LS}}$$
$$((p_{10} + q_{01})\omega_{LS} + (r_{10} + s_{01})\omega_{RS}),$$
$$C_2^2 = b_R(u_{10} + v_{01})\left(\arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right) - \frac{2\tau_{RS}}{\tau_{RS}^2 + 1}\right) + \frac{b_R}{2\omega_{RS}\omega_{LS}}((p_{10} + q_{01})\omega_{LS} + (r_{10}$$
$$+ s_{01})\omega_{RS})(1 + \log(2) + \log(\tau_{RS})),$$

and D_j^i, $i = 1, 2$ and $j = 1, 2$, depending on the parameters of system (3.1), whose expressions have been omitted for simplicity. Moreover, as

$$\lim_{h \to \tau_{RS}} \log(\tau_{RS} - h)(h - \tau_{RS})^j = 0, \quad j = 1, 2,$$

the factors $\log(\tau_{RS} - h)$ of the expansions in (3.30) and (3.31) can be disregarded in the study of the number of zeros. Therefore, we consider only the coefficients C_j^i, $i = 1, 2$ and $j = 0, 1, 2, 3$.

When $\tau_{RS} \neq \tau_{LS}$, for

$$p_{00} = -\frac{1}{b_L}\Big(b_L p_{10} - b_R(r_{00} - r_{10}) + 2b_L b_R v_{01}\Big), \quad q_{01} = -p_{10}, \quad s_{01} = -r_{10} \quad \text{and} \quad u_{10} = -v_{01}$$

we have that

$$C_0^1 = C_1^1 = C_2^1 = C_3^1 = 0 \quad \text{with} \quad \text{rank}\frac{\partial(C_0^1, C_1^1, C_2^1, C_3^1)}{\partial(p_{00}, q_{01}, s_{01}, u_{10})} = 4.$$

This implies, by Proposition 3, that C_0^1, C_1^1, C_2^1 and C_3^1 can be taken as free coefficients, satisfying

$$C_0^1 C_1^1 > 0, \quad C_1^1 C_2^1 > 0, \quad C_2^1 C_3^1 > 0 \quad \text{and} \quad |C_0^1| \ll |C_1^1| \ll |C_2^1| \ll |C_3^1| \ll 1,$$

such that function (3.30) has at least three positive simple zeros near $h = \tau_{RS}$. Therefore, the number of limit cycles from system (3.1) that can bifurcate of the periodic annulus near the homoclinic loop in $h = \tau_{RS}$, when $\tau_R \neq \tau_{LS}$, is at least three.

When $\tau_{RS} = \tau_{LS}$, for

$$
\begin{aligned}
p_{00} =& -p_{10} + \frac{b_R}{b_R}(r_{00} - r_{10} + b_L(u_{10} - v_{01})) - \frac{b_R \tau_{RS}}{2\omega_{LS}}(r_{10} + s_{01}) - \frac{b_R \tau_{RS}}{2\omega_{RS}}(p_{10} + q_{01}) \\
& - \frac{b_R}{2\tau_{RS}}(u_{10} + v_{01})(1 + \tau_{RS}^2) \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right), \\
s_{01} =& -r_{10} - \frac{\omega_{LS}}{\omega_{RS}}(p_{10} + q_{01}) + \frac{\omega_{LS}}{\tau_{RS}^2 \log(2\tau_{RS})}(u_{10} + v_{01})\Bigg(2(\tau_{RS} + \tau_{RS}^3) - (\tau_{RS}^2 - 1) \\
& \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right)\Bigg) - \frac{2\omega_{LS}\tau_{RS}}{\log(2\tau_{RS})}(u_{10} + v_{01}),
\end{aligned}
$$

we have that

$$
C_0^2 = C_1^2 = 0, \quad C_2^2 = \frac{b_R(u_{10} + v_{01})\lambda}{2\tau_{RS}^2(1 + \tau_{RS}^2)\log(2\tau_{RS})} \quad \text{and} \quad \text{rank}\frac{\partial(C_0^2, C_1^2, C_2^2)}{\partial(p_{00}, s_{01}, u_{10})} = 3,
$$

with

$$
\begin{aligned}
\lambda =& 2\tau_{RS}\big(1 + \tau_{RS} r^2 - (\ \tau_{RS}^2 - 1)\log(2\tau_{RS})\big) + (1 + \tau_{RS}^2)\big(1 - \tau_{RS}^2 + (1 + \tau_{RS}^2)\log(2\tau_{RS})\big) \\
& \arccos\left(\frac{\tau_{RS}^2 - 1}{\tau_{RS}^2 + 1}\right).
\end{aligned}
$$

Moreover, there are parameter values for system $(3.1)|_{\epsilon=0}$ such that $\lambda \neq 0$. For example, if $a_L = b_L = b_R = c_R = \beta_L = 1$, $a_R = c_L = 0$ and $\beta_R = -2$, we have that the left subsystem from $(3.1)|_{\epsilon=0}$ has a saddle at the point $(-2, 1)$, the right subsystem from $(3.1)|_{\epsilon=0}$ has a saddle at the point $(2, 0)$, $\tau_{RS} = 1$ and $\lambda = 4 + 2\pi \log(2)$. Hence, for $u_{10} \neq -v_{01}$, we have that $C_2^2 \neq 0$. This implies, by Proposition 3, that we can choose C_0^2, C_1^2, and C_2^2 as free coefficients, such that function (3.31) has exactly two positive simple zeros near $h = \tau_{RS}$. Therefore, the number of limit cycles from system (3.1) that can bifurcate of the periodic annulus near the heteroclinic orbit in $h = \tau_{RS}$, when $\tau_{RS} = \tau_{LS}$, is at least two.

Case CCS. Consider the Melnikov function given by Theorem 7. By Lemma 1, we can expand this function at $h = \tau_{RS}$ as

$$
M(h) = \sum_{j=0}^{3} C_j (h - \tau_{RS})^j + \sum_{j=1}^{2} D_j \log(\tau_{RS} - h)(h - \tau_{RS})^j + \mathcal{O}((h - \tau_{RS})^4), \tag{3.32}
$$

where

$$C_0 = \tau_{RS}\Bigg(\frac{2b_R}{b_L}(r_{10}-r_{00}) + 2(p_{00}+p_{10}+b_R(v_{01}-u_{10})) - \frac{b_R\tau_{LC}}{\omega_{LC}}(r_{10}+s_{01}) + \frac{b_R\tau_{RS}}{\omega_{RS}}(p_{10}$$
$$+ q_{01})\Bigg) + b_R(u_{10}+v_{01})(1+\tau_{RS}^2)(\arccos\left(\frac{\tau_{RS}^2-1}{\tau_{RS}^2+1}\right) + \frac{b_R}{2\omega_{LC}}(r_{10}+s_{01})(\tau_{LC}^2+\tau_{RS}^2)$$
$$(2\pi\mu_2 + (-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2-\tau_{RS}^2}{\tau_{LC}^2+\tau_{RS}^2}\right)\Bigg),$$
$$C_1 = \frac{2b_R}{b_L}(r_{10}-r_{00}) + 2(p_{00}+p_{10}-2b_Ru_{10}) + \frac{b_R\tau_{RS}}{\omega_{RS}}(p_{10}+q_{01}) + \frac{b_R}{\omega_{LC}}(r_{10}+s_{01})(((-1)^{\mu_2}$$
$$-1)\tau_{LC} + 2\pi\mu_2\tau_{RS}) + b_R\tau_{RS}\Bigg(2(u_{10}+v_{01})\arccos\left(\frac{\tau_{RS}^2-1}{\tau_{RS}^2+1}\right) + \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10}+s_{01})$$
$$\arccos\left(\frac{\tau_{LC}^2-\tau_{RS}^2}{\tau_L^2+\tau_{RS}^2}\right) + \frac{1}{\omega_{RS}}(p_{10}+q_{01})\log(2\tau_{RS})\Bigg),$$

$$C_2 = \frac{b_R}{2}\Bigg(2(u_{10}+v_{01})\Bigg(\arccos\left(\frac{\tau_{RS}^2-1}{\tau_{RS}^2+1}\right) - \frac{2\tau_{RS}}{1+\tau_{RS}^2}\Bigg) + \frac{1}{\omega_{LC}}(r_{10}+s_{01})\Bigg(2\pi\mu_2$$
$$+ \frac{2(-1)^{\mu_2}\tau_{LC}\tau_{RS}}{\tau_{LC}^2+\tau_{RS}^2} + (-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2-\tau_{RS}^2}{\tau_{LC}^2+\tau_{RS}^2}\right)\Bigg) + \frac{1}{\omega_{RS}}(p_{10}+q_{01})(1+\log(2)$$
$$+\log(\tau_{RS}))\Bigg),$$
$$C_3 = \frac{b_R}{8\omega_{RS}\tau_{RS}}(p_{10}+q_{01}) + \frac{2b_R}{3}\left(\frac{(-1)^{\mu_2}\tau_{LC}^3}{\omega_{LC}(\tau_{LC}^2+\tau_{RS}^2)^2}(r_{10}+s_{01}) - \frac{2}{(\tau_{RS}^2+1)^2}(u_{10}+v_{01})\right),$$

and D_j, $j=1,2$, depending on the parameters of system (3.1), whose expressions have been omitted for simplicity. As in the preview case, we consider only the coefficients C_j, $j=0,\ldots,3$. Moreover, when

$$p_{00} = -\frac{1}{b_L}\Big(b_Lp_{10} - b_R(r_{00}-r_{10}) + 2b_Lb_Rv_{01}\Big),\quad q_{01}=-p_{10},\quad s_{01}=-r_{10}\quad \text{and}\quad u_{10}=-v_{01}$$

we have that

$$C_0=C_1=C_2=C_3=0 \quad \text{with} \quad \text{rank}\frac{\partial(C_0,C_1,C_2,C_3)}{\partial(p_{00},q_{01},s_{01},u_{10})}=4.$$

This implies, by Proposition 3, that C_0, C_1, C_2 and C_3 can be taken as free coefficients such that function (3.32) has at least three positive simple zeros near $h=\tau_{RS}$. Therefore, the number of limit cycles from system (3.1) that can bifurcate of the periodic annulus near the homoclinic loop in $h=\tau_{RS}$, is at least three.

Case CCC. Consider the Melnikov function given by Theorem 8. By Lemma 1, we can expand this function at $h=\tau$, with $\tau\in(0,\infty)$, as

$$M(h)=\sum_{j=0}^{3}C_j(h-\tau)^j+\mathcal{O}((h-\tau)^4), \tag{3.33}$$

where

$$\begin{aligned}
C_0 &= 2(p_{00}+p_{10})\tau+\frac{2b_R\tau}{b_L}(r_{10}-r_{00})+2b_R(v_{01}-u_{10})\tau-\frac{\tau b_R\tau_{LC}}{\omega_{LC}}(r_{10}+s_{01})-\frac{\tau b_R\tau_{RC}}{\omega_{RC}}\\
&\quad(p_{10}+q_{01})+b_R(u_{10}+v_{01})(1+\tau^2)\arccos\left(\frac{\tau^2-1}{\tau^2+1}\right)+\frac{b_R(\tau^2+\tau_{LC}^2)(r_{10}+s_{01})}{2\omega_{LC}}\\
&\quad\left(2\pi\mu_2+(-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2-\tau^2}{\tau^2+\tau_{LC}^2}\right)\right)+\frac{b_R(\tau^2+\tau_{RC}^2)(p_{10}+q_{01})}{2\omega_{RC}}\Bigg(2\pi\mu_1+(-1)^{\mu_1}\\
&\quad\arccos\left(\frac{\tau_{RC}^2-\tau^2}{\tau^2+\tau_{RC}^2}\right)\Bigg),\\
C_1 &= 2(p_{00}+p_{10})+b_R\left(\frac{2}{b_L}(r_{10}-r_{00})-4u_{10}+\frac{(r_{10}+s_{01})}{\omega_{LC}}\Big(2\pi\mu_2\tau+((-1)^{\mu_2}-1)\tau_{LC}\Big)\right)\\
&\quad+\frac{b_R(p_{10}+q_{01})}{\omega_{LC}}\Big(2\pi\mu_1\tau+((-1)^{\mu_1}-1)\tau_{RC}\Big)+b_R\tau\Bigg(2(u_{10}+v_{01})\arccos\left(\frac{\tau^2-1}{\tau^2+1}\right)\\
&\quad+\frac{(-1)^{\mu_2}(r_{10}+s_{01})}{\omega_{LC}}\arccos\left(\frac{\tau_{LC}^2-\tau^2}{\tau^2+\tau_{LC}^2}\right)+\frac{(-1)^{\mu_1}(p_{10}+q_{01})}{\omega_{RC}}\arccos\left(\frac{\tau_{RC}^2-\tau^2}{\tau^2+\tau_{RC}^2}\right)\Bigg),\\
C_2 &= b_R\Bigg(\frac{\pi\mu_2(r_{10}+s_{01})}{\omega_{LC}}+\frac{\pi\mu_1(p_{10}+q_{01})}{\omega_{RC}}+\tau\Bigg(-\frac{2(u_{10}+v_{01})}{\tau^2+1}+\frac{(-1)^{\mu_2}\tau_{LC}(r_{10}+s_{01})}{\omega_{LC}(\tau^2+\tau_{LC}^2)}\\
&\quad+\frac{(-1)^{\mu_1}\tau_{RC}(p_{10}+q_{01})}{\omega_{RC}(\tau^2+\tau_{RC}^2)}\Bigg)\Bigg)+b_R(u_{10}+v_{01})\arccos\left(\frac{\tau^2-1}{\tau^2+1}\right)+\frac{b_R(-1)^{\mu_2}(r_{10}+s_{01})}{2\omega_{LC}}\\
&\quad\arccos\left(\frac{\tau_{LC}^2-\tau^2}{\tau^2+\tau_{LC}^2}\right)+\frac{b_R(-1)^{\mu_1}(p_{10}+q_{01})}{2\omega_{RC}}\arccos\left(\frac{\tau_{RC}^2-\tau^2}{\tau^2+\tau_{RC}^2}\right),\\
C_3 &= \frac{2b_R}{3}\left(-\frac{2(u_{10}+v_{01})}{(1+\tau^2)^2}+\frac{(-1)^{\mu_2}\tau_{LC}^3(r_{10}+s_{01})}{\omega_{LC}(\tau^2+\tau_{LC}^2)^2}+\frac{(-1)^{\mu_1}\tau_{RC}^3(p_{10}+q_{01})}{\omega_{RC}(\tau^2+\tau_{RC}^2)^2}\right).
\end{aligned}$$

Moreover, when

$$p_{00}=-\frac{1}{b_L}\Big(b_Lp_{10}-b_R(r_{00}-r_{10})+2b_Lb_Rv_{01}\Big),\quad q_{01}=-p_{10},\quad s_{01}=-r_{10}\quad\text{and}\quad u_{10}=-v_{01}$$

we have that

$$C_0=C_1=C_2=C_3=0\quad\text{with}\quad\operatorname{rank}\frac{\partial(C_0,C_1,C_2,C_3)}{\partial(p_{00},q_{01},s_{01},u_{10})}=4.$$

This implies, by Proposition 3, that C_0, C_1, C_2 and C_3 can be taken as free coefficients such that function (3.33) has at least three positive simple zeros near $h=\tau$. Therefore, the number of limit cycles from system (3.1) that can bifurcate of the periodic annulus in $h=\tau$ is at least three. □

3.6 Another method to prove Theorem 5

In the article II of the list of published and submitted papers derived from this thesis, which is available at the end of Introduction, we use another approach to prove Theorem 5. The idea is to apply a well-known result, which states that if a family of n real functions is linearly independent, then there exists a linear combination of these functions with at least $n-1$ zeros. In this section, we will prove Theorem 5 using this method, but first, let us recall some basic results from linear algebra.

Definition 11. *Let $\{f_0, f_1, \ldots, f_n\}$ be a set of real functions defined on a proper interval $I \subset \mathbb{R}$. We say that $\{f_0, f_1, \ldots, f_n\}$ is linearly independent if the unique solution of the equation*

$$\sum_{i=0}^{n} \alpha_i f_i(t) = 0,$$

for all $t \in I$, is $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$.

Proposition 4. *If $\{f_0, f_1, \ldots, f_n\}$ is linearly independent then there exist $t_1, t_2, \ldots, t_n \in I$, with $t_i \neq t_j$ for $i \neq j$, and $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, not all null, such that for every $j \in \{1, 2, \ldots, n\}$*

$$\sum_{i=0}^{n} \alpha_i f_i(t_j) = 0.$$

For a proof of Proposition 4, see Proposition 1 in [30].

Definition 12. *Let $f_0, f_1, \ldots, f_n$ be functions that have derivatives until order n on I. The continuous Wronskian of $\{f_0, f_1, \ldots, f_k\}$, $k = 0, \ldots, n$, at $x \in I$ is defined to be*

$$W_k(f_0, f_1, \ldots, f_k)(x) = \begin{vmatrix} f_0(x) & f_1(x) & \ldots & f_k(x) \\ f_0'(x) & f_1'(x) & \ldots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(x) & f_1^{(k)}(x) & \ldots & f_k^{(k)}(x) \end{vmatrix}.$$

Recall that if the Wronskian $W(f_0, f_1, \ldots, f_n)(x)$ is different of zero for some $x \in I$, then $\{f_0, f_1, \ldots, f_n\}$ is linearly independent on I. For more details, see [62, p. 124].

The next corollaries provide lower bounds for the number of limit cycles of system (3.1) that can bifurcate from a periodic annulus of system $(3.1)|_{\epsilon=0}$ in the cases SCS, CCS and CCC.

Corollary 2 (Case SCS with $\tau_{LS} \neq \tau_{RS}$). *Consider system* (3.1) *with* $a_L = b_L = b_R = c_R = 1$, $a_R = c_L = 0$, $\beta_L = 2$ *and* $\beta_R = -2$. *Then, for* $0 < \epsilon \ll 1$, *system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = b_L = b_R = c_R = 1$, $a_R = c_L = 0$, $\beta_L = 2$ and $\beta_R = -2$, the eigenvalues of the linear part of the left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are ± 1, $\pm i$ and ± 1, respectively, i.e., we have one center and two saddles. Moreover, the coordinates of the singular points on the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(-3, 2)$ and $(2, 0)$, respectively, and so the saddles are real. Note that $P_L^u = (-1, 2)$ and $P_R^s = (1, 1)$, i.e., the ordinates of this points are different and $\tau_{RS} = 1$. Therefore, the first order Melnikov function (3.8) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^S f_L^S(h), \quad \forall\, h \in (0, 1),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^S(h) = (h^2-1)\log\left(\frac{h+1}{1-h}\right), \quad f_L^S(h) = (h^2-4)\log\left(\frac{h+2}{2-h}\right),$$

$$k_0 = 2(p_{00} - r_{00} + 2r_{10} + s_{01} - u_{10} + v_{01}) + 3p_{10} + q_{01},$$
$$k_C^C = u_{10} + v_{01}, \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^S = \frac{1}{2}\Big(r_{10} + s_{01}\Big).$$

Consider the set of functions $\mathcal{F}_{SCS} = \{f_0, f_C^C, f_R^S, f_L^S\}$. Using the algebraic manipulator Mathematica (see [57]), we compute the Wronskian $W(f_0, f_C^C, f_R^S, f_L^S)(h)$ and evaluate in some point on the interval $(0, 1)$. More precisely, $W(f_0, f_C^C, f_R^S, f_L^S)(0.4) \approx 9.16568$. Then, the set of functions $\mathcal{F}_{SCS}$ is linearly independent on the interval $(0, 1)$ and, by Proposition 4, there are $h_i \in (0, 1)$, with $i = 1, 2, 3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1, 2, 3$, i.e., system (3.1) has at least three limit cycles. □

Corollary 3 (Case SCS with $\tau_{LS} = \tau_{RS}$). *Consider system* (3.1) *with* $a_L = b_L = b_R = c_R = \beta_L = 1$, $a_R = c_L = 0$ *and* $\beta_R = -2$. *Then, for* $0 < \epsilon \ll 1$, *system* (3.1) *has at least two limit cycles.*

Proof. For $a_L = b_L = b_R = c_R = \beta_L = 1$, $a_R = c_L = 0$ and $\beta_R = -2$, the eigenvalues of the linear part of the left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are ± 1, $\pm i$ and ± 1, respectively, i.e., we have one center and two saddles. Moreover, the coordinates of the singular points on the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(-2, 1)$ and $(2, 0)$, respectively, and so the saddles are real. Note that $P_L^u = (-1, 1)$ and $P_R^s = (1, 1)$, i.e., the ordinates of this points are the same and

$\tau_{RS} = 1$. Therefore, the first order Melnikov function (3.9) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h), \quad \forall\, h \in (0,1),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right), \quad f_R^S(h) = (h^2-1)\log\left(\frac{h+1}{1-h}\right),$$

$$k_0 = 2(p_{00} - r_{00} + 2r_{10} + s_{01} - u_{10} + v_{01}) + 3p_{10} + q_{01},$$
$$k_C^C = u_{10} + v_{01} \text{ and } k_R^S = \frac{1}{2}\Big(p_{10} + q_{01} + r_{10} + s_{01}\Big).$$

Consider the set of functions $\mathcal{F}_{SCS} = \{f_0, f_C^C, f_R^S\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^S)(h)$ and evaluate in some point on the interval $(0,1)$. More precisely, $W(f_0, f_C^C, f_R^S)(0.4) \approx -10.6955$. Then, the set of functions $\mathcal{F}_{SCS}$ is linearly independent on the interval $(0,1)$ and, by Proposition 4, there are $h_i \in (0,1)$, with $i = 1,2$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1,2$, i.e., system (3.1) has at least two limit cycles. □

Corollary 4 (Case CCS with $\mu_2 = 0$)**.** *Consider system* (3.1) *with* $a_L = \beta_L = b_R = c_R = 1$, $b_L = 2$, $c_L = -1$, $a_R = 0$ *and* $\beta_R = -2$. *Then for* $0 < \epsilon \ll 1$, *system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = \beta_L = b_R = c_R = 1$, $b_L = 2$, $c_L = -1$, $a_R = 0$ and $\beta_R = -2$, the eigenvalues of the linear part of left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are $\pm i$, $\pm i$ and ± 1, respectively, i.e., we have two centers and one saddle. Moreover, the coordinates of the singular point on the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(3,-2)$ and $(2,0)$, respectively, and so we have a virtual center and a real saddle. Note that $P_R^s = (1,1)$ and $\tau_{RS} = 1$. Therefore, the first order Melnikov function (3.15) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^C f_L^C(h), \quad \forall\, h \in (0,1),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^S(h) = (h^2-1)\log\left(\frac{h+1}{1-h}\right), \quad f_L^C(h) = (h^2+4)\arccos\left(\frac{4-h^2}{h^2+4}\right),$$

$$k_0 = 2(p_{00} - r_{00} + 2r_{10} + s_{01} - u_{10} + v_{01}) + 3p_{10} + q_{01},$$
$$k_C^C = u_{10} + v_{01}, \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^C = \frac{1}{2}\Big(r_{10} + s_{01}\Big).$$

Consider the set of functions $\mathcal{F}_{CCS} = \{f_0, f_C^C, f_R^S, f_L^C\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^S, f_L^C)(h)$ and evaluate in some point on the interval $(0,1)$. More precisely, $W(f_0, f_C^C, f_R^S, f_L^C)(0.4) \approx 13.25$. Then, the set of functions $\mathcal{F}_{CCS}$ is linearly independent on the interval $(0,1)$ and, by Proposition 4, there are $h_i \in (0,1)$, with $i = 1,2,3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1,2,3$, i.e., system (3.1) has at least three limit cycles. □

Corollary 5 (Case CCS with $\mu_2 = 1$)**.** *Consider system* (3.1) *with* $a_L = b_R = c_R = 1$, $b_L = 2$, $c_L = -1$, $a_R = 0$ *and* $\beta_R = \beta_L = -2$. *Then for* $0 < \epsilon << 1$, *system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = b_R = c_R = 1$, $b_L = 2$, $c_L = -1$, $a_R = 0$ and $\beta_R = \beta_L = -2$, the eigenvalues of the linear part of left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are $\pm i$, $\pm i$ and ± 1, respectively, i.e., we have two centers and one saddle. Moreover, the coordinates of the equilibrium point on the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(-3,1)$ and $(2,0)$, respectively, and so we have a real center and a real saddle. Note that $P_R^s = (1,1)$ and $\tau_{RS} = 1$. Therefore, the first order Melnikov function (3.15) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^S f_R^S(h) + k_L^C f_L^C(h), \quad \forall\, h \in (0,1),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^S(h) = (h^2-1)\log\left(\frac{h+1}{1-h}\right), \quad f_L^C(h) = (h^2+1)\left(2\pi - \arccos\left(\frac{1-h^2}{h^2+1}\right)\right),$$

$$k_0 = 2(p_{00} + r_{10} - u_{10} + v_{01}) + 3p_{10} + q_{01} - r_{00} + s_{01},$$
$$k_C^C = u_{10} + v_{01} \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^S = \frac{1}{2}\Big(r_{10} + s_{01}\Big).$$

Consider the set of functions $\mathcal{F}_{CCS} = \{f_0, f_C^C, f_R^S, f_L^C\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^S, f_L^C)(h)$ and evaluate in some point on the interval $(0,1)$. More precisely, $W(f_0, f_C^C, f_R^S, f_L^C)(0.2) \approx 4.26846$. Then, the set of functions $\mathcal{F}_{CCS}$ is linearly independent on the interval $(0,1)$ and, by Proposition 4, there are $h_i \in (0,1)$, with $i = 1,2,3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1,2,3$, i.e., system (3.1) has at least three limit cycles. □

Corollary 6 (Case CCC with $\mu_1 = \mu_2 = 0$)**.** *Consider system* (3.1) *with* $a_L = b_R = \beta_L = 1$, $b_L = 2$, $c_L = c_R = -1$ *and* $a_R = \beta_R = 0$. *Then, for* $0 < \epsilon << 1$, *system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = b_R = \beta_L = 1$, $b_L = 2$, $c_L = c_R = -1$ and $a_R = \beta_R = 0$, the eigenvalues of the linear part of the left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e., we have three centers. Moreover, the coordinates of the singular points of the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(3,-2)$ and $(0,0)$, respectively, and so the centers are virtual. Therefore, the first order Melnikov function (3.21) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^C f_R^C(h) + k_L^C f_L^C(h), \quad \forall\, h \in (0,\infty),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^C(h) = (h^2+1)\arccos\left(\frac{1-h^2}{h^2+1}\right), \quad f_L^C(h) = (h^2+4)\arccos\left(\frac{4-h^2}{h^2+4}\right),$$

$$k_0 = p_{10} - q_{01} - r_{00} - r_{10} - 2(s_{01} + u_{10} - v_{01} - p_{00}),$$
$$k_C^C = u_{10} + v_{01}, \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^S = \frac{1}{2}\Big(r_{10} + s_{01}\Big)$$

Consider the set of functions $\mathcal{F}_{CCC} = \{f_0, f_C^C, f_R^C, f_L^C\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^C, f_L^C)(h)$ and evaluate in some point on the interval $(0,\infty)$. More precisely, $W(f_0, f_C^C, f_R^C, f_L^C)(0.2) \approx -0.730377$. Then, the set of functions $\mathcal{F}_{CCC}$ is linearly independent on the interval $(0,\infty)$ and, by Proposition 4, there are $h_i \in (0,\infty)$, with $i = 1,2,3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1,2,3$, i.e., system (3.1) has at least three limit cycles. □

Corollary 7 (Case CCC with $\mu_1 = 0$ and $\mu_2 = 1$)**.** *Consider system* (3.1) *with* $a_L = b_R = 1$, $\beta_L = -3$, $b_L = 2$, $c_L = c_R = -1$ *and* $a_R = \beta_R = 0$. *Then, for* $0 < \epsilon << 1$, *system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = b_R = 1$, $\beta_L = -3$, $b_L = 2$, $c_L = c_R = -1$ and $a_R = \beta_R = 0$, the eigenvalues of the linear part of the left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e., we have three centers. Moreover, the coordinates of the equilibrium points of the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(-5,2)$ and $(0,0)$, respectively, and so we have a real center and a virtual center. Therefore, the first order Melnikov function (3.21) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^C f_R^C(h) + k_L^C f_L^C(h), \quad \forall\, h \in (0,\infty),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^C(h) = (h^2+1)\arccos\left(\frac{1-h^2}{h^2+1}\right), \quad f_L^C(h) = (h^2+4)\left(2\pi - \arccos\left(\frac{4-h^2}{h^2+4}\right)\right),$$

$$k_0 = p_{10} - q_{01} - r_{00} + 3r_{10} + 2(s_{01} - u_{10} + v_{01} + p_{00}),$$
$$k_C^C = u_{10} + v_{01}, \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^S = \frac{1}{2}\Big(r_{10} + s_{01}\Big)$$

Consider the set of functions $\mathcal{F}_{CCC} = \{f_0, f_C^C, f_R^C, f_L^C\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^C, f_L^C)(h)$ and evaluate in some point on the interval $(0,\infty)$. More precisely, $W(f_0, f_C^C, f_R^C, f_L^C)(0.5) \approx -599.176$. Then, the set of functions $\mathcal{F}_{CCC}$ is linearly independent on the interval $(0,\infty)$ and, by Proposition 4, there are $h_i \in (0,\infty)$, with $i = 1,2,3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1,2,3$, i.e., system (3.1) has at least three limit cycles. $\square$

Corollary 8 (Case CCC with $\mu_1 = \mu_2 = 1$)**.** *Consider system* (3.1) *with $a_L = b_R = 1$, $\beta_L = -3$, $b_L = \beta_R = 2$, $c_L = c_R = -1$ and $a_R = 0$. Then, for $0 < \epsilon << 1$, system* (3.1) *has at least three limit cycles.*

Proof. For $a_L = b_R = 1$, $\beta_L = -3$, $b_L = \beta_R = 2$, $c_L = c_R = -1$ and $a_R = 0$, the eigenvalues of the linear part of the left, central and right subsystems from $(3.1)|_{\epsilon=0}$ are $\pm i$, $\pm i$ and $\pm i$, respectively, i.e., we have three centers. Moreover, the coordinates of the singular points of the left and right subsystems from $(3.1)|_{\epsilon=0}$ are $(-5,2)$ and $(2,0)$, respectively, and so the centers are real. Therefore, the first order Melnikov function (3.21) becomes

$$M(h) = k_0 f_0(h) + k_C^C f_C^C(h) + k_R^C f_R^C(h) + k_L^C f_L^C(h), \quad \forall\, h \in (0,\infty),$$

with

$$f_0(h) = h, \quad f_C^C(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right),$$
$$f_R^C(h) = (h^2+1)\left(2\pi - \arccos\left(\frac{1-h^2}{h^2+1}\right)\right), \quad f_L^C(h) = (h^2+4)\left(2\pi - \arccos\left(\frac{4-h^2}{h^2+4}\right)\right),$$

$$k_0 = 3(p_{10} + r_{10}) + q_{01} - r_{00} + 2(s_{01} - u_{10} + v_{01} + p_{00}),$$
$$k_C^C = u_{10} + v_{01}, \quad k_R^S = \frac{1}{2}\Big(p_{10} + q_{01}\Big) \quad \text{and} \quad k_L^S = \frac{1}{2}\Big(r_{10} + s_{01}\Big)$$

Consider the set of functions $\mathcal{F}_{CCC} = \{f_0, f_C^C, f_R^C, f_L^C\}$. Using the algebraic manipulator Mathematica, we compute the Wronskian $W(f_0, f_C^C, f_R^C, f_L^C)(h)$ and evaluate in some point on the interval $(0, \infty)$. More precisely, $W(f_0, f_C^C, f_R^C, f_L^C)(0.5) \approx -599.176$. Then, the set of functions $\mathcal{F}_{CCC}$ is linearly independent on the interval $(0, \infty)$ and, by Proposition 4, there are $h_i \in (0, \infty)$, with $i = 1, 2, 3$, such that $M(h_i) = 0$. Therefore, for $0 < \epsilon << 1$, system (3.1) has a unique limit cycle near L_{h_i}, for each $i = 1, 2, 3$, i.e., system (3.1) has at least three limit cycles. $\square$

Finally, the proof of Theorem 5 is a straightforward consequence of Corollaries 2, 4–8.

Chapter

4

Bifurcation of limit cycles from a periodic annulus, by quadratic/cubic perturbations, from system $(1)|_{\epsilon=0}$ having a center in the central region

In this chapter, motivated by the results obtained in Chapter 3, we will study the number of limit cycles that can bifurcate from the periodic annulus of system $(3.1)|_{\epsilon=0}$, assuming the same hypotheses (Hi) of the previous chapter, for $i = 1, 2, 3$, but with the difference that the functions f and g are polynomial functions of degree n, for $n = 2, 3$. Denoting by $H(n)$ the number of limit cycles that can bifurcate from this periodic annulus, we will show that $H(2) \geq 4$ and $H(3) \geq 7$.

4.1 Preliminaries and main result

Consider discontinuous piecewise linear near-Hamiltonian differential systems with three zones, given by

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \epsilon g(x,y). \end{cases} \tag{4.1}$$

Let us assume the same hypotheses as in Chapter 3, i.e.,

(H1) The unperturbed central subsystem from $(4.1)|_{\epsilon=0}$ has a real center at the origin and the other unperturbed subsystems have centers or saddles.

(H2) The unperturbed system from $(4.1)|_{\epsilon=0}$ has only crossing points on the straight lines $x = \pm 1$, except by some tangent points.

(H3) The unperturbed system from $(4.1)|_{\epsilon=0}$ has a periodic annulus consisting of a family of crossing periodic orbits around the origin such that each orbit of this family passes through the three zones with clockwise orientation.

Hence, we can consider H in our normal form, given by Proposition 1 from Chapter 3, with $\beta_C = 0$, i.e.,

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_L xy + a_L y - \beta_L x, & x \leq -1, \\ H^C(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}, & -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_R xy - a_R y - \beta_R x, & x \geq 1. \end{cases} \tag{4.2}$$

What distinguishes our approach in this chapter from that carried out in Chapter 3 is that the functions f and g are given by

$$f(x,y) = \begin{cases} f_L(x,y) = \sum\limits_{i+j=0}^{n} r_{ij}x^i y^j, & x \leq -1, \\ f_C(x,y) = \sum\limits_{i+j=0}^{n} u_{ij}x^i y^j, & -1 \leq x \leq 1, \\ f_R(x,y) = \sum\limits_{i+j=0}^{n} p_{ij}x^i y^j, & x \geq 1, \end{cases} \tag{4.3}$$

$$g(x,y) = \begin{cases} g_L(x,y) = \sum\limits_{i+j=0}^{n} s_{ij}x^i y^j, & x \leq -1, \\ g_C(x,y) = \sum\limits_{i+j=0}^{n} v_{ij}x^i y^j, & -1 \leq x \leq 1, \\ g_R(x,y) = \sum\limits_{i+j=0}^{n} q_{ij}x^i y^j, & x \geq 1, \end{cases} \tag{4.4}$$

for $n = 2, 3$. Note that the case $n = 1$ was studied in Chapter 3.

The main result of this chapter is the following.

Theorem 9. *The number of limit cycles of system* (4.1), *satisfying the hypotheses* (Hi) *for* $i = 1, 2, 3$, *which can bifurcate from the periodic annulus of the unperturbed system* $(4.1)|_{\epsilon=0}$ *is at least four if* $n = 2$ *and seven if* $n = 3$.

	$\mathrm{S_R C_R S_R}$ with $\tau_L \neq \tau_R$	$\mathrm{S_R C_R S_R}$ with $\tau_L = \tau_R$	$\mathrm{C_V C_R S_R}$	$\mathrm{C_R C_R S_R}$	$\mathrm{C_V C_R C_V}$	$\mathrm{C_R C_R C_V}$	$\mathrm{C_R C_R C_R}$
a_L	1	1	1	1	1	1	1
b_L	1	1	2	2	2	2	2
c_L	0	0	-1	-1	-1	-1	-1
β_L	2	1	1	-2	1	-3	-3
a_R	0	0	0	0	0	0	0
b_R	1	1	1	1	1	1	1
c_R	1	1	1	1	-1	-1	-1
β_R	-2	-2	-2	-2	0	0	2

Table 4.1: Parameter values for which system $(4.1)|_{\epsilon=0}$ satisfies the hypothesis (H1). For the case $\mathrm{S}_R\mathrm{C}_R\mathrm{S}_R$, when $\tau_L \neq \tau_R$ system $(4.1)|_{\epsilon=0}$ has a homoclinic loop and when $\tau_L = \tau_R$ system $(4.1)|_{\epsilon=0}$ has two heteroclinic orbits.

As in Chapter 3, to prove Theorem 9, we study the number of zeros of the first order Melnikov function associated with system (4.1). To simplify the expression of the Melnikov function associated with system (4.1), we assume parameter values for system $(4.1)|_{\epsilon=0}$ satisfying the hypothesis (H1). More precisely, as in Chapter 2, we classify the systems that satisfy the hypothesis (H1) according to the configuration of their singular points, i.e., we have the following three classes of piecewise linear Hamiltonian systems: SCS, CCS and CCC. Moreover, we also use the capital letters R and V to denote the real and virtual singular points, respectively. So, for instance, we say that a system is of the type $\mathrm{C_R C_V S_R}$ if the left subsystem has a real center, the central subsystem has a virtual center and the right subsystem has a real saddle. Then, let us consider the parameter values for system $(4.1)|_{\epsilon=0}$ given by Table 4.1.

From Table 4.1, we can see that if system $(4.1)|_{\epsilon=0}$ is of the type $\mathrm{S_R C_R S_R}$ then the right subsystem has a saddle at the point $(2, 0)$ and the left subsystem has a saddle at the point $(-3, 2)$ (resp. $(-2, 1)$) when $\beta_L = 2$ (resp. when $\beta_L = 1$). Moreover, when $\beta_L = 2$ (resp. when $\beta_L = 1$) we have a homoclinic loop (resp. two heteroclinic orbits) passing through the points $P_R^u = (1, -1)$ and $P_R^s = (1, 1)$ (resp. $P_L^u = (-1, 1)$, $P_L^s = (-1, -1)$, $P_R^u = (1, -1)$ and $P_R^s = (1, 1)$). Now, if system $(4.1)|_{\epsilon=0}$ is of the type $\mathrm{C_V C_R S_R}$ (resp. $\mathrm{C_R C_R S_R}$) then the right subsystem has a saddle at the point $(2, 0)$ and the left subsystem has a virtual (resp. real) center at the point $(3, -2)$ (resp. $(-3, 1)$) when $\beta_L = 1$ (resp. when $\beta_L = -2$). Moreover, we have a homoclinic loop passing through the points $P_R^u = (1, -1)$ and $P_R^s = (1, 1)$. Finally, if system $(4.1)|_{\epsilon=0}$ is of the type CCC then the right subsystem has a virtual (resp. real) center at the point $(0, 0)$ (resp. $(2, 0)$) when $\beta_R = 0$ (resp. $\beta_R = 2$) and the left subsystem has a virtual (resp. real) center at the point $(3, 2)$ (resp. $(-5, 2)$) when $\beta_L = 1$ (resp. $\beta_L = -3$).

4.2 A classification to system $(4.1)|_{\epsilon=0}$

In order to computation the zeros of the first order Melnikov function M associated with system (4.1), that can be expressed as (see Theorem 1)

$$\begin{aligned} M(h) = {} & \frac{H_y^R(A)}{H_y^C(A)} I_C + \frac{H_y^R(A) H_y^C(A_3)}{H_y^C(A) H_y^L(A_3)} I_L + \frac{H_y^R(A) H_y^C(A_3) H_y^L(A_2)}{H_y^C(A) H_y^L(A_3) H_y^C(A_2)} \bar{I}_C \\ & + \frac{H_y^R(A) H_y^C(A_3) H_y^L(A_2) H_y^C(A_1)}{H_y^C(A) H_y^L(A_3) H_y^C(A_2) H_y^R(A_1)} I_R, \quad h \in J, \end{aligned} \tag{4.5}$$

where

$$I_C = \int_{\widehat{A_3 A}} g_C dx - f_C dy, \quad I_L = \int_{\widehat{A_2 A_3}} g_L dx - f_L dy,$$

$$\bar{I}_C = \int_{\widehat{A_1 A_2}} g_C dx - f_C dy, \quad I_R = \int_{\widehat{A A_1}} g_R dx - f_R dy,$$

it is necessary to find the open interval J where it is defined. For this, consider the following proposition which can be proved with the same steps of the proof of Proposition 2 from Chapter 3.

Proposition 5. *Consider system* (4.1) *assuming the hypotheses (Hi),* $i = 1, 2, 3$, *and the parameter values given by Table 4.1.*

(a) *If system* $(4.1)|_{\epsilon=0}$ *is either of the type SCS or CCS, then* $J = (0, 1)$, *and the periodic annuli are equivalent to one of the figures of Figure 3.2.*

(b) *If system* $(4.1)|_{\epsilon=0}$ *is of the type CCC, then* $J = (0, \infty)$, *and the periodic annuli are equivalent to one of the figures of Figure 3.3.*

As in Proposition 2, for $A(h) = (1, h)$, with $h \in (0, 1)$ or $h \in (0, \infty)$, system $(4.1)|_{\epsilon=0}$ has a family of crossing periodic orbits L_h that intersect the straight lines $x = \pm 1$ at four points. More precisely, by equations

$$H^R(A(h)) = H^R(A_1(h)),$$

$$H^C(A_1(h)) = H^C(A_2(h)),$$

$$H^L(A_2(h)) = H^L(A_3(h)),$$

$$H^C(A_3(h)) = H^C(A(h)),$$

for each $h \in (0, 1)$ or $h \in (0, \infty)$, we have the four points given by $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$, $A_3(h) = (-1, h)$ and a periodic orbit L_h passing through these points. Therefore, assuming the parameter values from Table 4.1, the coefficients that multiply the integrals of first order Melnikov function (4.5) associated with system (4.1) can be easily calculated. More precisely, we have the immediate corollary.

Corollary 9. *Let J be the interval of definition of Melnikov function (4.5). For $h \in J$,*

$$\frac{H_y^R(A)}{H_y^C(A)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} = 1$$

and

$$\frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} = \frac{1}{b_L},$$

with $b_L \in \{1,2\}$. Then, the first order Melnikov function associated with system (4.1) can be written as

$$\begin{aligned} M(h) = {} & \int_{\widehat{A_3A}} g_C dx - f_C dy + \frac{1}{b_L}\int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ & + \int_{\widehat{A_1A_2}} g_C dx - f_C dy + \int_{\widehat{AA_1}} g_R dx - f_R dy. \end{aligned}$$

4.3 Melnikov functions associated with system (4.1)

In what follows, we determinate the first order Melnikov function associated with system (4.1) when system $(4.1)|_{\epsilon=0}$ is of the type SCS, CCS and CCC. For this, we define the functions:

$$\begin{aligned}
& f_0(h) = h, \quad h \in (0,\infty), \\
& f_1(h) = (h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right), \quad h \in (0,\infty), \\
& f_2(h) = h^3, \quad h \in (0,\infty), \\
& f_3(h) = h^2(h^2+1)\arccos\left(\frac{h^2-1}{h^2+1}\right), \quad h \in (0,\infty), \\
& f_0^S(h) = (h^2-1)\log\left(\frac{1+h}{1-h}\right), \quad h \in (0,1), \\
& f_1^S(h) = (h^2-4)\log\left(\frac{2+h}{2-h}\right), \quad h \in (0,1), \\
& f_2^S(h) = h^2(h^2-1)\log\left(\frac{1+h}{1-h}\right), \quad h \in (0,1), \\
& f_3^S(h) = h^2(h^2-4)\log\left(\frac{2+h}{2-h}\right), \quad h \in (0,1), \\
& f_0^C(h) = (h^2+4)\left(2\pi\mu + (-1)^\mu \arccos\left(\frac{4-h^2}{4+h^2}\right)\right), \quad h \in (0,\infty), \\
& f_1^C(h) = h^2(h^2+4)\left(2\pi\mu + (-1)^\mu \arccos\left(\frac{4-h^2}{4+h^2}\right)\right), \quad h \in (0,\infty), \\
& f_2^C(h) = (h^2+1)\left(2\pi\lambda + (-1)^\lambda \arccos\left(\frac{1-h^2}{1+h^2}\right)\right), \quad h \in (0,\infty), \\
& f_3^C(h) = h^2(h^2+1)\left(2\pi\lambda + (-1)^\lambda \arccos\left(\frac{1-h^2}{1+h^2}\right)\right), \quad h \in (0,\infty),
\end{aligned} \tag{4.6}$$

with $\lambda, \mu \in \{0,1\}$.

Theorem 10 (Case SCS)**.** *Consider system* (4.1) *with* H *given in* (4.2)*, with* $a_L = b_L = b_R = c_R = 1$*,* $a_R = c_L = 0$*,* $\beta_R = -2$*,* $\beta_L \in \{1, 2\}$*, and* f*,* g *given in* (4.3) *and* (4.4)*, respectively. Then, the first order Melnikov functions associated with system* (4.1) *can be expressed when* $\beta_L = 2$ *(i.e., we have a homoclinic loop) as*

$$\begin{aligned} M_{12}(h) &= \sum_{i=0}^{2} k_{12}^{i} f_i(h) + \sum_{i=0}^{1} l_{12}^{i} f_i^{S}(h), \quad \textit{if} \quad n = 2, \qquad (4.7)\\ M_{13}(h) &= \sum_{i=0}^{3} k_{13}^{i} f_i(h) + \sum_{i=0}^{3} l_{13}^{i} f_i^{S}(h), \quad \textit{if} \quad n = 3, \qquad (4.8) \end{aligned}$$

and when $\beta_L = 1$ *(i.e., we have two heteroclinic orbits) as*

$$\begin{aligned} M_{22}(h) &= \sum_{i=0}^{2} k_{22}^{i} f_i(h) + l_{22}^{0} f_0^{S}(h), \qquad \textit{if} \quad n = 2, \qquad (4.9)\\ M_{23}(h) &= \sum_{i=0}^{3} k_{23}^{i} f_i(h) + l_{23}^{0} f_0^{S}(h) + l_{23}^{1} f_2^{S}(h), \quad \textit{if} \quad n = 3, \qquad (4.10) \end{aligned}$$

where the functions f_i *and* f_i^S*, for* $i = 0, \ldots, 3$ *are the ones defined in* (4.6) *and the coefficients* k_{ij}^m *and* l_{ij}^m*, for* $i = 1, 2$*,* $j = 2, 3$ *and* $m = 0, \ldots, 3$*, depend on the parameters of system* (4.1)*.*

Proof. Firstly, let us obtain the expressions of Melnikov functions (4.7) and (4.8). For this, consider system $(4.1)|_{\epsilon=0}$ with $a_L = b_L = b_R = c_R = 1$, $a_R = c_L = 0$, $\beta_R = -2$ and $\beta_L = 2$. The orbit $(x_R(t), y_R(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$\begin{aligned} x_R(t) &= \frac{e^t}{2}(h-1) - \frac{e^{-t}}{2}(h+1) + 2,\\ y_R(t) &= \frac{e^t}{2}(h-1) + \frac{e^{-t}}{2}(h+1). \end{aligned} \qquad (4.11)$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is given by

$$t_R = \log\left(\frac{1+h}{1-h}\right). \qquad (4.12)$$

Now, for g_R and f_R defined in (4.3) and (4.4), respectively, we have that

$$\begin{aligned} I_R^n &= \int_{\widehat{AA_1}} g_R dx - f_R dy\\ &= \int_0^{t_R} \big(g_R(x_R(t), y_R(t))x_R'(t) - f_R(x_R(t), y_R(t))y_R'(t)\big)dt\\ &= \int_0^{t_R} \bigg(\sum_{i+j=0}^{n} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{n} p_{ij} x_R^i(t) y_R^j(t) y_R'(t)\bigg)dt, \end{aligned} \qquad (4.13)$$

with $n = 2, 3$. Replacing (4.11) and (4.12) into (4.13), we obtain

$$
\begin{aligned}
I_R^2 =& (2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2q_{11}) f_0(h) + \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01} \\
&+2q_{11}\Big) f_0^S(h) + \frac{2}{3}\Big(p_{02} - 2p_{20} - q_{11}\Big) f_2(h),
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
I_R^3 =& \frac{1}{3}\Big(8p_{00} + 12p_{10} - p_{12} + 24p_{20} + 59p_{30} + 4q_{01} - 3q_{03}\Big) f_0(h) \\
&+\frac{1}{3}\Big(8q_{11} + 17q_{21}\Big) f_0(h) + \frac{1}{8}(4p_{10} - p_{12} + 16p_{20} + 51p_{30}) f_0^S(h) \\
&+\frac{1}{8}\Big(4q_{01} - 3q_{03} + 8q_{11} + 17q_{21}\Big) f_0^S(h) + \frac{1}{12}\Big(8p_{02} + 13p_{12}\Big) f_2(h) \\
&+\frac{1}{12}\Big(- 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21}\Big) f_2(h) \\
&+\frac{1}{8}\Big(p_{12} - 3p_{30} + 3q_{03} - q_{21}\Big) f_2^S(h).
\end{aligned} \tag{4.15}
$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C1}(0),\, y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \cos(t) - h\sin(t), \\
y_{C1}(t) &= -h\cos(t) - \sin(t).
\end{aligned} \tag{4.16}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is given by

$$
t_{C1} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right). \tag{4.17}
$$

Now, for g_C and f_C defined in (4.3) and (4.4), respectively, we have that

$$
\begin{aligned}
\bar{I}_C^n &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\
&= \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t)) x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t)) y'_{C1}(t)\big) dt \\
&= \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{n} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x'_{C1}(t) - \sum_{i+j=0}^{n} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y'_{C1}(t)\Bigg) dt,
\end{aligned} \tag{4.18}
$$

with $n = 2, 3$. Replacing (4.16) and (4.17) into (4.18), we obtain

$$\begin{aligned}\bar{I}_C^2 = &-\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01})f_0(h) - 2v_{02}\, h^2 \\ &+\frac{1}{2}\Big(u_{10} + v_{01}\Big)f_1(h),\end{aligned} \tag{4.19}$$

$$\begin{aligned}\bar{I}_C^3 = &-\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) + \frac{1}{4}\Big(-4u_{10} + u_{12} - 5u_{30} + 4v_{01} \\ &+3v_{03} + v_{21}\Big)f_0(h) - 2v_{02}h^2 - \frac{1}{4}\Big(u_{12} + 3u_{30} - 5v_{03} + v_{21}\Big)f_2(h) \\ &+\frac{1}{8}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big)f_1(h) + \frac{1}{8}\Big(u_{12} + 3(u_{30} \\ &+v_{03}) + v_{21}\Big)f_3(h).\end{aligned} \tag{4.20}$$

The orbit $(x_L(t), y_L(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_L(0),\, y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned} x_L(t) &= \frac{e^t}{2}(2-h) + \frac{e^{-t}}{2}(h+2) - 3, \\ y_L(t) &= -\frac{e^{-t}}{2}(h+2) + 2. \end{aligned} \tag{4.21}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is given by

$$t_L = \log\left(\frac{2+h}{2-h}\right). \tag{4.22}$$

Now, for g_L and f_L defined in (4.3) and (4.4), respectively, we have that

$$\begin{aligned} I_L^n &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ &= \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\big)dt \\ &= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{n} s_{ij} x_L^i(t) y_L^j(t) x_L'(t) - \sum_{i+j=0}^{n} r_{ij} x_L^i(t) y_L^j(t) y_L'(t)\Bigg)dt \end{aligned} \tag{4.23}$$

with $n = 2, 3$. Replacing (4.21) and (4.22) into (4.23), we obtain

$$\begin{aligned} I_L^2 &= 2(2r_{10} - r_{00} + 2r_{11} - 7r_{20} + s_{01} + 4s_{02} - 3s_{11}) f_0(h) \\ &\quad + \frac{1}{2}\Big(r_{10} + 2r_{11} - 6r_{20} + s_{01} + 4s_{02} - 3s_{11}\Big) f_1^S(h) \\ &\quad - \frac{2}{3}\Big(r_{02} + r_{11} - 2r_{20} + 2s_{02} - s_{11}\Big) f_2(h), \end{aligned} \tag{4.24}$$

$$\begin{aligned} I_L^3 &= 2(2r_{10} - r_{00} + 2r_{11} + 4r_{12} - 7r_{20} - 14r_{21} + 31r_{30}) f_0(h) \\ &\quad + 2(s_{01} + 4s_{02} + 12s_{03} - 3s_{11} - 14s_{12} + 10s_{21}) f_0(h) \\ &\quad + \frac{1}{2}\Big(- 3s_{11} - 14s_{12} + 10s_{21} + r_{10} + 2r_{11} + 4r_{12} - 6r_{20}\Big) f_1^S(h) \\ &\quad + \frac{1}{2}\Big(- 14r_{21} + 30r_{30} + s_{01} + 4s_{02} + 12s_{03}\Big) f_1^S(h) \\ &\quad + \frac{1}{6}\Big(- 8r_{20} - 34r_{21} + 63r_{30} + 8s_{02} + 24s_{03} - 4s_{11}\Big) f_2(h) \\ &\quad + \frac{1}{6}\Big(- 34s_{12} + 21s_{21} + 4r_{02} + 4r_{11} + 4r_{12}\Big) f_2(h) \\ &\quad + \frac{1}{8}\Big(2r_{21} - 3r_{30} + 2s_{12} - s_{21}\Big) f_3^S(h). \end{aligned} \tag{4.25}$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C2}(0), y_{C2}(0)) = (-1, h)$, is given by

$$\begin{aligned} x_{C2}(t) &= -\cos(t) + h\sin(t), \\ y_{C2}(t) &= h\cos(t) + \sin(t). \end{aligned} \tag{4.26}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right). \tag{4.27}$$

Now, for g_C and f_C defined in (4.3) and (4.4), respectively, we have that

$$\begin{aligned} I_C^n &= \int_{\widehat{A_3 A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t)) x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t)) y'_{C2}(t)\big) dt \\ &= \int_0^{t_{C2}} \left(\sum_{i+j=0}^{1} v_{ij} x_{C2}^i(t) y_{C2}^j(t) x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C2}^i(t) y_{C2}^j(t) y'_{C2}(t)\right) dt \end{aligned} \tag{4.28}$$

with $n = 2, 3$. Replacing (4.26) and (4.27) into (4.28), we obtain

$$
\begin{aligned}
I_C^2 = & \frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01})f_0(h) + 2v_{02}\,h^2 \\
& + \frac{1}{2}\Big(u_{10} + v_{01}\Big)f_1(h),
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
I_C^3 = & \frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) + \frac{1}{4}\Big(- 4u_{10} + u_{12} - 5u_{30} + 4v_{01} \\
& + 3v_{03} + v_{21}\Big)f_0(h) + 2v_{02}h^2 - \frac{1}{4}\Big(u_{12} + 3u_{30} - 5v_{03} + v_{21}\Big)f_2(h) \\
& + \frac{1}{8}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big)f_1(h) + \frac{1}{8}\Big(u_{12} + 3(u_{30} \\
& + v_{03}) + v_{21}\Big)f_3(h).
\end{aligned} \tag{4.30}
$$

Moreover, by Corollary 9, the first order Melnikov function associated with system (4.1) is given by

$$
M(h) = I_C^n + I_L^n + \bar{I}_C^n + I_R^n, \quad n = 2, 3. \tag{4.31}
$$

Therefore, if $n = 2$, replacing (4.14), (4.19), (4.24) and (4.29) into (4.31) we obtain (4.7), with

$$
\begin{aligned}
k_{12}^0 = & 2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2(q_{11} - r_{00} + 2r_{10} + 2r_{11} - 7r_{20} + s_{01} + 4s_{02} - 3s_{11} \\
& - u_{10} + v_{01}), \\
k_{12}^1 = & u_{10} + v_{01}, \\
k_{12}^2 = & -\frac{2}{3}\Big(2p_{20} - p_{02} + q_{11} + r_{02} + r_{11} - 2r_{20} + 2s_{02} - s_{11}\Big), \\
l_{12}^0 = & \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01} + 2q_{11}\Big), \\
l_{12}^1 = & \frac{1}{2}\Big(r_{10} + 2r_{11} - 6r_{20} + s_{01} + 4s_{02} - 3s_{11}\Big),
\end{aligned}
$$

Finally, if $n = 3$, replacing (4.15), (4.20), (4.25) and (4.30) into (4.31) we obtain (4.8), with

$$
\begin{aligned}
k_{13}^0 = & 2p_{00} + 3p_{10} + \frac{1}{4}\Big(- p_{12} + 24p_{20} + 59p_{30} + 4q_{01} - 3q_{03} + 8q_{11}\Big) \\
& + \frac{1}{4}\Big(17q_{21} + 2(-4r_{00} + 8r_{10} + 8r_{11} + 16r_{12} - 28r_{20} - 56r_{21})\Big) \\
& + \frac{1}{2}\Big(124r_{30} + 4s_{01} + 16s_{02} + 48s_{03} - 12s_{11} - 56s_{12} + 40s_{21}\Big) \\
& + \frac{1}{2}\Big(- 4u_{10} + u_{12} - 5u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\
k_{13}^1 = & \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big),
\end{aligned}
$$

$$
\begin{aligned}
k_{13}^2 =& \frac{1}{12}\Big(8p_{02} + 13p_{12} - 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21}\Big)\\
& - \frac{1}{6}\Big(4r_{02} + 4r_{11} + 4r_{12} - 8r_{20} - 34r_{21} + 63r_{30} + 8s_{02} + 4s_{03}\Big)\\
& - \frac{1}{6}\Big(- 4s_{11} - 34s_{12} + 3(7s_{21} + u_{12} + 3u_{30} - 5v_{03} + v_{21})\Big),\\
k_{13}^3 =& \frac{1}{4}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big),\\
l_{13}^0 =& \frac{1}{8}\Big(4p_{10} - p_{12} + 16p_{20} + 51p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21}\Big),\\
l_1^1 =& \frac{1}{2}\Big(r_{10} + 2r_{11} + 4r_{12} - 6r_{20} - 14r_{21} + 30r_{30} + s_{01} + 4s_{02}\Big)\\
& + \frac{1}{2}\Big(12s_{03} - 3s_{11} - 14s_{12} + 10s_{21}\Big),\\
l_{13}^2 =& \frac{1}{8}\Big(p_{12} - 3p_{30} + 3q_{03} - q_{21}\Big),\\
l_3^3 =& \frac{1}{8}\Big(2r_{21} - 3r_{30} + 2s_{12} - s_{21}\Big).
\end{aligned}
$$

With similar proofs, that will be omitted to simplifying the text, we can obtain the expressions of the Melnikov functions (4.9) and (4.10). Moreover, the coefficients $k_{i,j}^m$ and $l_{i,j}^m$ of the Melnikov function (4.9) are given by

$$
\begin{aligned}
k_{22}^0 =& 2(p_{00} + 3p_{20} + q_{11} - r_{00} - 3r_{20} + s_{02} - s_{11} - u_{10} + v_{01}) + 3(p_{10} + r_{10}) + r_{11},\\
k_{22}^1 =& u_{10} + v_{01},\\
k_{22}^2 =& -\frac{2}{3}\Big(2p_{20} - p_{02} + q_{11} + r_{02} + r_{11} - 2r_{20} + 2s_{02} - s_{11}\Big),\\
l_{22}^0 =& \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01} + 2q_{11} + r_{10} + r_{11} - 4r_{20} + s_{01} + 2s_{02} - 2s_{11}\Big),
\end{aligned}
$$

and of the Melnikov function (4.10) are given by

$$
\begin{aligned}
k_{23}^0 =& 2p_{00} + 3p_{10} - \frac{p_{12}}{4} + 6p_{20} + \frac{59p_{30}}{4} + q_{01} - \frac{3q_{03}}{4} + 2q_{11} + \frac{17q_{21}}{4} - 2r_{00} + 3r_{10} + r_{11}\\
& + r_{12} - 6r_{20} - \frac{9r_{21}}{2} + \frac{59r_{30}}{4} + s_{01} + 2s_{02} + 3s_{03} - 2s_{11} - \frac{9s_{12}}{2} + \frac{17s_{21}}{4} + \frac{1}{2}\Big(u_{12}\\
& - 4u_{10} - 5u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big),\\
k_{23}^1 =& \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big),\\
k_{23}^2 =& \frac{1}{12}\Big(8p_{02} + 13p_{12} - 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21} - 8r_{02} - 8r_{11} + 16r_{20}\\
& + 42r_{21} - 87r_{30} - 16s_{02} - 24s_{03} + 8s_{11} + 42s_{12} - 29s_{21} - 6(u_{12} + 3u_{30} - 5v_{03} + v_{21})\Big),\\
k_{23}^3 =& \frac{1}{4}\Big(u_{12} + v_{21} + 3(u_{30} + v_{03})\Big),\\
l_{23}^0 =& \frac{1}{8}\Big(4p_{10} - p_{12} + 16p_{20} + 51p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21} + 4r_{10} + 4r_{11} + 4r_{12}\\
& - 16r_{20} - 18r_{21} + 51r_{30} + 4s_{01} + 8s_{02} + 12s_{03} - 8s_{11} - 18s_{12} + 17s_{21}\Big),\\
l_{23}^1 =& \frac{1}{8}\Big(p_{12} - (3p_{30} - 3q_{03} + q_{21} - 2r_{21} + 3r_{30} - 2s_{12} + s_{21})\Big).
\end{aligned}
$$

□

The following two theorems provide expressions for the first order Melnikov functions associated with system (4.1) when system $(4.1)|_{\epsilon=0}$ is of the types CCS and CCC. The proofs of these results are similar to the proof of Theorem 10 and are presented in the Appendix 7.1 and 7.2, respectively, to simplify the text.

Theorem 11 (Case CCS). *Consider system* (4.1) *with* H *given in* (4.2), *with* $a_L = b_R = c_R = 1$, $b_L = 2$, $a_R = 0$, $c_L = -1$, $\beta_R = -2$, $\beta_L \in \{-2, 1\}$, f *and* g *given in* (4.3) *and* (4.4), *respectively. Then, the first order Melnikov functions associated with system* (4.1) *can be expressed when* $\beta_L = 1$ *(i.e., system* $(4.1)|_{\epsilon=0}$ *is of the type* $C_V C_R S_R$*) as*

$$
\begin{aligned}
M_{12}(h) &= \sum_{i=0}^{2} k_{12}^{i} f_i(h) + l_{12}^{0} f_0^S(h) + m_{12}^{0} f_0^C(h), && \textit{if} \quad n = 2, && (4.32)\\
M_{13}(h) &= \sum_{i=0}^{3} k_{13}^{i} f_i(h) + l_{13}^{0} f_0^S(h) + l_{13}^{1} f_2^S(h) + \sum_{i=0}^{1} m_{13}^{i} f_i^C(h), && \textit{if} \quad n = 3, && (4.33)
\end{aligned}
$$

with $\mu = \lambda = 0$, *and when* $\beta_L = -2$ *(i.e., system* $(4.1)|_{\epsilon=0}$ *is of the type* $C_R C_R S_R$*) as*

$$
\begin{aligned}
M_{22}(h) &= \sum_{i=0}^{2} k_{22}^{i} f_i(h) + l_{22}^{0} f_0^S(h) + m_{22}^{2} f_2^C(h), && \textit{if} \quad n = 2, && (4.34)\\
M_{23}(h) &= \sum_{i=0}^{3} k_{23}^{i} f_i(h) + l_{23}^{0} f_0^S(h) + l_{23}^{1} f_2^S(h) + \sum_{i=2}^{3} m_{23}^{i} f_i^C(h), && \textit{if} \quad n = 3, && (4.35)
\end{aligned}
$$

with $\mu = 0$ *and* $\lambda = 1$. *Here the functions* f_i, f_i^S *and* f_i^C, *for* $i = 0, \ldots, 3$ *are the ones defined in* (4.6) *and the coefficients* k_{ij}^{α}, l_{ij}^{α} *and* m_{ij}^{α}, *for* $i = 1, 2$, $j = 2, 3$ *and* $\alpha = 0, \ldots, 3$, *depend on the parameters of system* (4.1).

Theorem 12 (Case CCC). *Consider system* (4.1) *with* H *given in* (4.2), *with* $a_L = b_R = 1$, $b_L = 2$, $a_R = 0$, $c_L = c_R = -1$, $\beta_L \in \{-3, 1\}$, $\beta_R \in \{0, 2\}$, f *and* g *given in* (4.3) *and* (4.4), *respectively. Then, the first order Melnikov functions associated with system* (4.1) *can be expressed when* $\beta_L = 1$ *and* $\beta_R = 0$ *(i.e., system* $(4.1)|_{\epsilon=0}$ *is the type* $C_V C_R C_V$*) as*

$$
\begin{aligned}
M_{12}(h) &= \sum_{i=0}^{2} k_{12}^{i} f_i(h) + l_{12}^{0} f_0^C(h) + l_{12}^{2} f_2^C(h), && \textit{if} \quad n = 2, && (4.36)\\
M_{13}(h) &= \sum_{i=0}^{3} k_{13}^{i} f_i(h) + \sum_{i=0}^{3} l_{13}^{i} f_i^C(h), && \textit{if} \quad n = 3, && (4.37)
\end{aligned}
$$

with $\mu = \lambda = 0$, *and when* $\beta_L = -3$ *and* $\beta_R = 0$ *(i.e., system* $(4.1)|_{\epsilon=0}$ *is of the type* $C_R C_R C_V$*)*

as

$$M_{22}(h) = \sum_{i=0}^{2} k_{22}^{i} f_i(h) + l_{22}^{0} f_0^C(h) + l_{22}^{2} f_2^C(h), \quad if \quad n = 2, \tag{4.38}$$

$$M_{23}(h) = \sum_{i=0}^{3} k_{23}^{i} f_i(h) + \sum_{i=0}^{3} l_{23}^{i} f_i^C(h), \quad if \quad n = 3, \tag{4.39}$$

with $\mu = 1$ *and* $\lambda = 0$*, and when* $\beta_L = -3$ *and* $\beta_R = 0$ *(i.e. system* $(4.1)|_{\epsilon=0}$ *is of the type* $C_R C_R C_R$*) as*

$$M_{32}(h) = \sum_{i=0}^{2} k_{32}^{i} f_i(h) + l_{32}^{0} f_0^C(h) + l_{32}^{2} f_2^C(h), \quad if \quad n = 2, \tag{4.40}$$

$$M_{33}(h) = \sum_{i=0}^{3} k_{33}^{i} f_i(h) + \sum_{i=0}^{3} l_{33}^{i} f_i^C(h), \quad if \quad n = 3, \tag{4.41}$$

with $\mu = \lambda = 1$*. Here the functions* f_i *and* f_i^C*, for* $i = 0, \ldots, 3$ *are the ones defined in* (4.6) *and the coefficients* k_{ij}^m *and* l_{ij}^m*, for* $i = 1, 2, 3$*,* $j = 2, 3$ *and* $m = 0, \ldots, 3$*, depend on the parameters of system* (4.1)*.*

4.4 Proof of Theorem 9

To prove Theorem 9 we use Proposition 3 and Lemma 1, from Chapter 3.

Proof of Theorem 9. In order to prove Theorem 9, let us consider the Melnikov functions associated with system (4.1) given by Theorems 10, 11 and 12. Our study will be concentrated in a neighborhood of the orbit L_h, with $h = 1$ for the cases SCS, CCS and $h = \tau$, $\tau \in (0, \infty)$, for the case CCC, since to estimate the zeros of the Melnikov functions we consider their expansions at the point corresponding to these orbits. We begin the proof with the case where system $(4.1)|_{\epsilon=0}$ is of the type SCS. For this case, we have two subcases. The first one is when we have two heteroclinic orbits and the second one is when we have a homoclinic loop. The second case is when system $(4.1)|_{\epsilon=0}$ is of the type CCS. For this case, again we have two subcases, which are $\mathrm{C_V C_R S_R}$ and $\mathrm{C_R C_R S_R}$. Finally, the third case is when system $(4.1)|_{\epsilon=0}$ is of the type CCC. For this case, we have three subcases, which are $\mathrm{C_V C_R C_V}$, $\mathrm{C_R C_R C_V}$ and $\mathrm{C_R C_R C_R}$.

Case SCS with heteroclinic orbits. Consider the Melnikov functions M_{22} and M_{23} given by

Theorem 10. By Lemma 1, we can expand these functions at $h = 1$ as

$$M_{22}(h) = \sum_{j=0}^{3} C_{22}^{j}(h-1)^j + \sum_{j=1}^{2} D_{22}^{j} \log(1-h)(h-1)^j + \mathcal{O}((h-1)^4), \tag{4.42}$$

$$M_{23}(h) = \sum_{j=0}^{5} C_{23}^{j}(h-1)^j + \sum_{j=1}^{4} D_{23}^{j} \log(1-h)(h-1)^j + \mathcal{O}((h-1)^6), \tag{4.43}$$

where

$$\begin{aligned} C_{22}^0 = {}& 2p_{00} + \frac{1}{3}\Big(2p_{02} + 9p_{10} + 14p_{20} + 3q_{01} + 4q_{11} - 6r_{00} - 2r_{02} + 9r_{10} + r_{11} - 14r_{20} \\ & + 3s_{01} + 2s_{02} - 4s_{11}\Big) + (\pi - 2)u_{10} + (2+\pi)v_{01}, \\ C_{22}^1 = {}& 2p_{00} + 2p_{02} + 3p_{10} + q_{01} - 2r_{00} - 2r_{02} + 3r_{10} - r_{11} - 2r_{20} + s_{01} - 2s_{02} - 4u_{10} \\ & + \pi u_{10} + \pi v_{01} + (p_{10} + q_{01} + r_{10} + r_{11} - 4r_{20} + s_{01}) \log(2) + (q_{11} + s_{02} - s_{11}) \\ & \log(4) + p_{20}(2 + \log(16)), \end{aligned}$$

$$\begin{aligned} C_{22}^2 = {}& \frac{1}{2}\Big(4p_{02} + p_{10} + q_{01} - 2q_{11} - 4r_{02} + r_{10} - 3r_{11} + 4r_{20} + s_{01} - 6s_{02} + 2s_{11} + (\pi - 2) \\ & (u_{10} + v_{01}) + (p_{10} + q_{01} + r_{10} + r_{11} - 4r_{20} + s_{01}) \log(2) + (q_{11} + s_{02} - s_{11}) \log(4) \\ & + p_{20}(\log(16) - 4)\Big), \\ C_{22}^3 = {}& \frac{1}{24}\Big(16p_{02} + 3p_{10} - 20p_{20} + 3q_{01} - 10q_{11} - 16r_{02} + 3r_{10} - 13r_{11} + 20r_{20} + 3s_{01} \\ & - 26s_{02} + 10s_{11} - 8(u_{10} + v_{01})\Big), \\ C_{23}^0 = {}& \frac{1}{6}\Big(12p_{00} + 4p_{02} + 18p_{10} + 5p_{12} + 28p_{20} + 45p_{30} + 6q_{01} + 3q_{03} + 8q_{11} + 11q_{21} \\ & - 12r_{00} - 4r_{02} + 18r_{10} + 2r_{11} + 6r_{12} - 28r_{20} - 6r_{21} + 45r_{30} + 6s_{01} + 4s_{02} + 6s_{03} \\ & - 8s_{11} - 6s_{12} + 11s_{21} + 3(4v_{01} - 8u_{30} + 8v_{03} + 2(\pi - 2)u_{10} + \pi(u_{12} + 3u_{30} + 2v_{01} \\ & + 3v_{03} + v_{21}))\Big), \end{aligned}$$

$$\begin{aligned} C_{23}^1 = {}& 2p_{00} + 2p_{02} + 3p_{10} + 3p_{12} + 2p_{20} - 7p_{30} + q_{01} + 3q_{03} - 3q_{21} - 2r_{00} - 2r_{02} + 3r_{10} \\ & - r_{11} + r_{12} - 2r_{20} + 6r_{21} - 7r_{30} + s_{01} + 6s_{12} - 3s_{21} + (\pi - 4)u_{10} - 2(u_{12} + 5u_{30} \\ & - 3v_{03} + v_{21}) + \pi(u_{12} + 3u_{30} + v_{01} + 3v_{03} + v_{21}) + (p_{10} + 12p_{30} + q_{01} + r_{10} + r_{11} \\ & + r_{12} - 4r_{20} - 4r_{21} + 12r_{30} + s_{01} - 4s_{12}) \log(2) + s_{02}(\log(4) - 2) + q_{11} \log(4) \\ & - s_{11} \log(4) + s_{03}(\log(8) - 3) + (p_{20} + q_{21} + s_{21}) \log(16), \end{aligned}$$

$$
\begin{aligned}
C_{23}^2 = \frac{1}{4}\Big(&8p_{02} + 2p_{10} + 13p_{12} - 8p_{20} - 63p_{30} + 2q_{01} + 15q_{03} - 4q_{11} - 21q_{21} - 8r_{02} + 2r_{10} \\
&- 6r_{11} + 2r_{12} + 8r_{20} + 34r_{21} - 63r_{30} + 2s_{01} - 12s_{02} - 18s_{03} + 4s_{11} + 34s_{12} - 21s_{21} \\
&+ 2(\pi - 2)u_{10} + 2\pi(2u_{12} + 6u_{30} + v_{01} + 6v_{03} + 2v_{21}) - 4(3u_{12} + 9u_{30} + v_{01} - 3v_{03} \\
&+ 3v_{21}) + (p_{10} + p_{12} + 4p_{20} + 9p_{30} + q_{01} + r_{10} + r_{11} + r_{12} - 4r_{20} - 2r_{21} + 9r_{30} + s_{01} \\
&- 2(s_{11} + s_{12}))\log(4) + (q_{11} + s_{02})\log(16) + (q_{03} + q_{21} + s_{03} + s_{21})\log(64)\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{23}^3 = \frac{1}{24}\Big(&16p_{02} + 3p_{10} - 20p_{20} - 156p_{30} + 3q_{01} + 48q_{03} - 10q_{11} - 52q_{21} - 16r_{02} + 3r_{10} \\
&- 13r_{11} + 3r_{12} + 20r_{20} + 84r_{21} - 156r_{30} + 3s_{01} - 26s_{02} - 39s_{03} + 10s_{11} + 84s_{12} \\
&- 52s_{21} - 8u_{10} + 4(3\pi - 10)u_{12} + 12\pi(v_{21} + 3(u_{30} + v_{03})) - 8(15u_{30} + v_{01} \\
&+ 3v_{03} + 5v_{21}) - 12(3p_{30} - 3q_{03} + q_{21} - 2r_{21} + 3r_{30} - 2s_{12} + s_{21})\log(2) + 4p_{12} \\
&(8 + \log(8))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{23}^4 = \frac{1}{48}\Big(&4r_{20} - p_{10} - 4p_{20} - q_{01} - 2q_{11} - 13q_{21} - r_{10} - r_{11} - r_{12} + 22r_{21} - 39r_{30} - s_{01} \\
&- 2s_{02} - 3s_{03} + 2s_{11} + 22s_{12} - 13s_{21} + 8u_{10} + 2(3\pi - 8)u_{12} + 6(3\pi - 8)u_{30} + 8v_{01} \\
&+ 2(3\pi - 8)(3v_{03} + v_{21}) - 6(q_{21} - 2r_{21} + 3r_{30} - 2s_{12} + s_{21})\log(2) + 9q_{03}(3 + \log(4)) \\
&+ p_{12}(9 + \log(64)) - 3p_{30}(13 + \log(64))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{23}^5 = \frac{1}{960}\Big(&5p_{10} + 20p_{12} + 20p_{20} + 5q_{01} + 60q_{03} + 10q_{11} + 5r_{10} + 5r_{11} + 5r_{12} - 20r_{20} \\
&+ 20r_{21} + 5s_{01} + 10s_{02} + 15s_{03} - 10s_{11} + 20s_{12} - 32(2u_{10} + u_{12} + 3u_{30} + 2v_{01} \\
&+ 3v_{03} + v_{21})\Big),
\end{aligned}
$$

and D_{2i}^j, $i = 2, 3$ and $j = 1, \ldots, 4$, depend on the parameters of system (4.1), whose expressions have been omitted for simplicity. As in the case $n = 1$ from Chapter 3, we consider only the coefficients C_{n1}^i, $n = 2, 3$ and $i = 0, \ldots, 5$. Moreover, as

$$\lim_{h \to 1} \log(1 - h)(h - 1)^j = 0, \quad j = 1, \ldots, 4,$$

the factors $\log(\tau_{RS} - h)$ of the expansions in (4.42) and (4.43) can be disregarded in the study of the number of zeros. Therefore, we consider only the coefficients C_{2i}^j, $i = 2, 3$ and $j = 0, \ldots, 5$.

For equation (4.42), when

$$
\begin{aligned}
p_{00} &= \frac{1}{6}\Big(6r_{00} - 2p_{02} - 9p_{10} - 14p_{20} - 3q_{01} - 4q_{11} + 2r_{02} - 9r_{10} - r_{11} + 14r_{20} - 3s_{01} \\
&\quad - 2s_{02} + 4s_{11} + 6u_{10} - 3\pi u_{10} - 6v_{01} - 3\pi v_{01}\Big), \\
p_{10} &= \frac{1}{\log(8)}\Big(8p_{20} - 4p_{02} + 4q_{11} + 4r_{02} + 4r_{11} - 8r_{20} + 8s_{02} - 4s_{11} + 6(u_{10} + v_{01}) \\
&\quad - (4p_{20} + q_{01} + r_{10} + r_{11} + s_{01})\log(8) - (q_{11} - 2r_{20} + s_{02} - s_{11})\log(64)\Big), \\
p_{20} &= \frac{1}{8(2\log(2) - 1)}\Big(4q_{11} - 4p_{02} + 4r_{02} + 4r_{11} - 8r_{20} + 8s_{02} - 4s_{11} + 6u_{10} + 6v_{01} \\
&\quad + \log(2)(-8q_{11} - 8r_{02} - 8r_{11} + 16r_{20} - 16s_{02} + 8s_{11}) + 4p_{02}\log(4) + \log(8) \\
&\quad (\pi u_{10} + \pi v_{01})\Big), \\
u_{10} &\neq -v_{01},
\end{aligned}
$$

we have that

$$
C_{22}^0 = C_{22}^1 = C_{22}^2 = 0, \quad C_{22}^3 \neq 0 \quad \text{with} \quad \text{rank}\frac{\partial(C_{22}^0, C_{22}^1, C_{22}^2, C_{22}^3)}{\partial(p_{00}, p_{10}, p_{20}, u_{10})} = 4.
$$

This implies, by Proposition 3, we can choose $C_{22}^0, C_{22}^1, C_{22}^2$ and C_{22}^3 satisfying

$$
C_{22}^0 C_{22}^1 > 0, \quad C_{22}^1 C_{22}^2 > 0, \quad C_{22}^2 C_{22}^3 > 0 \quad \text{and} \quad |C_{22}^0| \ll |C_{22}^1| \ll |C_{22}^2| \ll |C_{22}^3| \ll 1,
$$

such that function (4.42) has exactly three positive simple zeros near $h = 1$. Therefore, the number of limit cycles from system (4.1) when $n = 2$, that can bifurcate from the periodic annulus near the heteroclinic orbit in $h = 1$, is at least three.

For equation (4.43), when

$$
\begin{aligned}
r_{00} &= p_{00} + p_{10} + p_{20} + p_{30} + r_{10} - r_{20} + r_{30} - 2(u_{10} + u_{30}), \\
r_{02} &= \frac{1}{2}\Big(2p_{02} + p_{10} + q_{01} - 6q_{03} + r_{10} - r_{11} + r_{12} + s_{01} - 2s_{02} - 3s_{03} + 12v_{03}\Big), \\
s_{11} &= \frac{1}{2}\Big(p_{10} + 4p_{12} + 4p_{20} + q_{01} + 12q_{03} + 2q_{11} + r_{10} + r_{11} + r_{12} - 4r_{20} + 4r_{21} + s_{01} \\
&\quad + 2s_{02} + 3s_{03} + 4s_{12}\Big), \\
s_{21} &= p_{12} - 3p_{30} + 3q_{03} - q_{21} + 2r_{21} - 3r_{30} + 2s_{12}, \\
v_{01} &= -u_{10}, \\
v_{21} &= -u_{12} - 3(u_{30} + v_{03}).
\end{aligned}
$$

we have that

$$C_{23}^{0}=C_{232}^{1}=C_{23}^{2}=C_{23}^{3}=C_{23}^{4}=C_{23}^{5}=0 \quad \text{with} \quad \text{rank}\frac{\partial(C_{23}^{0},C_{23}^{1},C_{23}^{2},C_{23}^{3},C_{23}^{4},C_{23}^{5})}{\partial(r_{00},r_{02},s_{11},s_{21},v_{01},v_{21})}=6.$$

This implies, by Proposition 3, we can choose $C_{23}^{0}, C_{23}^{1}, C_{23}^{2}, C_{23}^{3}, C_{23}^{4}$ and C_{23}^{5} such that function (4.43) has at least five positive simple zeros near $h=1$. Therefore, the number of limit cycles from system (4.1), when $n=3$, that can bifurcate from the periodic annulus near the heteroclinic orbit in $h=1$, is at least five.

Case SCS with homoclinic loop. Consider the Melnikov functions M_{12} and M_{13} given by Theorem 10. By Lemma 1, we can expand these functions at $h=1$ as

$$M_{12}(h) = \sum_{j=0}^{4} C_{12}^{j}(h-1)^{j}+\sum_{j=1}^{2} D_{12}^{j}\log(1-h)(h-1)^{j}+\mathcal{O}((h-1)^{5}), \qquad (4.44)$$

$$M_{13}(h) = \sum_{j=0}^{7} C_{13}^{j}(h-1)^{j}+\sum_{j=1}^{4} D_{13}^{j}\log(1-h)(h-1)^{j}+\mathcal{O}((h-1)^{8}), \qquad (4.45)$$

where the coefficients C_{1i}^{j}, $i=2,3$ and $0=1,\ldots,7$, that depend on the parameters of system (4.1), are listed in Appendix 7.6. Moreover, as in the previous case, the factors of the expansions in (4.44) and (4.45) that have $\log(1-h)$ can be disregarded in the study of the number of zeros. Therefore, we consider only the coefficients C_{1i}^{j}, $i=2,3$ and $j=0,\ldots,7$.

For equation (4.44), it is possible choose parameters values p_{00}, p_{10}, p_{20}, s_{01} and u_{10}, that are listed in Appendix 7.10, such that

$$C_{12}^{0}=C_{12}^{1}=C_{12}^{2}=C_{12}^{3}=0, \quad C_{12}^{4}\neq 0 \quad \text{with} \quad \text{rank}\frac{\partial(C_{12}^{0},C_{12}^{1},C_{12}^{2},C_{12}^{3},C_{12}^{4})}{\partial(p_{00},p_{10},p_{20},s_{01},u_{10})}=5.$$

As in the previous cases, by Proposition 3, we can choose $C_{12}^{0}, C_{12}^{1}, C_{12}^{2}, C_{12}^{3}$ and C_{12}^{4} such that function (4.44) has exactly four positive simple zeros near $h=1$. Therefore, the number of limit cycles from system (4.1), when $n=2$, that can bifurcate from the periodic annulus, near the homoclinic orbit in $h=1$, is at least four.

For equation (4.45), it is possible choose parameters values q_{01}, q_{21}, r_{00}, r_{02}, s_{11}, s_{21}, v_{01} and v_{21}, that are listed in Appendix 7.11, such that

$$C_{13}^{0}=C_{13}^{1}=C_{13}^{2}=C_{13}^{3}=C_{13}^{4}=C_{13}^{5}=C_{13}^{6}=C_{13}^{7}=0$$

with

$$\text{rank}\frac{\partial(C_{13}^{0},C_{13}^{1},C_{13}^{2},C_{13}^{3},C_{13}^{4},C_{13}^{5},C_{13}^{6},C_{13}^{7})}{\partial(q_{01},q_{21},r_{00},r_{02},s_{11},s_{21},v_{01},v_{21})}=8.$$

As in the previous cases, by Proposition 3, we can choose $C_{13}^0, C_{13}^1, C_{13}^2, C_{13}^3, C_{13}^5, C_{13}^6$ and C_{13}^7 such that function (4.45) has at least seven positive simple zeros near $h = 1$. Therefore, the number of limit cycles from system (4.1), when $n = 3$, that can bifurcate from the periodic annulus, near the homoclinic orbit in $h = 1$, is at least seven.

Case CCS. Consider the Melnikov functions given by Theorem 11. By Lemma 1, we can expand these functions at $h = 1$ as

$$M_{12}(h) = \sum_{j=0}^{4} C_{12}^j (h-1)^j + \sum_{j=1}^{2} D_{12}^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^5), \tag{4.46}$$

$$M_{13}(h) = \sum_{j=0}^{7} C_{13}^j (h-1)^j + \sum_{j=1}^{4} D_{13}^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^8), \tag{4.47}$$

$$M_{22}(h) = \sum_{j=0}^{4} C_{22}^j (h-1)^j + \sum_{j=1}^{2} D_{22}^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^5), \tag{4.48}$$

$$M_{23}(h) = \sum_{j=0}^{7} C_{23}^j (h-1)^j + \sum_{j=1}^{4} D_{23}^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^8), \tag{4.49}$$

where the coefficients C_{ij}^k, $i = 1, 2$, $j = 2, 3$ and $k = 0, \ldots, 7$, that depend on the parameters of system (4.1), are listed in Appendix 7.7. Moreover, as in the previous cases, the factors of the expansions in (4.46) – (4.49) that have $\log(1-h)$ can be disregarded in the study of the number of zeros. Therefore, we consider only the coefficients C_{ij}^k, $i = 1, 2$, $j = 2, 3$ and $k = 0, \ldots, 7$.

For equations (4.46) and (4.48), it is possible choose parameters values p_{00}, p_{02}, q_{11}, v_{01} and s_{11}, that are listed in Appendix 7.12, such that

$$C_{i2}^0 = C_{i2}^1 = C_{i2}^2 = C_{i2}^3 = 0, \; C_{i2}^4 \neq 0$$

with

$$\operatorname{rank} \frac{\partial(C_{i2}^0, C_{i2}^1, C_{i2}^2, C_{i2}^3, C_{i2}^4)}{\partial(p_{00}, p_{02}, q_{11}, v_{01}, s_{11})} = 5, \quad \text{for} \;\; i = 1, 2.$$

As in the previous cases, by Proposition 3, we can choose $C_{i2}^0, C_{i2}^1, C_{i2}^2, C_{i2}^3$ and C_{i2}^4, for $i = 1, 2$, such that functions (4.46) and (4.48) have exactly four positive simple zeros near $h = 1$. Therefore, the number of limit cycles from system (4.1), when $n = 2$, that can bifurcate from the periodic annulus, near the homoclinic orbit in $h = 1$, is at least four.

For equations (4.47) and (4.49), it is possible choose parameters values q_{01}, q_{21}, r_{00}, r_{02}, s_{01}, s_{21}, v_{01} and v_{21}, that are listed in Appendix 7.12, such that

$$C_{i3}^0 = C_{i3}^1 = C_{i3}^2 = C_{i3}^3 = C_{i3}^4 = C_{i3}^5 = C_{i3}^6 = C_{i3}^7 = 0$$

with

$$\text{rank}\frac{\partial(C_{i3}^0, C_{i3}^1, C_{i3}^2, C_{i3}^3, C_{i3}^4, C_{i3}^5, C_{i3}^6, C_{i3}^7)}{\partial(q_{11}, q_{21}, r_{00}, r_{02}, s_{01}, s_{21}, v_{01}, v_{21})} = 8, \quad \text{for } \ i = 1, 2.$$

As in the previous cases, by Proposition 3, we can choose $C_{i3}^0, C_{i3}^1, C_{i3}^2, C_{i3}^3, C_{i3}^4, C_{i3}^5, C_{i3}^6$ and C_{i3}^7, for $i = 1, 2$, such that functions (4.47) and (4.49) have at least seven simple zeros near $h = 1$. Therefore, the number of limit cycles from system (4.1), when $n = 3$, that can bifurcate from the periodic annulus, near the homoclinic orbit in $h = 1$, is at least seven.

Case CCC. Consider the Melnikov functions given by Theorem 12. By Lemma 1, we can expand these functions at $h = \tau$, for $\tau \in (0, \infty)$, as

$$M_{12}(h) = \sum_{j=0}^{4} C_{12}^j (h-\tau)^j + \mathcal{O}((h-\tau)^5), \tag{4.50}$$

$$M_{13}(h) = \sum_{j=0}^{7} C_{13}^j (h-\tau)^j + \mathcal{O}((h-\tau)^8), \tag{4.51}$$

$$M_{22}(h) = \sum_{j=0}^{4} C_{22}^j (h-\tau)^j + \mathcal{O}((h-\tau)^5), \tag{4.52}$$

$$M_{23}(h) = \sum_{j=0}^{7} C_{23}^j (h-\tau)^j + \mathcal{O}((h-\tau)^8), \tag{4.53}$$

$$M_{32}(h) = \sum_{j=0}^{4} C_{32}^j (h-\tau)^j + \mathcal{O}((h-\tau)^5), \tag{4.54}$$

$$M_{33}(h) = \sum_{j=0}^{7} C_{33}^j (h-\tau)^j + \mathcal{O}((h-\tau)^8), \tag{4.55}$$

where the coefficients C_{ij}^k, $i = 1, 2$, $j = 2, 3$ and $k = 0, \dots, 7$, that depend on the parameters of system (4.1), have their expressions omitted to simplify the text.

For equations (4.50), (4.52) and (4.54), it is possible choose parameters values q_{01}, r_{00}, r_{02}, s_{11} and v_{01}, such that

$$C_{i2}^0 = C_{i2}^1 = C_{i2}^2 = C_{i2}^3 = C_{i2}^4 = 0$$

with

$$\text{rank}\frac{\partial(C_{i2}^0, C_{i2}^1, C_{i2}^2, C_{i2}^3, C_{i2}^4)}{\partial(q_{01}, r_{00}, r_{02}, s_{11}, v_{01})} = 5, \quad \text{for } \ i = 1, 2, 3.$$

As in the previous cases, by Proposition 3, we can choose $C_{i2}^0, C_{i2}^1, C_{i2}^2, C_{i2}^3$ and C_{i2}^4, for $i = 1, 2, 3$, such that functions (4.50), (4.52) and (4.54) have at least four positive simple zeros near $h = \tau$. Therefore, the number of limit cycles from system (4.1), when $n = 2$, that can bifurcate from the periodic annulus, near the orbit in $h = \tau$, is at least four.

For equations (4.51), (4.53) and (4.55), due to the complexity of the calculation, we consider $\tau = 1$. Thus, it is possible choose parameters values q_{01}, q_{21}, r_{00}, r_{02}, s_{01}, s_{21}, v_{01} and v_{21}, such that

$$C^0_{i3} = C^1_{i3} = C^2_{i3} = C^3_{i3} = C^4_{i3} = C^5_{i3} = C^6_{i3} = C^7_{i3} = 0$$

with

$$\text{rank}\frac{\partial(C^0_{i3}, C^1_{i3}, C^2_{i3}, C^3_{i3}, C^4_{i3}, C^5_{i3}, C^6_{i3}, C^7_{i3})}{\partial(q_{01}, q_{21}, r_{00}, r_{02}, s_{01}, s_{21}, v_{01}, v_{21})} = 8, \quad \text{for} \quad i = 1, 2, 3.$$

As in the previous cases, by Proposition 3, we can choose $C^0_{i3}, C^1_{i3}, C^2_{i3}, C^3_{i3}, C^4_{i3}, C^5_{i3}, C^6_{i3}$ and C^7_{i3}, for $i = 1, 2, 3$, such that functions (4.51), (4.53) and (4.55) have at least seven simple zeros near $h = 1$. Therefore, the number of limit cycles from system (4.1), when $n = 3$, that can bifurcate from the periodic annulus, near the orbit in $h = 1$, is at least seven. $\square$

Chapter

5

Persistence of periodic solutions from periodic annulus formed by a center and two saddles of system $(1)|_{\epsilon=0}$

In Chapter 3, we proved that if the central subsystem from system $(3.1)|_{\epsilon=0}$ has a real center at the origin and the right or left subsystems have centers or saddles, then we have at least three limit cycles bifurcating from the periodic annulus by linear perturbations. A natural question is if this number of limit cycles changes when we take the center of the central subsystem from system $(3.1)|_{\epsilon=0}$ not necessarily at the origin. The main objective of this chapter is to answer this question for the case where system $(3.1)|_{\epsilon=0}$ is of the type SCS (we choose the case SCS because it has more interesting configuration, like heteroclinic orbits, etc, and apparently it has more potential to generate limit cycles than the other cases SCC and CCC). We will prove that if the central subsystem has a real or boundary center then the maximum number of limit cycles that bifurcate from the periodic annulus by linear perturbations is at least six. Now, if the central subsystem has a virtual center then we have at least four limit cycles bifurcating from the periodic annulus by linear perturbations.

5.1 Preliminaries and main result

In this chapter, let us consider discontinuous piecewise linear near-Hamiltonian differential systems with three zones, given by

$$\begin{cases} \dot{x} = H_y(x,y) + \epsilon f(x,y), \\ \dot{y} = -H_x(x,y) + \epsilon g(x,y), \end{cases} \tag{5.1}$$

with

$$f(x,y) = \begin{cases} f_L(x,y) = r_{10}x + r_{01}y + r_{00}, & x \leq -1, \\ f_C(x,y) = u_{10}x + u_{01}y + u_{00}, & -1 \leq x \leq 1, \\ f_R(x,y) = p_{10}x + p_{01}y + p_{00}, & x \geq 1, \end{cases} \tag{5.2}$$

$$g(x,y) = \begin{cases} g_L(x,y) = s_{10}x + s_{01}y + s_{00}, & x \leq -1, \\ g_C(x,y) = v_{10}x + v_{01}y + v_{00}, & -1 \leq x \leq 1, \\ g_R(x,y) = q_{10}x + q_{01}y + q_{00}, & x \geq 1. \end{cases} \tag{5.3}$$

We assume the hypotheses:

(H1) The unperturbed central subsystem from $(5.1)|_{\epsilon=0}$ has a center and the other unperturbed subsystems have only saddles.

(H2) The unperturbed system $(5.1)|_{\epsilon=0}$ has only crossing points on the straight lines $x = \pm 1$, except by some tangent points.

(H3) The unperturbed system $(5.1)|_{\epsilon=0}$ has two periodic annulus. The first one, consisting of a family of crossing periodic orbits passing through three zones and, the second one, consisting of a family of crossing periodic orbits passing through two zones, such that each orbit of these families has clockwise orientation (see Figure 5.1).

Thus, we can consider H in the normal form, given by Proposition 1 from Chapter 3, i.e.,

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_Lxy + a_Ly - \beta_Lx, & x \leq -1, \\ H^C(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2} - \beta_Cx, & -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_Rxy - a_Ry - \beta_Rx, & x \geq 1. \end{cases} \tag{5.4}$$

Note that by hypothesis (H1), the center of the central subsystem from $(4.1)|_{\epsilon=0}$ is not necessarily

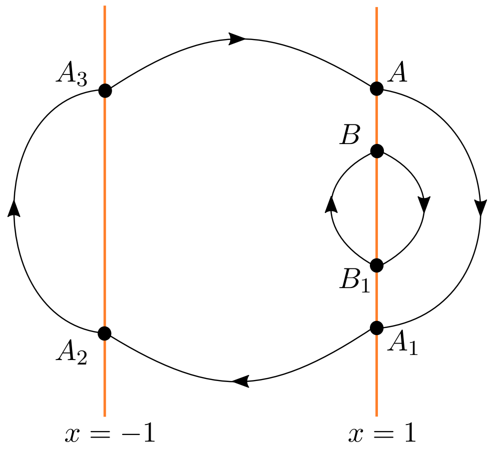


Figure 5.1: Crossing periodic orbits of system $(5.1)|_{\epsilon=0}$.

at the origin. As the case $\beta_C = 0$ studied in Chapter 3, we consider $\beta_C \neq 0$. In fact, by Remark 1, we can assume without loss of generality that $\beta_C > 0$.

The main result of this chapter is the following.

Theorem 13. *The maximum number of limit cycles of system* (5.1)*, satisfying hypotheses* (Hi)*, for* $i = 1, 2, 3$*, which can bifurcate from the periodic annuli of the unperturbed system* $(5.1)|_{\epsilon=0}$*, is either at least six, if the central subsystem has a real/boundary center, or at least four, if the central subsystem has a virtual center. Moreover, in the first case, we have four limit cycles passing through the three zones and two passing through two zones. In the second one, we have three passing through the three zones and one passing through two zones.*

To prove Theorem 13, we study the number of zeros of the first order Melnikov function associated to system (5.1). Our study is concentrated in a neighborhood of the periodic orbit that simultaneously bounded the periodic annulus from the two and three zones, since to estimate the zeros of the Melnikov function we consider its expansion at the point corresponding to this orbit. Moreover, we provide a detailed analytical method to study the number of simultaneous zeros from Melnikov functions defined in two or three zones (see Proposition 7).

5.2 A classification to system $(5.1)|_{\epsilon=0}$

In order to computation the zeros of the first order Melnikov functions M_0 and M_1 associated to system (5.1) that, similarly as in Theorem 1 but doing some adaptations, are expressed as

$$M_0(h) = \frac{H_y^R(A)}{H_y^C(A)} I_C^0 + \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} I_L^0 + \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} \overline{I}_C^0$$
$$+ \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} I_R^0, \quad h \in J_0,$$

and

$$M_1(h) = \frac{H_y^R(B)}{H_y^C(B)} I_C^1 + \frac{H_y^R(B) H_y^C(B_1)}{H_y^C(B) H_y^R(B_1)} I_R^1, \quad h \in J_1,$$

with

$$I_C^0 = \int_{\widehat{A_3A}} g_C dx - f_C dy, \quad I_L^0 = \int_{\widehat{A_2A_3}} g_L dx - f_L dy, \quad \bar{I}_C^0 = \int_{\widehat{A_1A_2}} g_C dx - f_C dy,$$

$$I_R^0 = \int_{\widehat{AA_1}} g_C dx - f_C dy, \quad I_C^1 = \int_{\widehat{B_1B}} g_C dx - f_C dy \quad \text{and} \quad I_R^1 = \int_{\widehat{BB_1}} g_R dx - f_R dy,$$

it is necessary to find the open intervals J_0 and J_1, where they are defined. For this, we have the following proposition.

Proposition 6. *Consider system* $(5.1)|_{\epsilon=0}$ *satisfying the hypotheses (Hi),* $i = 1, 2, 3$. *Then* $J_0 = (2\sqrt{\beta_C}, \tau_R)$ *and* $J_1 = (0, 2\sqrt{\beta_C})$, *where* $\tau_R = (a_R^2 - b_R\beta_R - \omega_R^2)/(b_R\omega_R)$ *with* $\omega_R = \sqrt{a_R^2 + b_R c_R}$, *and the periodic annuli are equivalent to one of the Figures 5.4–5.5.*

Proof. Firstly, we can note that if the saddles of the right or left subsystem from $(5.1)|_{\epsilon=0}$ are virtual or boundary, then we have not periodic orbits passing through the three zones. Let W_R^u and W_R^s (resp. W_L^u and W_L^s) be the unstable and stable separatrices of the saddles of the right (resp. left) subsystems from $(5.1)|_{\epsilon=0}$, respectively. Denote by $P_L^i = W_L^i \cap \{(-1, y) : y \in \mathbb{R}\}$ and $P_R^i = W_R^i \cap \{(1, y) : y \in \mathbb{R}\}$, for $i = u, s$. After some computation, it is possible to show that

$$P_L^u = (-1, \tau_L), \quad P_L^s = (-1, -\tau_L), \quad P_R^u = (1, -\tau_R), \quad P_R^s = (1, \tau_R),$$

where $\tau_R = (a_R^2 - b_R\beta_R - \omega_R^2)/b_R\omega_R$, $\tau_L = (a_L^2 + b_L\beta_L - \omega_L^2)/b_L\omega_L$, $\omega_R = \sqrt{a_R^2 + b_R c_R}$ and $\omega_L = \sqrt{a_L^2 + b_L c_L}$. Moreover, we have a symmetry between the points P_L^u and P_L^s (resp. P_R^u and P_R^s) with respect to x-axis.

As the vector field X_C associated with the central subsystem from $(5.1)|_{\epsilon=0}$ is $X_C(x, y) = (y, -x + \beta_C)$, with $\beta_C > 0$, then we have a periodic orbit L_0^1 tangent to straight line $x = -1$ in the point $P_L = (-1, 0)$ that intercepts crosswise the straight line $x = 1$ at two points, $P_R^1 = (1, 2\sqrt{\beta_C})$ and $P_R^2 = (1, -2\sqrt{\beta_C})$. If $0 < \beta_C < 1$, there is a periodic orbit L_0^2 tangent to straight line $x = 1$ in the point $P_R = (1, 0)$ contained in the region bounded by L_0^1. Moreover, the periodic orbit of central subsystem from $(5.1)|_{\epsilon=0}$ with initial condition at the point P_R^u intercepts the straight line $x = -1$ for positive time at two points $P_L^2 = (-1, -\sqrt{\tau_R^2 - 4\beta_C})$ and $P_L^1 = (-1, \sqrt{\tau_R^2 - 4\beta_C})$. Note that this periodic orbit coincides with the periodic orbit L_0^1 when $\beta_C = \tau_R^2/4$ (see Figure 5.2) or it does not intercept the straight line $x = -1$ when $\tau_R^2/4 < \beta_C$ (see Figure 5.3). In these cases, the orbit of system $(5.1)|_{\epsilon=0}$ with initial condition at the point $A(h) = (1, h)$, $h \in (0, \tau_R)$, does not intersect crosswise the straight line $x = -1$ for positive times, i.e., system $(5.1)|_{\epsilon=0}$ has no periodic orbits passing through the three zones.

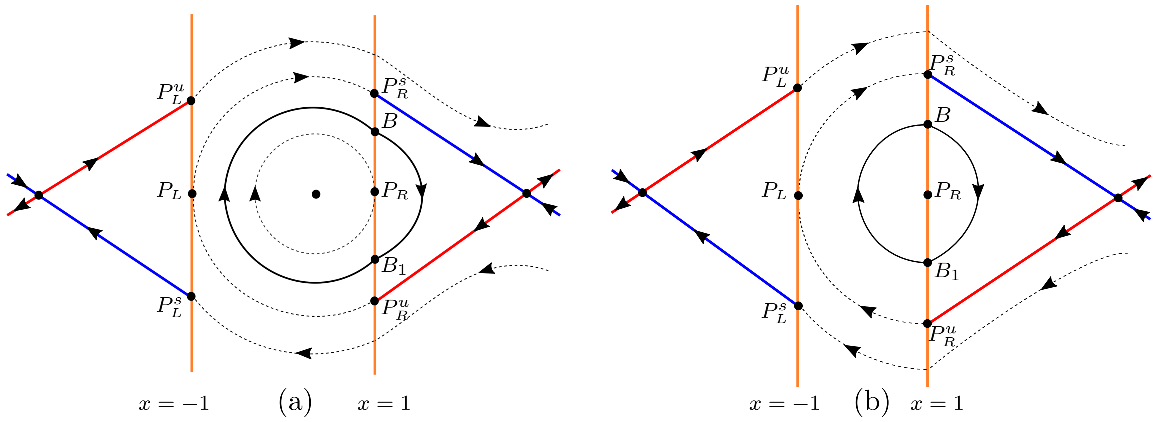


Figure 5.2: Phase portraits of system $(5.1)|_{\epsilon=0}$ with (a) $\beta_C = \tau_R^2/4 < 1$, (b) $\beta_C = \tau_R^2/4 \geq 1$.

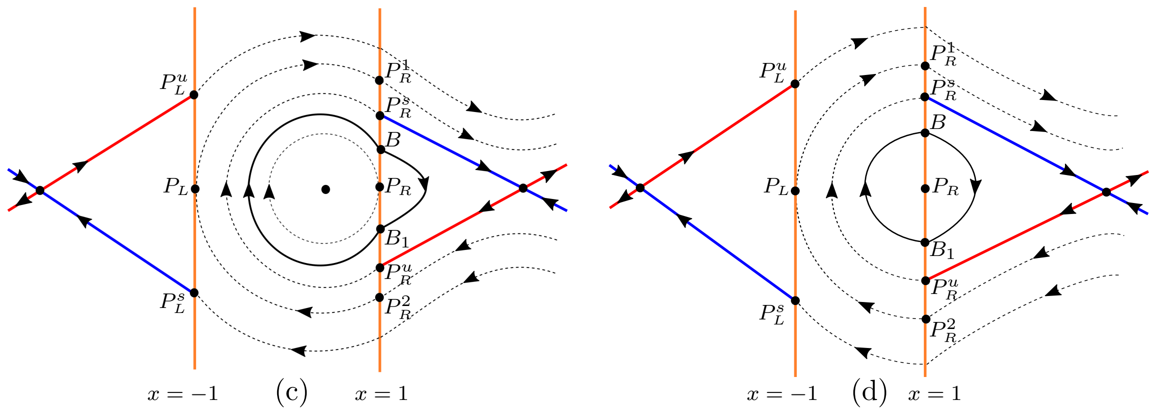


Figure 5.3: Phase portraits of system $(5.1)|_{\epsilon=0}$ with (c) $\tau_R^2/4 < \beta_C < 1$, (d) $\tau_R^2/4 < \beta_C$ and $\beta_C \geq 1$.

Suppose now that $\beta_C < \tau_R^2/4$. Then, by a reflection around the y-axis if necessary, we can assume without loss of generality that $\sqrt{\tau_R^2 - 4\beta_C} \leq \tau_L$. Moreover, if $\sqrt{\tau_R^2 - 4\beta_C} < \tau_L$ system $(5.1)|_{\epsilon=0}$ has a homoclinic loop passing through the points P_R^s, P_R^u, P_L^1 and P_L^2 (see Figure 5.4) and if $\sqrt{\tau_R^2 - 4\beta_C} = \tau_L$ system $(5.1)|_{\epsilon=0}$ has heteroclinic orbits passing through the points P_R^s, P_R^u, P_L^s and P_L^u (see Figure 5.5).

Consider an initial point of the form $A(h) = (1, h)$, with $h \in (2\sqrt{\beta_C}, \tau_R)$. By the hypothesis (H2), system $(5.1)|_{\epsilon=0}$ has a family of crossing periodic orbits that intersect the straight lines $x = \pm 1$ at four points, $A(h)$, $A_1(h) = (1, a_1(h))$, with $a_1(h) < h$, and $A_2(h) = (-1, a_2(h))$,

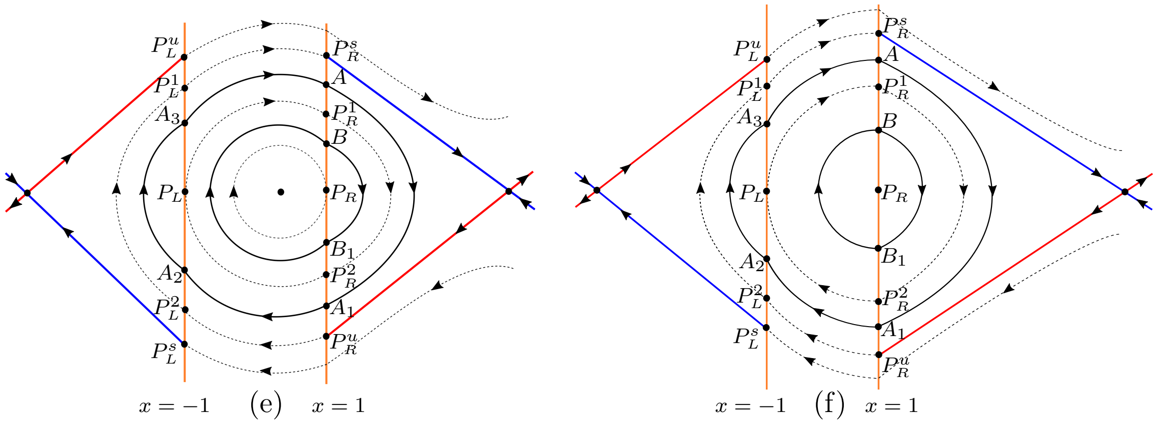


Figure 5.4: Phase portraits of system $(5.1)|_{\epsilon=0}$ with (e) $0 < \beta_C < 1$, $\beta_C < \tau_R^2/4$ and $\sqrt{\tau_R^2 - 4\beta_C} < \tau_L$, (f) $1 \leq \beta_C < \tau_R^2/4$ and $\sqrt{\tau_R^2 - 4\beta_C} < \tau_L$.

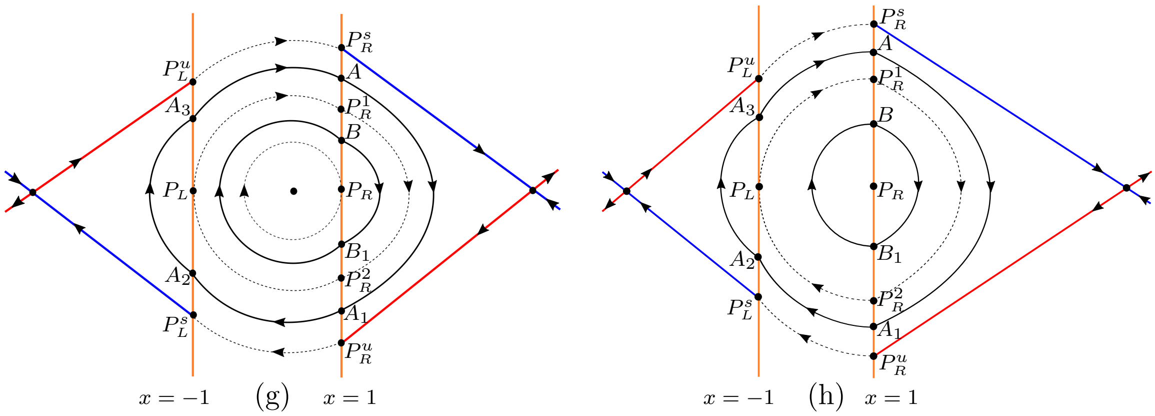


Figure 5.5: Phase portraits of system $(5.1)|_{\epsilon=0}$ with (g) $0 < \beta_C < 1$, $\beta_C < \tau_R^2/4$ and $\sqrt{\tau_R^2 - 4\beta_C} = \tau_L$, (h) $1 \leq \beta_C < \tau_R^2/4$ and $\sqrt{\tau_R^2 - 4\beta_C} = \tau_L$.

$A_3(h) = (-1, a_3(h))$, with $a_2(h) < a_3(h)$ satisfying

$$
\begin{aligned}
H^R(A(h)) &= H^R(A_1(h)),\\
H^C(A_1(h)) &= H^C(A_2(h)),\\
H^L(A_2(h)) &= H^L(A_3(h)),\\
H^C(A_3(h)) &= H^C(A(h)).
\end{aligned}
$$

More precisely, we have the equations

$$
\begin{aligned}
&\frac{b_R}{2}(h - a_1(h))(h + a_1(h)) = 0,\\
&\frac{1}{2}((a_1(h) - a_2(h))(a_1(h) + a_2(h)) - 4\beta_C) = 0,\\
&\frac{b_L}{2}(a_2(h) - a_3(h))(a_2(h) + a_3(h)) = 0,\\
&\frac{1}{2}((a_3(h) - h)(a_3(h) + h) + 4\beta_C) = 0.
\end{aligned}
$$

As $a_1(h) < h$, $a_2(h) < a_3(h)$, $b_R > 0$ and $b_L > 0$, the only solution of the above system is $a_1(h) = -h$, $a_2(h) = -\sqrt{h^2 - 4\beta_C}$ and $a_3(h) = \sqrt{h^2 - 4\beta_C}$, i.e., we have the four points given by $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -\sqrt{h^2 - 4\beta_C})$ and $A_3(h) = (-1, \sqrt{h^2 - 4\beta_C})$. Moreover, system $(5.1)|_{\epsilon=0}$ has a periodic orbit L_h passing through these points, for all $h \in (2\sqrt{\beta_C}, \tau_R)$.

Now, consider an initial point of the form $B(h) = (1, h)$, with $h \in (0, 2\sqrt{\beta_C})$. By hypotheses (H2), system $(5.1)|_{\epsilon=0}$ has a family of crossing periodic orbits that intersect the straight line $x = 1$ at two points, $B(h)$ and $B_1(h) = (1, b_1(h))$, with $b_1(h) < h$ satisfying

$$\begin{aligned} H^R(B(h)) &= H^R(B_1(h)), \\ H^C(B_1(h)) &= H^C(B(h)). \end{aligned}$$

More precisely, we have the equations

$$\begin{aligned} &\frac{b_R}{2}(h - b_1(h))(h + b_1(h)) = 0, \\ &\frac{1}{2}(b_1(h) - h)(b_1(h) + h) = 0. \end{aligned}$$

As $b_1(h) < h$ and $b_R > 0$, the only solution of the above system is $b_1(h) = -h$, i.e., we have the two points given by $B(h) = (1, h)$ and $B_1(h) = (1, -h)$. Moreover, system $(5.1)|_{\epsilon=0}$ has a periodic orbit L_h passing through these points, for all $h \in (0, 2\sqrt{\beta_C})$. If $h \in [\tau_R, \infty)$ then the orbit of system $(5.1)|_{\epsilon=0}$ with initial condition at $A(h)$ does not return to the straight line $x = 1$ for positive times, i.e., system $(5.1)|_{\epsilon=0}$ has no periodic orbit passing through the point $A(h)$. Therefore, system $(5.1)|_{\epsilon=0}$ has a periodic annulus, formed by the periodic orbits L_h, for $h \in (0, 2\sqrt{\beta_C})$, passing by two zones and bounded by the periodic orbit L_0^1, when $\beta_C \geq 1$, or the two periodic orbits L_0^1 and L_0^2, when $0 < \beta_C < 1$. Now, assuming that $\beta_C < \tau_R^2/4$, system $(5.1)|_{\epsilon=0}$ has a periodic annulus, formed by the periodic orbits L_h, for $h \in (2\sqrt{\beta_C}, \tau_R)$, passing by three zones and bounded by the periodic orbit L_0^1 and a homoclinic loop when $\sqrt{\tau_R^2 - 4\beta_C} < \tau_L$ or by two heteroclinic orbits when $\sqrt{\tau_R^2 - 4\beta_C} = \tau_L$. This completes the proof. $\square$

As, for $h \in (2\sqrt{\beta_C}, \tau_R)$, $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -\sqrt{h^2 - 4\beta_C})$, $A_3(h) = (-1, \sqrt{h^2 - 4\beta_C})$ and, for $h \in (0, 2\sqrt{\beta_C})$, $B(h) = (1, h)$, $B_1(h) = (1, -h)$, the coefficients that multiply the integrals of first order Melnikov functions (5.5) and (5.6) associated with system (5.1) can be easily calculated. More precisely, we have the immediate corollary.

Corollary 10. *For $J_0 = (2\sqrt{\beta_C}, \tau_R)$ and $J_1 = (0, 2\sqrt{\beta_C})$, the interval of definition of Melnikov* *functions given by* (5.5) *and* (5.6)*, respectively, we have that*

$$\frac{H_y^R(A)}{H_y^C(A)} = \frac{H_y^R(B)}{H_y^C(B)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} = b_R, \quad \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} = \frac{b_R}{b_L},$$

and

$$\frac{H_y^R(B)H_y^C(B_1)}{H_y^C(B)H_y^L(B_1)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} = 1.$$

Then, the first order Melnikov functions associated to system (5.1) *can be written as*

$$\begin{aligned} M_0(h) = b_R \int_{\widehat{A_3A}} g_C dx - f_C dy + \frac{b_R}{b_L} \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ + b_R \int_{\widehat{A_1A_2}} g_C dx - f_C dy + \int_{\widehat{AA_1}} g_R dx - f_R dy \end{aligned} \tag{5.5}$$

and

$$M_1(h) = b_R \int_{\widehat{B_1B}} g_C dx - f_C dy + \int_{\widehat{BB_1}} g_R dx - f_R dy. \tag{5.6}$$

5.3 Melnikov functions associated with system (5.1)

In order to prove Theorem 13, we simplify the expression to the first order Melnikov functions M_0 and M_1 associated to system (5.1). For this, we define the following functions:

$$\begin{aligned}
f_1(h) &= \sqrt{h^2 - 4\beta_C}, \\
f_2(h) &= -\left(\frac{h^2 + 2 - hf_1(h) - 2\beta_C}{h^2 + (\beta_C - 1)^2}\right), \\
f_3(h) &= \sqrt{1 - \left(\frac{hf_1(h) - 1 + \beta_C^2}{h^2 + (\beta_C - 1)^2}\right)^2}, \\
f_4(h) &= \left(\frac{hf_1(h) - 1 + \beta_C^2}{h^2 + (\beta_C - 1)^2}\right), \\
f_0^R(h) &= (h - \tau_R)(h + \tau_R) \log\left(\frac{h + \tau_R}{\tau_R - h}\right), \\
f_0^L(h) &= (h^2 - 4\beta_C - \tau_L^2) \log\left(\frac{f_1(h) + \tau_L}{\tau_L - f_1(h)}\right), \\
f_0^C(h) &= (h^2 + (\beta_C - 1)^2) \arccos(f_4(h)), \\
f_1^C(h) &= (h^2 + (\beta_C - 1)^2)\left(2\pi - \arccos\left(\frac{1 - h^2 - 2\beta_C + \beta_C^2}{h^2 + (\beta_C - 1)^2}\right)\right),
\end{aligned} \tag{5.7}$$

with $h \in (0, \tau_R)$ and $0 < \beta_C < \tau_R^2/4$.

Theorem 14. *The first order Melnikov functions* M_0 *and* M_1 *given by equations* (5.5) *and* (5.6)*,* *respectively, can be expressed as*

$$\begin{aligned} M_0(h) = &k_0^0 h + (k_1^0 + k_1^1 h) f_2(h) + (k_2^0 + k_2^1 h) f_3(h) + (k_3^0 + k_3^1 h + k_3^2 h^2) \\ & f_3(h)^2 + (k_4^0 + k_4^1 h + k_4^2 h^2) f_3(h) f_4(h) + k_5^0 f_1(h) + k_6^0 f_1(h) f_2(h) \\ & + k_7^0 f_1(h) f_3(h) + k_8^0 f_1(h) f_3(h)^2 + k_9^0 f_1(h) f_3(h) f_4(h) + k_{10}^0 f_0^R(h) \\ & + k_{11}^0 f_0^L(h) + k_{12}^0 f_0^C(h), \quad h \in \left(2\sqrt{\beta_C}, \tau_R\right) \end{aligned} \tag{5.8}$$

and

$$M_1(h) = k_{13}^0 h + k_{14}^0 f_0^R(h) + k_{15}^0 f_1^C(h), \quad h \in \left(0, 2\sqrt{\beta_C}\right). \tag{5.9}$$

The functions f_i, f_0^R, f_0^L, f_j^C, $i = 1, \ldots, 4$ *and* $j = 0, 1$, *are given in* (5.7). *Here the coefficients* k_i^j, $i = 0, \ldots, 15$ *and* $j = 0, 1, 2$, *depend on the parameters of system* (5.1).

Proof. Firstly, let us simplify the Melnikov function M_0 given by equation (5.5). For this, consider the orbit $(x_R(t), y_R(t))$ of system $(5.1)|_{\epsilon=0}$, such that $(x_R(0),\, y_R(0)) = (1, h)$, given by

$$\begin{aligned} x_R(t) = & -\frac{e^{-t\omega_R}}{2\omega_R}\Big(b_R h - b_R e^{2t\omega_R} h - 2e^{t\omega_R}\omega_R + b_R\tau_R - 2b_R e^{t\omega_R}\tau_R + b_R e^{2t\omega_R}\tau_R\Big) \\ y_R(t) = & -\frac{e^{-t\omega_R}}{2\omega_R}\Big(- a_R h + a_R e^{2t\omega_R} h - \omega_R h - e^{2t\omega_R}\omega_R h - a_R\tau_R + 2a_R e^{t\omega_R}\tau_R \\ & - a_R e^{2t\omega_R}\tau_R - \omega_R\tau_R + e^{2t\omega_R}\omega_R\tau_R\Big). \end{aligned}$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$t_R = \frac{1}{\omega_R}\log\left(\frac{h + \tau_R}{\tau_R - h}\right).$$

Now, for f_R and g_R defined in (5.2) and (5.3), respectively, we have

$$\begin{aligned} I_R^0 = & \int_{\widehat{AA_1}} g_R dx - f_R dy \\ = & \int_0^{t_R} \big(g_R(x_R(t), y_R(t)) x_R'(t) - f_R(x_R(t), y_R(t)) y_R'(t)\big) dt \\ = & \int_0^{t_R} \Bigg(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y_R'(t)\Bigg) dt \\ = & \alpha_1 h + \alpha_2 f_0^R(h) \end{aligned} \tag{5.10}$$

with

$$\alpha_1 = \frac{1}{\omega_R}\Big(2(p_{00} + p_{10})\omega_R + b_R(p_{10} + q_{01})\tau_R\Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_R}\Big(p_{10} + q_{01}\Big).$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(5.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \beta_C + (1-\beta_C)\cos(t) - h\sin(t),\\
y_{C1}(t) &= -h\cos(t) + (\beta_C - 1)\sin(t).
\end{aligned}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -\sqrt{h^2 - 4\beta_C})$, is

$$t_{C1} = \arccos(f_4(h)).$$

Now, for f_C and g_C defined in (5.2) and (5.3), respectively, we obtain

$$
\begin{aligned}
\bar{I}_C^0 &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy\\
&= \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t))x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t))y'_{C1}(t)\big)dt\\
&= \int_0^{t_{C1}} \left(\sum_{i+j=0}^{1} v_{ij}x^i_{C1}(t)y^j_{C1}(t)x'_{C1}(t) - \sum_{i+j=0}^{1} u_{ij}x^i_{C1}(t)y^j_{C1}(t)y'_{C1}(t)\right)dt \qquad (5.11)\\
&= (\alpha_3 + \alpha_4 h)f_2(h) + (\alpha_5 + \alpha_6 h + \alpha_7 h^2)f_3(h)f_4(h) + (\alpha_8 + \alpha_9 h)f_3(h)\\
&\quad + (\alpha_{10} + \alpha_{11}h + \alpha_{12}h^2)f_3(h)^2 + \alpha_{13}f_0^C(h),
\end{aligned}
$$

with

$$
\begin{aligned}
\alpha_3 &= -(\beta_C - 1)(v_{00} + v_{10}\beta_C),\\
\alpha_4 &= u_{00} + u_{10}\beta_C,\\
\alpha_5 &= \frac{1}{2}\Big(u_{10} - v_{01}\Big)(\beta_C - 1)^2,\\
\alpha_6 &= (u_{01} + v_{10})(\beta_C - 1),\\
\alpha_7 &= \frac{1}{2}\Big(v_{01} - u_{10}\Big),\\
\alpha_8 &= -(u_{00}(\beta_C - 1) + u_{10}(\beta_C - 1)\beta_C),
\end{aligned}
\qquad
\begin{aligned}
\alpha_9 &= -(v_{00} + v_{10}\beta_C),\\
\alpha_{10} &= \frac{1}{2}\Big(- u_{01} - v_{10}(\beta_C - 1)^2 + 2u_{01}\beta_C - u_{01}\beta_C^2\Big),\\
\alpha_{11} &= (u_{10} - v_{01})(\beta_C - 1),\\
\alpha_{12} &= \frac{1}{2}\Big(u_{01} + v_{10}\Big),\\
\alpha_{13} &= \frac{1}{2}\Big(u_{10} + v_{01}\Big).
\end{aligned}
$$

The orbit $(x_L(t), y_L(t))$ of system $(5.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -\sqrt{h^2 - 4\beta_C})$, is given by

$$
\begin{aligned}
x_L(t) &= \frac{e^{-t\omega_L}}{2\omega_L}\Big(b_L f_1(h) - b_L e^{2t\omega_L} f_1(h) - 2e^{t\omega_L}\omega_L + b_L\tau_L - 2b_L e^{t\omega_L}\tau_L + b_L e^{2t\omega_L}\tau_L\Big),\\
y_L(t) &= \frac{e^{-t\omega_L}}{2\omega_L}\Big(- a_L f_1(h) + a_L e^{2t\omega_L} f_1(h) - \omega_L f_1(h) - e^{2t\omega_L}\omega_L f_1(h) - a_L\tau_L + 2a_L e^{t\omega_L}\tau_L\\
&\quad - a_L e^{2t\omega_L}\tau_L - \omega_L\tau_L + e^{2t\omega_L}\omega_L\tau_L\Big).
\end{aligned}
$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -\sqrt{h^2 - 4\beta_C})$ to $A_3(h) =$

$(-1, \sqrt{h^2 - 4\beta_C})$, is

$$t_L = \frac{1}{\omega_L} \log\left(\frac{f_1(h) + \tau_L}{\tau_L - f_1(h)}\right).$$

Now, for f_L and g_L defined in (5.2) and (5.3), respectively, we have

$$\begin{aligned} I_L^0 &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ &= \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\big)dt \\ &= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{1} s_{ij} x_L^i(t) y_L^j(t) x_L'(t) - \sum_{i+j=0}^{1} r_{ij} x_L^i(t) y_L^j(t) y_L'(t)\Bigg)dt \\ &= \alpha_{14} f_1(h) + \alpha_{15} f_0^L(h), \end{aligned} \tag{5.12}$$

with

$$\alpha_{14} = \frac{1}{\omega_L}\Big(-2r_{00}\omega_L + 2r_{10}\omega_L + b_L(r_{10} + s_{01})\tau_L\Big) \quad \text{and} \quad \alpha_{15} = \frac{b_L}{2\omega_L}\Big(r_{10} + s_{01}\Big).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(5.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\ y_{C2}(0)) = (-1, \sqrt{h^2 - 4\beta_C})$, is given by

$$\begin{aligned} x_{C2}(t) &= \beta_C - (1 + \beta_C)\cos(t) + f_1(h)\sin(t), \\ y_{C2}(t) &= f_1(h)\cos(t) + (1 + \beta_C)\sin(t). \end{aligned}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, \sqrt{h^2 - 4\beta_C})$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos(f_4(h)).$$

Now, for f_C and g_C defined in (5.2) and (5.3), respectively, we obtain

$$\begin{aligned} I_C^0 &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t))x_{C2}'(t) - f_C(x_{C2}(t), y_{C2}(t))y_{C2}'(t)\big)dt \\ &= \int_0^{t_{C2}} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_{C2}^i(t) y_{C2}^j(t) x_{C2}'(t) - \sum_{i+j=0}^{1} u_{ij} x_{C2}^i(t) y_{C2}^j(t) y_{C2}'(t)\Bigg)dt \\ &= \alpha_{16} f_2(h) + \alpha_{17} f_1(h) f_2(h) + \alpha_{18} f_3(h) + \alpha_{19} f_1(h) f_3(h) \\ &\quad + (\alpha_{20} + \alpha_{21} h^2) f_3(h)^2 + \alpha_{22} f_1(h) f_3(h) + (\alpha_{23} + \alpha_{24} h^2) \\ &\quad f_3(h) f_4(h) + \alpha_{25} f_1(h) f_3(h) f_4(h) + \alpha_{13} f_0^C(h), \end{aligned} \tag{5.13}$$

with

$$\alpha_{13} = \frac{1}{2}\Big(u_{10} + v_{01}\Big),$$
$$\alpha_{16} = -(1+\beta_C)(v_{00} + v_{10}\beta_C),$$
$$\alpha_{17} = -(u_{00} + u_{10}\beta_C),$$
$$\alpha_{18} = -(1+\beta_C)(u_{00} + u_{10}\beta_C),$$
$$\alpha_{19} = (v_{00} + v_{10}\beta_C),$$
$$\alpha_{20} = \frac{1}{2}\Big(u_{01} + v_{10}\Big)(-1 - 6\beta_C - \beta_C^2),$$
$$\alpha_{21} = \frac{1}{2}\Big(u_{01} + v_{10}\Big),$$
$$\alpha_{22} = -(u_{10} - v_{01})(1+\beta_C),$$
$$\alpha_{23} = -\frac{1}{2}\Big(u_{10} - v_{01}\Big)(-1 - 6\beta_C - \beta_C^2),$$
$$\alpha_{24} = -\frac{1}{2}\Big(u_{10} - v_{01}\Big),$$
$$\alpha_{25} = -(u_{01} + v_{10})(1+\beta_C).$$

Hence, by Corollary 10, the first order Melnikov function M_0 associated to system (5.1) is given by

$$M_0(h) = b_R I_C^0 + \frac{b_R}{b_L} I_L^0 + b_R \bar{I}_C^0 + I_R^0. \tag{5.14}$$

Replacing (5.10), (5.11), (5.12) and (5.13) into (5.14) we obtain (5.8), with

$$k_0^0 = \alpha_1,$$
$$k_1^0 = b_R(\alpha_3 + \alpha_{16}),$$
$$k_1^1 = b_R\alpha_4,$$
$$k_2^0 = b_R(\alpha_8 + \alpha_{18}),$$
$$k_2^1 = b_R\alpha_9,$$
$$k_3^0 = b_R(\alpha_{10} + \alpha_{20}),$$
$$k_3^1 = b_R\alpha_{11},$$
$$k_3^2 = b_R(\alpha_{12} + \alpha_{21}),$$
$$k_4^0 = b_R(\alpha_5 + \alpha_{23}),$$
$$k_4^1 = b_R\alpha_6,$$
$$k_4^2 = b_R(\alpha_7 + \alpha_{24}),$$
$$k_5^0 = \frac{b_R}{b_L}\alpha_{14},$$
$$k_6^0 = b_R\alpha_{17},$$
$$k_7^0 = b_R\alpha_{19},$$
$$k_8^0 = b_R\alpha_{22},$$
$$k_9^0 = b_R\alpha_{25},$$
$$k_{10}^0 = \alpha_2,$$
$$k_{11}^0 = \frac{b_R}{b_L}\alpha_{15},$$
$$k_{12}^0 = 2b_R\alpha_{13}.$$

Now, let us simplify the Melnikov function M_1 given by (5.6). Similarly as in previous case, we have that

$$I_R^1 = \alpha_1 h + \alpha_2 f_0^R(h). \tag{5.15}$$

The orbit $(x_C(t), y_C(t))$ of system $(5.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (1, -h)$, is given by

$$x_C(t) = \beta_C + (1-\beta_C)\cos(t) - h\sin(t),$$
$$y_C(t) = -h\cos(t) + (\beta_C - 1)\sin(t).$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $B_1(h) = (1, -h)$ to $B(h) = (1, h)$, is

$$t_C = 2\pi - \arccos\left(\frac{1 - h^2 - 2\beta_C + \beta_C^2}{h^2 + 1 - 2\beta_C + \beta_C^2}\right).$$

Now, for f_C and g_C defined in (5.2) and (5.3), respectively, we obtain

$$
\begin{aligned}
I_C^1 &= \int_{\widehat{B_1 B}} g_C dx - f_C dy \\
&= \int_0^{t_C} \big(g_C(x_C(t), y_C(t)) x'_{C2}(t) - f_C(x_C(t), y_C(t)) y'_C(t)\big) dt \\
&= \int_0^{t_C} \left(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x'_C(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y'_C(t) \right) dt \\
&= \alpha_{26} h + \alpha_{27} f_1^C(h),
\end{aligned}
\tag{5.16}
$$

with

$$
\alpha_{26} = 2u_{00} + u_{10} - v_{01} + (u_{10} + v_{01})\beta_C \quad \text{and} \quad \alpha_{27} = \frac{1}{2}(u_{10} + v_{01}).
$$

Hence, by Corollary 10, the first order Melnikov function M_1 associated to system (5.1) is given by

$$
M_0(h) = b_R I_C^1 + I_R^1. \tag{5.17}
$$

Replacing (5.15) and (5.16) into (5.17) we obtain (5.9), with

$$
k_{13}^0 = \alpha_1 - b_R \alpha_{26}, \quad k_{14}^0 = \alpha_2 \quad \text{and} \quad k_{15}^0 = b_R \alpha_{27}.
$$

□

5.4 Proof of Theorem 13

To prove Theorem 13, we study the number of simultaneous zeros of the Melnikov functions M_0 and M_1 associated with system (5.1). Our study focuses on a neighborhood of the periodic orbit that simultaneously bounds the periodic annulus of the two and three zones, since to estimate the zeros of the Melnikov functions we consider their expansions at the point corresponding to this orbit. For this, we need of the following result.

Consider the functions $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ given by

$$
F(h) = \sum_{j=0}^{n} \left(C_j(\delta)(h-\tau)^j + D_{2j+1}(\delta)(h-\tau)^{\frac{2j+1}{2}} \right) + \mathcal{O}((h-\tau)^{n+1}), \tag{5.18}
$$

and

$$
G(h) = \sum_{j=0}^{n} C_j(\delta)(h-\tau)^j + \mathcal{O}((h-\tau)^{n+1}), \tag{5.19}
$$

with $\tau \geq 0$ and the coefficients $C_j(\delta)$ and $D_{2j+1}(\delta)$, $j = 0, \ldots, n$, are smooth functions depending on the parameters $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. Then we have the following proposition.

Proposition 7. *Suppose that there exist an integer $k \geq 1$ and $\tilde{\delta} \in \mathbb{R}^m$ with $m \geq 2k+1$ such that*

$$C_j(\tilde{\delta}) = D_{2j+1}(\tilde{\delta}) = 0, \quad j = 0, \ldots, k-1, \tag{5.20}$$

and

$$\operatorname{rank} \frac{\partial(C_0, D_1, \ldots, D_{2k-1}, C_k)}{\partial(\delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m)}(\tilde{\delta}) = 2k+1$$

or

$$C_j(\tilde{\delta}) = D_{2j+1}(\tilde{\delta}) = C_{k-1}(\tilde{\delta}) = 0, \quad j = 0, \ldots, k-2, \tag{5.21}$$

and

$$\operatorname{rank} \frac{\partial(C_0, D_1, \ldots, C_{k-1}, D_{2k-1})}{\partial(\delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m)}(\tilde{\delta}) = 2k.$$

Then, if condition (5.20) *holds, functions* (5.18) *and* (5.19) *have at least* $2k$ *and* k *real zeros, respectively, in a neighborhood of $h = \tau$ for all δ near $\tilde{\delta}$. Moreover, function* (5.19) *has exactly k real simple zeros, in a neighborhood of $h = \tau$ for all δ near $\tilde{\delta}$, when $C_k(\tilde{\delta}) \neq 0$. Now, if condition* (5.21) *holds, functions* (5.18) *and* (5.19) *have at least* $2k-1$ *and* $k-1$ *real zeros, respectively, in a neighborhood of $h = \tau$ for all δ near $\tilde{\delta}$.*

Proof. We prove only the case where condition (5.20) holds. The proof is similar when we have the condition (5.21). By the condition (5.20) we can assume that

$$\det \frac{\partial(C_0, D_1, \ldots, D_{2k-1}, C_k)}{\partial(\delta_1, \delta_2, \ldots, \delta_{2k}, \delta_{2k+1})}(\tilde{\delta}) \neq 0,$$

Then the change of parameters $\tilde{C}_j = C_j(\delta_1, \ldots, \delta_{2k+1}, \tilde{\delta}_{2k+2}, \ldots, \tilde{\delta}_m)$, $j = 0, \ldots, k$, and $\tilde{D}_{2j+1} = D_{2j+1}(\delta_1, \ldots, \delta_{2k+1}, \tilde{\delta}_{2k+2}, \ldots, \tilde{\delta}_m)$, $j = 0, \ldots, k-1$, has inverse $\delta_i(\tilde{C}_0, \tilde{D}_1, \ldots, \tilde{D}_{2k-1}, \tilde{C}_k)$, $i = 1, \ldots, 2k+1$, and we can rewrite (5.18) and (5.19) as

$$\begin{aligned}
F(h) &= \tilde{C}_0 + \tilde{D}_1(h-\tau)^{\frac{1}{2}} + \cdots + \tilde{D}_{2k-1}(h-\tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h-\tau)^k + \mathcal{O}((h-\tau)^{k+1}), \\
G(h) &= \tilde{C}_0 + \tilde{C}_1(h-\tau) + \cdots + \tilde{C}_k(h-\tau)^k + \mathcal{O}((h-\tau)^{k+1})
\end{aligned}$$

with $\tilde{C}_j(\tilde{\delta}) = \tilde{D}_{2j+1}(\tilde{\delta}) = 0$, $j = 0, \ldots, k-1$.

Suppose that k is even (if k is odd the proof is similar) and let $0 < |\tilde{C}_k - C_k(\tilde{\delta})| \ll 1$, $0 < h_1^F - \tau \ll 1$ and $0 < \tau - h_1^G \ll 1$ such that

$$\tilde{C}_k(h_1^F - \tau)^k > 0 \quad \text{and} \quad \tilde{C}_k(h_1^G - \tau)^k > 0.$$

Take $\tilde{D}_{2k-1}$ such that $|\tilde{D}_{2k-1}| \ll \tilde{C}_k$, $\tilde{D}_{2k-1}\tilde{C}_k < 0$ and

$$\tilde{D}_{2k-1}(h_1^F - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h_1^F - \tau)^k > 0.$$

As $\tilde{D}_{2k-1} < 0$, we can choose h_2^F, such that $0 < h_2^F - \tau \ll h_1^F - \tau \ll 1$ and

$$\tilde{D}_{2k-1}(h_2^F - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h_2^F - \tau)^k < 0.$$

Therefore, the equation

$$\tilde{D}_{2k-1}(h - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h - \tau)^k = 0$$

has a zero $\tilde{h}_1^F$, with $h_2^F < \tilde{h}_1^F < h_1^F$. Now, take $\tilde{C}_{k-1}$ such that $|\tilde{C}_{k-1}| \ll |\tilde{D}_{2k-1}| \ll \tilde{C}_k$, $\tilde{C}_{k-1}\tilde{D}_{2k-1} < 0$,

$$\tilde{C}_{k-1}(h_2^F - \tau)^{k-1} + \tilde{D}_{2k-1}(h_2^F - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h_2^F - \tau)^k < 0$$

and

$$\tilde{C}_{k-1}(h_1^G - \tau)^{k-1} + \tilde{C}_k(h_1^G - \tau)^k > 0.$$

As $\tilde{C}_{k-1} > 0$, we can choose h_3^F and h_2^G, such that $0 < h_3^F - \tau \ll h_2^F - \tau \ll 1$, $0 < \tau - h_2^G \ll \tau - h_1^G \ll 1$,

$$\tilde{C}_{k-1}(h_3^F - \tau)^{k-1} + \tilde{D}_{2k-1}(h_3^F - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h_3^F - \tau)^k > 0$$

and

$$\tilde{C}_{k-1}(h_2^G - \tau)^{k-1} + \tilde{C}_k(h_2^G - \tau)^k < 0.$$

Therefore, the equation

$$\tilde{C}_{k-1}(h - \tau)^{k-1} + \tilde{D}_{2k-1}(h - \tau)^{\frac{2k-1}{2}} + \tilde{C}_k(h - \tau)^k = 0$$

has one more zero $\tilde{h}_2^F$, with $h_3^F < \tilde{h}_2^F < h_2^F < \tilde{h}_1^F < h_1^F$, and the equation

$$\tilde{C}_{k-1}(h - \tau)^{k-1} + \tilde{C}_k(h - \tau)^k = 0$$

has a zero $\tilde{h}_1^G$, with $h_1^G < \tilde{h}_1^G < h_2^G$. Continuing with this reasoning, there are $\tilde{C}_0, \tilde{D}_1, \dots,$ $\tilde{D}_{2k-1}, \tilde{C}_k$, such that

$$\tilde{C}_0\tilde{D}_1 < 0,\ \tilde{D}_1\tilde{C}_2 < 0, \dots,\ \tilde{D}_{2k-1}\tilde{C}_k < 0, \quad |\tilde{C}_0| \ll |\tilde{D}_1| \ll \dots \ll |\tilde{D}_{2k-1}| \ll \tilde{C}_k,$$

and functions (5.18) and (5.19) have $2k$ and k real simple zeros near τ, respectively. Note that we could have done all of the above computations by adding at each step the remainders of functions (5.18) and (5.19) (indicated by the Landau Symbols). Moreover, if $\tilde{C}_k(\tilde{\delta}) \neq 0$ and $\tilde{C}_j(\tilde{\delta}) = \tilde{D}_{2j+1}(\tilde{\delta}) = 0$, $j = 0, \ldots, k-1$, by Rolle's theorem, for all δ near $\tilde{\delta}$ we can choose the $\tilde{C}_i$ and h_j^G, with $i = 0, \ldots, k$ and $j = 1, \ldots, k$, such that equation (5.19) have exactly k real simple zero near $h = \tau$. □

Now, we are able to prove Theorem 13.

Proof of Theorem 13. In order to prove Theorem 13, let us consider the Melnikov functions associated to system (5.1) with f and g given by (5.2) and (5.3), respectively, and H in the normal form given by (5.4). Our study will be concentrated in a neighborhood of the orbit L_h with $h = 2\sqrt{\beta_C}$ (i.e., the orbit L_0^1), since to estimate the zeros of the Melnikov functions we consider their expansions at the point corresponding to this orbit. More precisely, consider the Melnikov functions M_0 and M_1 given by Theorem 14. Note that, the function M_0 is not differentiable at $h = 2\sqrt{\beta_C}$. Thus, for obtain the expansion of this function at $h = 2\sqrt{\beta_C}$ we need to do a change of variables $h = u^2 + 2\sqrt{\beta_C}$ and, after compute its Taylor's expansion at $u = 0$, we return to the original variable by doing the change $u = \sqrt{h - 2\sqrt{\beta_C}}$. Then the expansion of functions M_0 and M_1 at $h = 2\sqrt{\beta_C}$ are given by

$$\begin{aligned} M_0(h) = C_{00} + D_1\big(h - 2\sqrt{\beta_C}\big)^{\frac{1}{2}} + C_{10}\big(h - 2\sqrt{\beta_C}\big) + D_3\big(h - 2\sqrt{\beta_C}\big)^{\frac{3}{2}} \\ + C_{20}\big(h - 2\sqrt{\beta_C}\big)^2 + \mathcal{O}\big(\big(h - 2\sqrt{\beta_C}\big)^3\big), \quad 0 < h - 2\sqrt{\beta_C} \ll 1 \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} M_1(h) = C_{01} + C_{11}\big(h - 2\sqrt{\beta_C}\big) + C_{21}\big(h - 2\sqrt{\beta_C}\big)^2 + \mathcal{O}\big(\big(h - 2\sqrt{\beta_C}\big)^3\big), \\ 0 < 2\sqrt{\beta_C} - h \ll 1 \end{aligned} \tag{5.23}$$

where

$$
\begin{aligned}
C_{00} &= \frac{1}{2\omega_R}\Bigg(4\sqrt{\beta_C}((2p_{00}+2p_{10}-b_R(2u_{00}+u_{10}-v_{01}+(u_{10}+v_{01})\beta_C))\omega_C+b_R(p_{10}\\
&\quad +q_{01})\tau_R)+2b_R(u_{10}+v_{01})(1+\beta_C)^2\omega_R\arccos\left(\frac{\beta_C-1}{\beta_C+1}\right)+b_R(p_{10}+q_{01})(4\beta_C\\
&\quad -\tau_R^2)\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right)\Bigg),\\
D_1 &= \frac{4b_R\sqrt[4]{\beta_C}}{b_L}\Bigg(r_{10}-r_{00}+b_L(u_{00}-u_{10})\Bigg),\\
C_{10} &= 2\Bigg(p_{00}+p_{10}-b_R(u_{00}+u_{10})+\frac{b_R\sqrt{\beta_C}}{\omega_R}\Bigg(2(u_{10}+v_{01})\omega_R\arccos\left(\frac{\beta_C-1}{\beta_C+1}\right)\\
&\quad +(p_{10}+q_{01})\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right)\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
D_3 &= \frac{1}{6b_L\sqrt[4]{\beta_C}(1+\beta_C)\omega_L\tau_L}\Bigg(-3b_R(r_{00}-r_{10})(1+\beta_C)\omega_L\tau_L+b_Lb_R(32(r_{10}+s_{01})\beta_C\\
&\quad (1+\beta_C)+(-3u_{10}-35u_{10}\beta_C-32v_{01}\beta_C+3u_{00}(1+\beta_C))\omega_L\tau_L)\Bigg),\\
C_{20} &= b_R(u_{10}+v_{01})\arccos\left(\frac{\beta_C-1}{\beta_C+1}\right)+\frac{b_R}{2\omega_R}\Bigg(\frac{4\sqrt{\beta_C}}{(1+\beta_C)^2(4\beta_C-\tau_R^2)}\Bigg(4(u_{10}+v_{01})\\
&\quad (\beta_C-1)\beta_C\omega_R-(p_{10}+q_{01})(1+\beta_C)^2\tau_R-(u_{10}+v_{01})(\beta_C-1)\omega_R\tau_R^2\Bigg)\\
&\quad +(p_{10}+q_{01})\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right)\Bigg),\\
C_{01} &= \frac{1}{2}\Bigg(8(p_{00}+p_{10})\sqrt{\beta_C}-4b_R\sqrt{\beta_C}(2u_{00}+u_{10}-v_{01}+(u_{10}+v_{01})\beta_C)+\frac{4b_R\sqrt{\beta_C}\tau_R}{\omega_R}\\
&\quad \Bigg(p_{10}+q_{01}\Bigg)+b_R(u_{10}+v_{01})(1+\beta_C)^2\Bigg(2\pi-\arccos\left(\frac{1-6\beta_C+\beta_C^2}{(1+\beta_C)^2}\right)\Bigg)+\frac{b_R}{\omega_R}\\
&\quad (p_{10}+q_{01})(2\sqrt{\beta_C}-\tau_R)(2\sqrt{\beta_C}+\tau_R)\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right)\Bigg),\\
C_{11} &= 2(p_{00}+p_{10})-b_R(2u_{00}+u_{10}-v_{01}+(u_{10}+v_{01})\beta_C)+\frac{b_R\tau_R}{\omega_R}\Bigg(p_{10}+q_{01}\Bigg)\\
&\quad +b_R(u_{10}+v_{01})\Bigg(-\sqrt{(\beta_L-1)^2}+2\sqrt{\beta_C}\Bigg(2\pi-\arccos\left(\frac{1-6\beta_C+\beta_C^2}{(1+\beta_C)^2}\right)\Bigg)\Bigg)\\
&\quad -\frac{b_R}{\omega_R}\Bigg(p_{10}+q_{01}\Bigg)\Bigg(\tau_R-2\sqrt{\beta_C}\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right)\Bigg),\\
C_{21} &= b_R\pi(u_{10}+v_{01})+2b_R\sqrt{\beta_C}\Bigg(\frac{\tau_R}{\omega_R(\tau_R^2-4\beta_C)}(p_{10}+q_{01})-\frac{\sqrt{(\beta_C-1)^2}}{(\beta_C^2+1)}(u_{10}+v_{01})\Bigg)\\
&\quad -\frac{b_R}{2}(u_{10}+v_{01})\arccos\left(\frac{1-6\beta_C+\beta_C^2}{(1+\beta_C)^2}\right)+\frac{b_R}{2\omega_R}(p_{10}+q_{01})\log\left(-\frac{2\sqrt{\beta_C}+\tau_R}{2\sqrt{\beta_C}-\tau_R}\right).
\end{aligned}
$$

Further, we have that

$$
\begin{aligned}
C_{00}-C_{01} &= -\frac{b_R}{2}\Big(u_{10}+v_{01}\Big)(1+\beta_C)^2\,\Phi(\beta_C),\\
C_{10}-C_{11} &= \begin{cases} -b_R(u_{10}+v_{01})\sqrt{\beta_C}\,\Phi(\beta_C), & \text{if} \quad 0<\beta_C\le 1,\\[2ex] 2b_R(u_{10}+v_{01})(\beta_C-1-\sqrt{\beta_C}\Phi(\beta_C)), & \text{if} \quad \beta_C>1,\end{cases}\\
C_{20}-C_{21} &= \begin{cases} -\dfrac{b_R}{2}\Big(u_{10}+v_{01}\Big)\,\Phi(\beta_C), \quad \text{if} \quad 0<\beta_C\le 1,\\[2ex] \dfrac{b_R}{2(\beta_C+1)^2}\Big(u_{10}+v_{01}\Big)\Big(8(\beta_C-1)\sqrt{\beta_C}-(\beta_C+1)^2\Phi(\beta_C)\Big),\\[2ex] \text{if} \quad \beta_C>1,\end{cases}
\end{aligned}
\tag{5.24}
$$

with

$$
\Phi(\beta_C)=2\pi-2\arccos\left(\frac{\beta_C-1}{\beta_C+1}\right)-\arccos\left(\frac{1-6\beta_C+\beta_C^2}{(1+\beta_C)^2}\right).
$$

In this way, we divide the proof of Theorem 13 in two cases. The first one is when $0<\beta_C\le 1$ (i.e., the central subsystem has a real or boundary center) and the second one is when $\beta_C>1$ (i.e., the central subsystem has a virtual center).

Case $0<\beta_C\le 1$: In this case we have that $\Phi(1)=0$ and, after some computation, it is possible to show that $\Phi'(\beta_C)=0$. Therefore, $\Phi(\beta_C)\equiv 0$ for $\beta_C\in(0,1]$ and, by equations (5.24), $C_{00}=C_{01}$, $C_{10}=C_{11}$ and $C_{20}=C_{21}$, i.e we can rewrite equation (5.23) as

$$
M_1(h)=C_{00}+C_{10}\big(h-2\sqrt{\beta_C}\big)+C_{20}\big(h-2\sqrt{\beta_C}\big)^2+\mathcal{O}\big(\big(h-2\sqrt{\beta_C}\big)^3\big).
$$

Moreover, when

$$
p_{00}=q_{01}+b_R(u_{00}-v_{01}), \quad r_{10}=-s_{01}, \quad u_{10}=-v_{01},
$$
$$
r_{00}=-s_{01}+b_L(u_{00}+v_{01}) \quad \text{and} \quad p_{10}=-q_{01},
$$

we have that

$$
C_{00}=D_1=C_{10}=D_3=C_{20}=0 \quad \text{with} \quad \operatorname{rank}\frac{\partial(C_{00},D_1,C_{10},D_3,C_{20})}{\partial(p_{00},r_{10},u_{10},r_{00},p_{10})}=5.
$$

This implies, by Proposition 7, that C_{00}, D_1, C_{10}, D_3 and C_{20} can be taken as free coefficients, satisfying

$$
C_{00}D_1<0, \quad D_1C_{10}<0, \quad C_{10}D_3<0, \quad D_3C_{20}<0
$$

and

$$|C_{00}| \ll |D_1| \ll |C_{10}| \ll |D_3| \ll |C_{20}|,$$

such that functions (5.22) and (5.23) have at least four and two positive simple zeros near $h = 2\sqrt{\beta_C}$, respectively.

Now, when

$$\begin{aligned}
p_{00} = {} & \frac{1}{2\omega_R\lambda_1}\Bigg(4\sqrt{\beta_C}((p_{10} + b_R(v_{01} - u_{00}))(\beta_C - 1)\omega_R - b_R(p_{10} + q_{01})\tau_R) - 2((p_{10} + b_R \\
& (v_{01} - u_{00}))(1 + (\beta_C - 6)\beta_C)\omega_R - 4b_R(p_{10} + q_{01})\beta_C\tau_R) \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right) - b_R \\
& (p_{10} + q_{01})\Bigg(- 4\beta_C^2 - \tau_R^2 + 2\sqrt{\beta_C}((\beta_C - 1)^2 + \tau_R^2) \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right)\Bigg), \\
& \log\left(\frac{2\sqrt{\beta_C} + \tau_R}{\tau_R - 2\sqrt{\beta_C}}\right)\Bigg), \\
r_{10} = {} & -\frac{1}{2\omega_R\lambda_1(\beta_C + 1)}\Bigg(4\sqrt{\beta_C}(s_{01}(\beta_C^2 - 1)\omega_R - (p_{10} + q_{01})\omega_L\tau_L\tau_R) - 2s_{01}(1 + \beta_C)(1 + (\beta_C \\
& - 6)\beta_C)\omega_R \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right) + (p_{10} + q_{01})\omega_L\tau_L(4\beta_C + \tau_R^2) \log\left(\frac{2\sqrt{\beta_C} + \tau_R}{\tau_R - 2\sqrt{\beta_C}}\right)\Bigg),
\end{aligned}$$

$$\begin{aligned}
u_{10} = {} & -\frac{1}{2\omega_R\lambda_1}\Bigg(4\sqrt{\beta_C}(v_{01}(\beta_C - 1)\omega_R - (p_{10} + q_{01})\tau_R) - 2v_{01}(1 + (\beta_C - 6)\beta_C)\omega_R \\
& \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right) + (p_{10} + q_{01})(4\beta_C + \tau_R^2) \log\left(\frac{2\sqrt{\beta_C} + \tau_R}{\tau_R - 2\sqrt{\beta_C}}\right)\Bigg), \\
r_{00} = {} & \frac{1}{2\omega_R\lambda_1(\beta_C + 1)}\Bigg(4\sqrt{\beta_C}(-(s_{01} - b_L(u_{00} + v_{01}))(\beta_C^2 - 1)\omega_R - (p_{10} + q_{01})(b_L + b_L\beta_C \\
& - \omega_L\tau_L)\tau_R) + 2(s_{01} - b_L(u_{00} + v_{01}))(1 + \beta_C)(1 + (\beta_C - 6)\beta_C)\omega_R \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right) \\
& + (p_{10} + q_{01})(b_L + b_L\beta_C - \omega_L\tau_L)(4\beta_C + \tau_R^2) \log\left(\frac{2\sqrt{\beta_C} + \tau_R}{\tau_R - 2\sqrt{\beta_C}}\right)\Bigg),
\end{aligned}$$

with

$$\lambda_1 = 2(\beta_C - 1)\sqrt{\beta_C} - (1 + (\beta_C - 6)\beta_C) \arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right),$$

we have that $C_{00} = D_1 = C_{10} = D_3 = 0$,

$$C_{20} = -\frac{b_C(p_{10} + q_{01})\lambda_2}{2\omega_R(1 + \beta_C)^2(4\beta_C - \tau_R^2)\lambda_1} \quad \text{and} \quad \operatorname{rank}\frac{\partial(C_{00}, D_1, C_{10}, D_3, C_{20})}{\partial(p_{00}, r_{10}, u_{10}, r_{00}, p_{10})} = 5,$$

with

$$\lambda_2 = 8(\beta_C - 1)\beta_C\tau_R((\beta_C - 1)^2 + \tau_R^2) - 4\sqrt{\beta_C}(1+\beta_C)^2(\beta_C - 1 - \tau_R)(\beta_C + \tau_R - 1)\tau_R$$
$$\arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right) + (4\beta_C - \tau_R^2)\Bigg(- 2(\ \beta_C - 1)\sqrt{\beta_C}((\beta_C - 1)^2 - \tau_R^2) + (1+\beta_C)^2$$
$$((\beta_C - 1)^2 + \tau_R^2)\arccos\left(\frac{\beta_C - 1}{\beta_C + 1}\right)\log\left(\frac{2\sqrt{\beta_C} + \tau_R}{\tau_R - 2\sqrt{\beta_C}}\right)\Bigg).$$

It is possible to show that the equation $\lambda_1 = 0$ has a unique zero in the interval $(0, 1]$. Using the Newton's method we obtain that the zero is $\beta_C \approx 0.23031022687$. Moreover, there are parameter values for system (5.1) such that $\lambda_2 \neq 0$. For example, if $a_L = b_L = b_R = c_R = 1$, $c_L = a_R = 0$, $\beta_R = -4$, $\beta_L \geq \sqrt{9 - 4\beta_C}$ and $\beta_C \in \{1/2, 1\}$, we have that the left subsystem from $(5.1)|_{\epsilon=0}$ has a saddle at the point $(-1-\beta_L, \beta_L)$, the right subsystem from $(5.1)|_{\epsilon=0}$ has a saddle at the point $(4, 0)$, $\lambda_2 \approx 223.526$ when $\beta_C = 1$ and $\lambda_2 \approx 23.0517$ when $\beta_C = 1/2$. Hence, for $p_{10} \neq -q_{01}$, we have that $C_{20} \neq 0$. This implies, by Proposition 7, that we can take C_{00}, D_1, C_{10}, D_3 and C_{20} as free coefficients, such that function (5.22) has at least four positive simple zeros near $h = 2\sqrt{\beta_C}$ and function (5.23) has exactly two positive zeros near $h = 2\sqrt{\beta_C}$. Therefore, the maximum number of limit cycles from system (5.1) that can bifurcate of the periodic annulus near $h = 2\sqrt{\beta_C}$ when $0 < \beta_C \leq 1$ is at least six.

Case $\beta_C > 1$: In this case, if we take $v_{01} = -u_{10}$ in (5.24), we obtain $C_{00} = C_{01}$, $C_{10} = C_{11}$ and $C_{20} = C_{21}$. Moreover, when

$$p_{00} = q_{01} + b_R(u_{00} + u_{10}), \quad r_{10} = r_{00} + b_L(u_{10} - u_{00}),$$
$$p_{10} = -q_{01} \quad \text{and} \quad u_{10} = -\frac{1}{b_L}(r_{00} + s_{01}) + u_{00},$$

we have that

$$C_{00} = D_1 = C_{10} = D_3 = 0 \quad \text{with} \quad \text{rank}\frac{\partial(C_{00}, D_1, C_{10}, D_3)}{\partial(p_{00}, r_{10}, p_{10}, u_{10})} = 4.$$

This implies, by Proposition 7, that we can take C_{00}, D_1, C_{10} and D_3 as free coefficients, such that functions (5.22) and (5.23) have at least three and one positive simple zeros near $h = 2\sqrt{\beta_C}$, respectively. Therefore, the number of limit cycles from system (5.1) that can bifurcate of the periodic annulus near $h = 2\sqrt{\beta_C}$ when $\beta_C > 1$ is at least four. □

Chapter

6

Persistence of periodic solutions from system $(1)|_{\epsilon=0}$ having a saddle in the central region

In [61], Zhang *et al.* showed that at least five limit cycles can bifurcate simultaneously of three periodic annuli, by linear perturbations, from a piecewise linear Hamiltonian differential system, symmetrical with respect to the origin, of the type CSC. Motivated by this result, in this chapter we estimate the number of limit cycles that can bifurcate from a periodic annulus, assuming that the central subsystem has a real saddle and the other subsystems have centers or saddles, without imposing restrictive hypotheses like symmetries. More precisely, the lower bound for the number of limit cycles that can bifurcate from the periodic annulus of this kind of piecewise systems, by linear perturbations, is at least six.

6.1 Preliminaries and main result

For this purpose, let us consider a discontinuous piecewise linear near-Hamiltonian differential systems with three zones, given by

$$\begin{cases} \dot{x} = H_y(x, y) + \epsilon f(x, y), \\ \dot{y} = -H_x(x, y) + \epsilon g(x, y), \end{cases} \tag{6.1}$$

with

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, & x \leq -1, \\ H^C(x,y) = \dfrac{b_C}{2}y^2 - \dfrac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, & -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, & x \geq 1, \end{cases}$$

$$f(x,y) = \begin{cases} f_L(x,y) = r_{10}x + r_{01}y + r_{00}, & x \leq -1, \\ f_C(x,y) = u_{10}x + u_{01}y + u_{00}, & -1 \leq x \leq 1, \\ f_R(x,y) = p_{10}x + p_{01}y + p_{00}, & x \geq 1, \end{cases} \tag{6.2}$$

$$g(x,y) = \begin{cases} g_L(x,y) = s_{10}x + s_{01}y + s_{00}, & x \leq -1, \\ g_C(x,y) = v_{10}x + v_{01}y + v_{00}, & -1 \leq x \leq 1, \\ g_R(x,y) = q_{10}x + q_{01}y + q_{00}, & x \geq 1. \end{cases} \tag{6.3}$$

In addition, let us assume that system $(6.1)|_{\epsilon=0}$ satisfies the following hypotheses:

(H1) The unperturbed central subsystem from $(6.1)|_{\epsilon=0}$ has a real saddle and the other unperturbed subsystems from $(6.1)|_{\epsilon=0}$ have centers or saddles.

(H2) The unperturbed system $(6.1)|_{\epsilon=0}$ has only crossing points on the straight lines $x = \pm 1$, except by some tangent points.

(H3) The unperturbed system $(6.1)|_{\epsilon=0}$ has three periodic annuli, one consisting of a family of crossing periodic orbits passing through three zones and the other two consisting of families of crossing periodic orbits passing through two zones, such that each orbit of these families has clockwise orientation (see Figure 6.1).

By hypothesis (H1), we can classify system $(6.1)|_{\epsilon=0}$ according to the configuration of their singular points (without taking into account if the singular points are real or virtual). More precisely, we have the following three classes of piecewise linear Hamiltonian systems: SSS, CSS and CSC.

We will use the vector $(k, l, m) \in \mathbb{Z}^3$ to denote the limit cycle configurations of system (6.1), where its components k, l and m indicate the number of limit cycles passing through three zones, the number of limit cycles passing through two zones and surrounding the tangent point of the central subsystem with the switching line $x = 1$ and the number of limit cycles surrounding the tangent point of the central subsystem with the switching line $x = -1$, respectively.

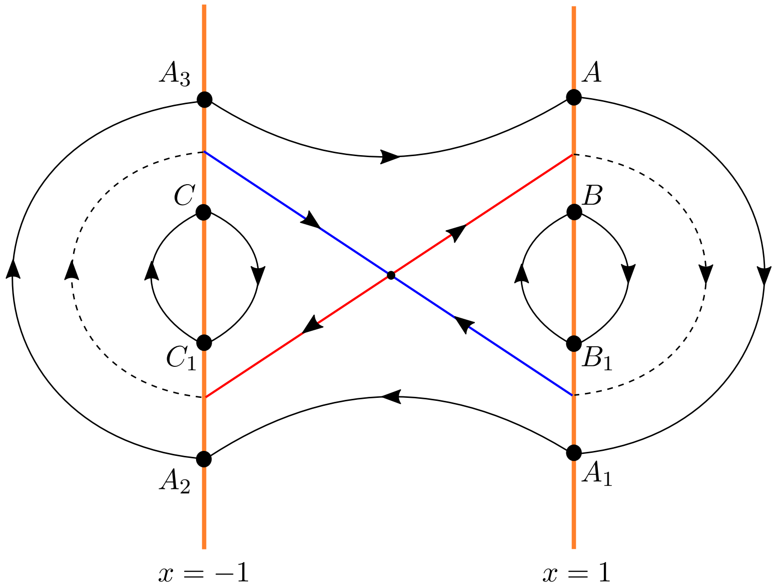


Figure 6.1: The crossing periodic orbits of system $(6.1)|_{\epsilon=0}$.

Thus, the main result of this chapter is the following.

Theorem 15. *The lower bound for the limit cycles of system* (6.1)*, satisfying hypotheses* (Hi)*, for* $i = 1, 2, 3$*, which can bifurcate simultaneously from the periodic annuli of the unperturbed system* $(6.1)|_{\epsilon=0}$ *is at least five, in the case SSS, and at least six, in the cases CSS and CSC. Moreover, the following configurations of limit cycles are achievable:* $(2, 2, 1)$ *or* $(2, 1, 2)$ *for the case SSS and* $(2, 2, 2)$ *for the cases CSS and CSC.*

Theorem 15 gives us a larger lower bound for the number of limit cycles than the one obtained in the paper [61]. We would like to point out that in [61] the authors perturb a particular system which is symmetrical with respect to the origin. In this chapter, we work with perturbations of a piecewise linear systems determined by the hypotheses (Hi), for $i = 1, 2, 3$, without imposing restrictive hypotheses like symmetries. Furthermore, in [61] the authors only deal in detail the Melnikov function associated with the right subsystem and do not explain clearly how to ensure the simultaneous occurrence of zeros in the three Melnikov functions involved in the problem. Thus, as in the previous chapter, we provide a detailed analytical method to study the number of simultaneous zeros from Melnikov functions defined in two or three zones (see Proposition 10). It is worth noting that Theorem 15 also deals with cases not yet addressed in the literature, i.e., the cases CSS and SSS.

6.2 A normal form to system $(6.1)|_{\epsilon=0}$

As in Chapter 3, we will do a continuous linear change of variables that keeps invariant the straight lines $x = \pm 1$, in order to reduce the number of parameters of system $(6.1)|_{\epsilon=0}$. More precisely, following the steps of the proof of Proposition 1, and doing the obvious adaptations, we have the following result.

Proposition 8. *Suppose that the central subsystem from* $(6.1)|_{\epsilon=0}$ *has a saddle and the other two subsystems have saddles or centers. Moreover, suppose also that system* $(6.1)|_{\epsilon=0}$ *satisfies the hypothesis* (H2)*. Then, after a linear change of variables and a rescaling of the independent variable, we can written system* $(6.1)|_{\epsilon=0}$ *with*

$$H(x,y) = \begin{cases} H^L(x,y) = \dfrac{b_L}{2}y^2 - \dfrac{c_L}{2}x^2 + a_Lxy + a_Ly - \beta_Lx, & x \leq -1, \\ H^C(x,y) = \dfrac{y^2}{2} - \dfrac{x^2}{2} - \beta_Cx, & -1 \leq x \leq 1, \\ H^R(x,y) = \dfrac{b_R}{2}y^2 - \dfrac{c_R}{2}x^2 + a_Rxy - a_Ry - \beta_Rx, & x \geq 1. \end{cases} \tag{6.4}$$

Remark 2. *In what follows, we will consider system* (6.1) *with* $f(x,y)$ *and* $g(x,y)$ *given by* (6.2) *and* (6.3)*, respectively, and* $H(x,y)$ *given by* (6.4) *with* $\beta_C = 0$*. This extra restriction implies that the singular point of the central subsystem from* $(6.1)|_{\epsilon=0}$ *is at the origin and it simplifies the expressions of the Melnikov functions, which becomes the computation more easy. Moreover, this normal form for system* $(3.1)|_{\epsilon=0}$*, with* $\beta_C = 0$*, still satisfies the hypotesis (Hi),* $i = 1,2,3$*, and so we can use it in the proof of Theorem 15.*

6.3 A classification to system $(6.1)|_{\epsilon=0}$

As in previous chapters, to study the zeros of the first order Melnikov functions associated with system (6.1), that are expressed by (see Theorem 1)

$$\begin{aligned} M_0(h) = {} & \frac{H_y^R(A)}{H_y^C(A)}I_C^0 + \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)}I_L^0 + \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)}\bar{I}_C^0 \\ & + \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)}I_R^0, \quad h \in J_0, \end{aligned} \tag{6.5}$$

$$M_1(h) = \frac{H_y^R(B)}{H_y^C(B)}I_C^1 + \frac{H_y^R(B)H_y^C(B_1)}{H_y^C(B)H_y^R(B_1)}I_R^1, \quad h \in J_1, \tag{6.6}$$

and

$$M_2(h) = \frac{H_y^C(C)}{H_y^L(C)}I_L^2 + \frac{H_y^C(C)H_y^L(C_1)}{H_y^L(C)H_y^CC_1)}I_C^2, \quad h \in J_2, \tag{6.7}$$

where

$$I_C^0 = \int_{\widehat{A_3A}} g_Cdx - f_Cdy, \quad I_L^0 = \int_{\widehat{A_2A_3}} g_Ldx - f_Ldy, \quad \bar{I}_C^0 = \int_{\widehat{A_1A_2}} g_Cdx - f_Cdy,$$

$$I_R^0 = \int_{\widehat{AA_1}} g_R dx - f_R dy, \quad I_C^1 = \int_{\widehat{B_1B}} g_C dx - f_C dy, \quad I_R^1 = \int_{\widehat{BB_1}} g_R dx - f_R dy,$$

$$I_L^2 = \int_{\widehat{C_1C}} g_L dx - f_L dy \quad \text{and} \quad I_C^2 = \int_{\widehat{CC_1}} g_C dx - f_C dy,$$

it is necessary to find the open intervals J_i, $i = 0, 1, 2$, where (6.5)-(6.7) are defined. For this, consider the following proposition.

Proposition 9. *Consider system* (6.1) *assuming the hypotheses (Hi),* $i = 1, 2, 3$.

(a) *If system* $(6.1)|_{\epsilon=0}$ *is of the type SSS or CSS, then* $J_0 = (1, \tau_{RS})$ *and* $J_1 = J_2 = (0, 1)$, *where* $\tau_{RS} = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/(b_R\omega_{RS})$ *with* $\omega_{RS} = \sqrt{a_R^2 + b_R c_R}$, *and the periodic annuli are equivalent to one of Figure 6.2–6.5.*

(b) *If system* $(6.1)|_{\epsilon=0}$ *is of the type CSC, then* $J_0 = (1, \infty)$ *and* $J_1 = J_2 = (0, 1)$, *and the periodic annuli are equivalent to one of Figure 6.6.*

Proof. Firstly, suppose that system $(6.1)|_{\epsilon=0}$ is of the type SSS or CSS. Note that, if the saddles from the right or left subsystems are virtual or boundary then we do not have periodic orbits passing through the three zones. Denote by W_L^u, W_C^u and W_R^u (resp. W_L^s, W_C^s and W_R^s) the unstable (resp. stable) separatrices of the saddles of the left, central and right subsystems from $(6.1)|_{\epsilon=0}$, respectively. Let $P_L^i = W_L^i \cap \{(-1, y) : y \in \mathbb{R}\}$, $P_R^i = W_R^i \cap \{(1, y) : y \in \mathbb{R}\}$, $P_C^i = W_C^i \cap \{(-1, y) : y \in \mathbb{R}\}$ and $\tilde{P}_C^i = W_C^i \cap \{(1, y) : y \in \mathbb{R}\}$, $i = u, s$. After some computation, it is possible to show that

$$P_L^u = (-1, \tau_{LS}), \quad P_L^s = (-1, -\tau_{LS}), \quad P_R^u = (1, -\tau_{RS}), \quad P_R^s = (1, \tau_{RS}),$$

$$P_C^u = (-1, -1), \quad P_C^s = (-1, 1), \quad \tilde{P}_C^u = (1, 1), \quad \tilde{P}_C^s = (1, -1),$$

where $\tau_{RS} = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/b_R\omega_{RS}$, $\tau_{LS} = (a_L^2 + b_L\beta_L - \omega_{LS}^2)/b_L\omega_{LS}$, $\omega_{RS} = \sqrt{a_R^2 + b_R c_R}$ and $\omega_{LS} = \sqrt{a_L^2 + b_L c_L}$. Moreover, we have a symmetry between the points P_L^u and P_L^s (resp. P_R^u and P_R^s) with respect to the x-axis. Define by τ the smallest ordinate value between the points P_R^s and P_L^u, i.e., $\tau = \min\{\tau_{LS}, \tau_{RS}\}$. Then, by a reflection around the y-axis if necessary, we can assume that $\tau = \tau_{RS}$.

Note that, if system $(6.1)|_{\epsilon=0}$ is of the type SSS and the ordinates of the points P_R^s and P_L^u are distinct, i.e., $\tau_{RS} \neq \tau_{LS}$ (see Figure 6.2) or if system $(6.1)|_{\epsilon=0}$ is of the type CSS (see Figure 6.4–6.5), then we have a homoclinic loop passing through the points P_R^s and P_R^u. Otherwise, if system $(6.1)|_{\epsilon=0}$ is of the type SSS and the points P_R^s and P_L^u have the same ordinate, i.e., $\tau_{RS} = \tau_{LS}$ (see Figure 6.3), then we have heteroclinic orbits passing through the points P_R^s, P_R^u, P_L^s and P_L^u. Moreover, the central subsystem from $(6.1)|_{\epsilon=0}$ has two tangent points $P_R = (1, 0)$

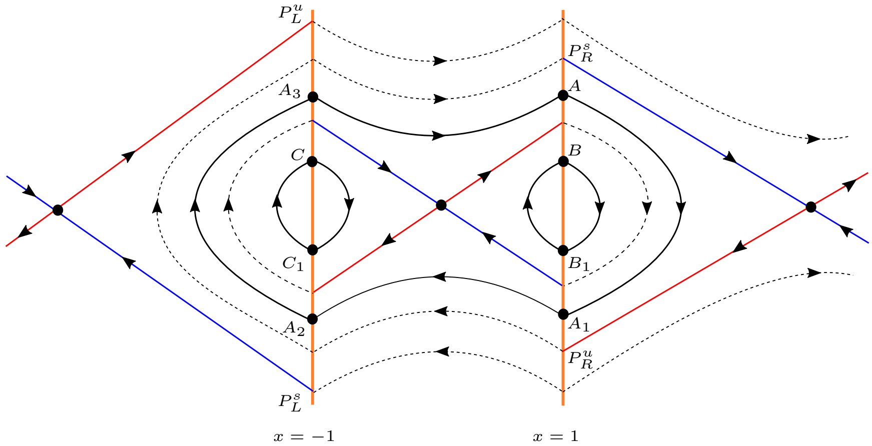


Figure 6.2: Phase portrait of system $(6.1)|_{\epsilon=0}$ of the type SSS with $\tau_{RS} \neq \tau_{LS}$.

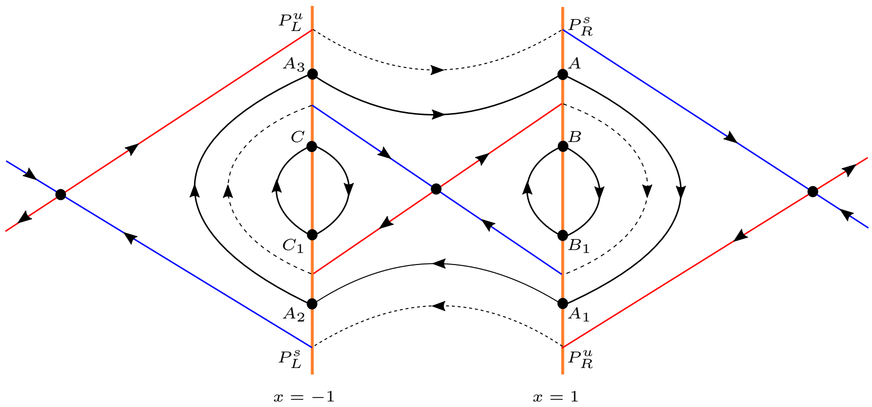


Figure 6.3: Phase portrait of system $(6.1)|_{\epsilon=0}$ of the type SSS with $\tau_{RS} = \tau_{LS}$.

and $P_L = (-1, 0)$. Figures 6.2–6.5 show the possible phase portraits of system $(6.1)|_{\epsilon=0}$ of the type SSS and CSS.

Consider an initial point of the form $A(h) = (1, h)$, with $h \in J_0 = (1, \tau_{RS})$. By the hypothesis (H3), system $(6.1)|_{\epsilon=0}$ has a family of crossing periodic orbits that intersect the straight lines $x = \pm 1$ at four points, $A(h)$, $A_1(h) = (1, a_1(h))$, with $a_1(h) < h$, $A_2(h) = (-1, a_2(h))$ and $A_3(h) = (-1, a_3(h))$, with $a_2(h) < a_3(h)$, satisfying

$$
\begin{aligned}
H^R(A(h)) &= H^R(A_1(h)), \\
H^C(A_1(h)) &= H^C(A_2(h)), \\
H^L(A_2(h)) &= H^L(A_3(h)), \\
H^C(A_3(h)) &= H^C(A(h)),
\end{aligned}
$$

where H^R, H^C and H^L are given by the normal form in Proposition 8. More precisely, we have

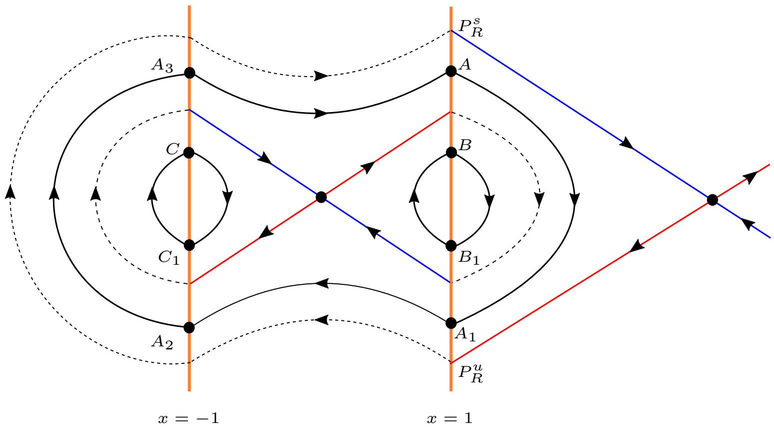


Figure 6.4: Phase portrait of system $(6.1)|_{\epsilon=0}$ of the type CSS with a virtual center in the left subsystem.

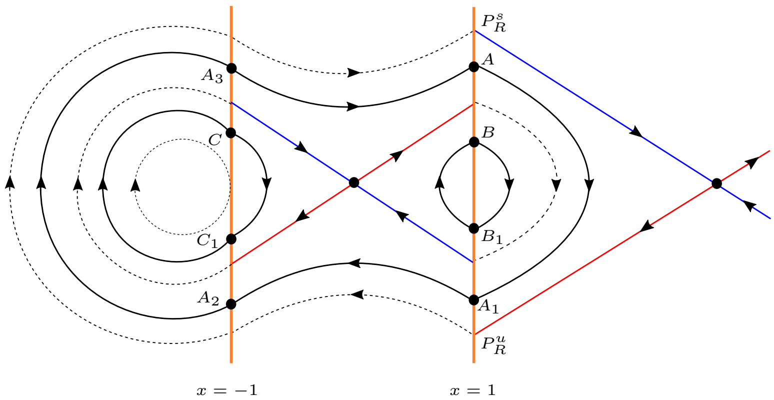


Figure 6.5: Phase portrait of system $(6.1)|_{\epsilon=0}$ of the type CSS with a real center in the left subsystem.

the equations

$$\begin{aligned}
&\frac{b_R}{2}(h - a_1(h))(h + a_1(h)) = 0,\\
&\frac{1}{2}(a_1(h) - a_2(h))(a_1(h) + a_2(h)) = 0,\\
&\frac{b_L}{2}(a_2(h) - a_3(h))(a_2(h) + a_3(h)) = 0,\\
&\frac{1}{2}(a_3(h) - h)(a_3(h) + h) = 0.
\end{aligned}$$

As $a_1(h) < h$, $a_2(h) < a_3(h)$, $b_R > 0$ and $b_L > 0$, the only solution of the above system is $a_1(h) = a_2(h) = -h$ and $a_3(h) = h$, i.e., we have the four points given by $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Moreover, for each $h \in J_0$, system $(6.1)|_{\epsilon=0}$ has a periodic orbit $L_h^0 = \widehat{AA_1} \cup \widehat{A_1A_2} \cup \widehat{A_2A_3} \cup \widehat{A_3A}$ passing through these points.

Similarly to the previous case, consider two initial points of the form $B(h) = (1, h)$, with $h \in J_1 = (0, 1)$, and $C(h) = (-1, h)$, with $h \in J_2 = (0, 1)$. By hypothesis (H3), system $(6.1)|_{\epsilon=0}$ has two families of crossing periodic orbits passing through the two zones with clockwise orientation. The first one, intersect the straight line $x = 1$ at two points, $B(h)$ and $B_1(h) = (1, b_1(h))$, with $b_1(h) < h$. The second one, intersect the straight line $x = -1$ at two points, $C(h)$ and $C_1(h) = (-1, c_1(h))$, with $c_1(h) < h$. Moreover, these families of crossing periodic orbits satisfy the following equations

$$H^R(B(h)) = H^R(B_1(h)),$$
$$H^C(B_1(h)) = H^C(B(h)),$$
$$H^C(C(h)) = H^C(C_1(h)),$$
$$H^L(C_1(h)) = H^L(C(h)).$$

More precisely, we have the equations

$$\begin{aligned}
&\frac{b_R}{2}(h - b_1(h))(h + b_1(h)) = 0,\\
&\frac{1}{2}(b_1(h) - h)(b_1(h) + h) = 0,\\
&\frac{1}{2}(h - c_1(h))(h + c_1(h)) = 0,\\
&\frac{b_L}{2}(c_1(h) - h)(c_1(h) + h) = 0.
\end{aligned}$$

As $b_1(h) < h$, $c_1(h) < h$, $b_R > 0$ and $b_L > 0$, the only solution of the above system is $b_1(h) = c_1(h) = -h$, i.e., we have the four points given by $B(h) = (1, h)$, $B_1(h) = (1, -h)$, $C(h) = (-1, h)$ and $C_1(h) = (-1, -h)$. Moreover, system $(6.1)|_{\epsilon=0}$ has two periodic orbits $L_h^1 = \widehat{BB_1} \cup \widehat{B_1B}$, with $h \in J_1$, and $L_h^2 = \widehat{CC_1} \cup \widehat{C_1C}$, with $h \in J_2$, passing through these points. If $h \in [\tau_{RS}, \infty)$ then the orbit of system $(6.1)|_{\epsilon=0}$ with initial condition at $A(h)$ does not return to the straight line $x = 1$ for positive times, i.e., system $(6.1)|_{\epsilon=0}$ has no periodic orbit passing through the point $A(h)$. Therefore, if $h \in (1, \tau_{RS})$ system $(6.1)|_{\epsilon=0}$ has a periodic annulus, formed by the periodic orbits L_h^0, bounded by one double homoclinic loop at $h = 1$ and either a homoclinic loop (see Figures 6.2–6.5) or two heteroclinic orbits (see Figure 6.3) at $h = \tau_{RS}$. Now, if $h \in (0, 1)$ system $(6.1)|_{\epsilon=0}$ has two periodic annuli, formed by the periodic orbits L_h^i, $i = 1, 2$, and bounded by homoclinic loops. Therefore, item (a) is proved.

To prove item (b), suppose that system $(6.1)|_{\epsilon=0}$ is of the type CSC. As in the previous cases, for each $h \in (1, \infty)$ we have a periodic orbit L_h^0 passing through the points $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$ and $A_3(h) = (-1, h)$. Therefore, system $(6.1)|_{\epsilon=0}$ has a continuum of periodic orbits formed by the periodic orbits L_h^0, with $h \in (1, \infty)$, bounded by one double homoclinic loop (see Fig. 6.6 (a)–(c)). Now, if $h \in (0, 1)$ system $(6.1)|_{\epsilon=0}$ has two periodic annuli, formed by the periodic orbits L_h^i, $i = 1, 2$, with L_h^1 passing through the points $B(h) = (1, h)$, $B_1(h) = (1, -h)$ and L_h^2 passing through the points $C(h) = (-1, h)$, $C_1(h) = (-1, -h)$, and bounded by the homoclinic loops. Figure 6.6 show the possible phase portraits of system $(6.1)|_{\epsilon=0}$ of the type CSC. $\square$

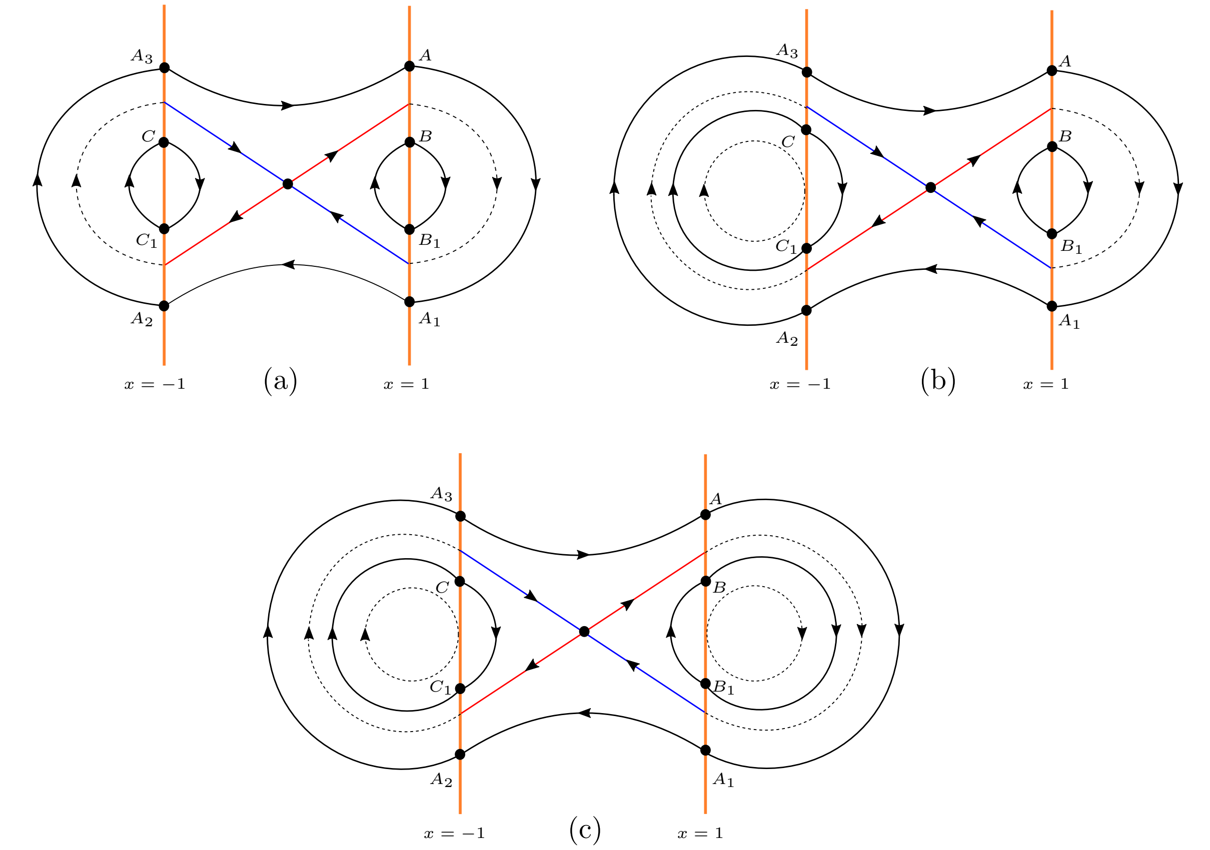


Figure 6.6: Phase portraits of system $(6.1)|_{\epsilon=0}$ of the type CSC when (a) left and right subsystems have virtual centers; (b) left subsystem has a real center and right subsystems has a virtual center; (c) left and right subsystems have real centers.

As, for $h \in J_0$, $A(h) = (1, h)$, $A_1(h) = (1, -h)$, $A_2(h) = (-1, -h)$, $A_3(h) = (-1, h)$ and, for $h \in J_i$, $i = 1, 2$, $B(h) = (1, h)$, $B_1(h) = (1, -h)$, $C(h) = (-1, h)$ and $C_1(h) = (-1, -h)$, then the coefficients that multiply the integrals of first order Melnikov functions (6.5), (6.6) and (6.7) associated with system (6.1) can be easily calculated. More precisely, we have the immediate corollary.

Corollary 11. *For $h \in J_0$, we have that*

$$\frac{H_y^R(A)}{H_y^C(A)} = \frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)} = b_R, \quad \frac{H_y^R(A)H_y^C(A_3)}{H_y^C(A)H_y^L(A_3)} = \frac{b_R}{b_L},$$

$$\frac{H_y^R(A)H_y^C(A_3)H_y^L(A_2)H_y^C(A_1)}{H_y^C(A)H_y^L(A_3)H_y^C(A_2)H_y^R(A_1)} = 1,$$

and for $h \in J_i$, $i = 1, 2$, we have that

$$\frac{H_y^R(B)}{H_y^C(B)} = b_R, \quad \frac{H_y^C(C)}{H_y^L(C)} = \frac{1}{b_L}, \quad \frac{H_y^R(B)H_y^C(B_1)}{H_y^C(B)H_y^R(B_1)} = \frac{H_y^C(C)H_y^L(C_1)}{H_y^L(C)H_y^C C_1)} = 1.$$

Then, the first order Melnikov functions associated to system (6.1) *can be rewritten as*

$$\begin{aligned} M_0(h) &= b_R \int_{\widehat{A_3A}} g_C dx - f_C dy + \frac{b_R}{b_L} \int_{\widehat{A_2A_3}} g_L dx - f_L dy \qquad (6.8)\\ &+ b_R \int_{\widehat{A_1A_2}} g_C dx - f_C dy + \int_{\widehat{AA_1}} g_R dx - f_R dy, \\ M_1(h) &= b_R \int_{\widehat{B_1B}} g_C dx - f_C dy + \int_{\widehat{BB_1}} g_R dx - f_R dy, \qquad (6.9)\\ M_2(h) &= \frac{1}{b_L} \int_{\widehat{C_1C}} g_L dx - f_L dy + \int_{\widehat{CC_1}} g_C dx - f_C dy. \qquad (6.10) \end{aligned}$$

6.4 Melnikov functions associated with system (6.1)

In order to prove Theorem 15, we will simplify the expressions of the first order Melnikov functions associated with system (6.1). For this, we define the following functions:

$$\begin{aligned} &f_0(h) = h, \quad h \in (0,\infty),\\ &f_C^0(h) = (h^2-1)\log\left(\frac{h+1}{h-1}\right), \quad h \in (1,\infty),\\ &f_C(h) = (h^2-1)\log\left(\frac{h+1}{1-h}\right), \quad h \in (0,1),\\ &f_{RC}(h) = (h^2+\tau_{RC}^2)\left(2\pi\mu_1 + (-1)^{\mu_1}\arccos\left(\frac{\tau_{RC}^2-h^2}{h^2+\tau_{RC}^2}\right)\right), \quad h \in (0,\infty), \qquad (6.11)\\ &f_{LC}(h) = (h^2+\tau_{LC}^2)\left(2\pi\mu_2 + (-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2-h^2}{h^2+\tau_{LC}^2}\right)\right), \quad h \in (0,\infty),\\ &f_{RS}(h) = (h^2-\tau_{RS}^2)\log\left(\frac{h+\tau_{RS}}{\tau_{RS}-h}\right), \quad h \in (0,1)\cup(1,\tau_{RS}),\\ &f_{LS}(h) = (h^2-\tau_{LS}^2)\log\left(\frac{h+\tau_{LS}}{\tau_{LS}-h}\right), \quad h \in (0,1)\cup(1,\tau_{RS}), \end{aligned}$$

with $\tau_{RC} = (a_R^2 - b_R\beta_R + \omega_{RC}^2)/b_R\omega_{RC}$, $\tau_{LC} = (a_L^2 + b_L\beta_L + \omega_{LC}^2)/b_L\omega_{LC}$, $\tau_{RS} = (a_R^2 - b_R\beta_R - \omega_{RS}^2)/b_R\omega_{RS}$, $\tau_{LS} = (a_L^2 + b_L\beta_L - \omega_{LS}^2)/b_L\omega_{LS}$, $\omega_{iC} = \sqrt{-a_i^2 - b_ic_i}$ and $\omega_{iS} = \sqrt{a_i^2 + b_ic_i}$, for $i = L, R$.

The following three theorems provide expressions for the first order Melnikov functions associated with system (6.1) when system $(6.1)|_{\epsilon=0}$ is of the type SSS, CSS and CSC. The proofs of these results are similar to the proof of Theorem 14, doing the obvious adaptations, and are presented in Appendix 7.3, 7.4 and 7.5, respectively, to simplify the text.

Theorem 16. *Suppose that system* $(6.1)|_{\epsilon=0}$ *is of the type SSS. Then, the first order Melnikov functions given by* (6.8), (6.9) *and* (6.10) *associated with system* (6.1) *can be expressed, when* *we have homoclinic loop, i.e.,* $\tau_{LS} \neq \tau_{RS}$*, as*

$$M_0(h) = k_0^0 f_0(h) + k_0^1 f_C^0(h) + k_0^2 f_{RS}(h) + k_0^3 f_{LS}(h), \quad h \in (1, \tau_{RS}), \tag{6.12}$$

$$M_1(h) = k_1^0 f_0(h) + k_1^1 f_C(h) + k_1^2 f_{RS}(h), \quad h \in (0, 1), \tag{6.13}$$

$$M_2(h) = k_2^0 f_0(h) + k_2^1 f_C(h) + k_2^2 f_{LS}(h), \quad h \in (0, 1), \tag{6.14}$$

and, when we have two heteroclinic orbits, i.e., $\tau_{LS} = \tau_{RS}$*, as*

$$M_0(h) = \tilde{k}_0^0 f_0(h) + \tilde{k}_0^1 f_C^0(h) + \tilde{k}_0^2 f_{RS}(h), \quad h \in (1, \tau_{RS}), \tag{6.15}$$

$$M_1(h) = \tilde{k}_1^0 f_0(h) + \tilde{k}_1^1 f_C(h) + \tilde{k}_1^2 f_{RS}(h), \quad h \in (0, 1), \tag{6.16}$$

$$M_2(h) = \tilde{k}_2^0 f_0(h) + \tilde{k}_2^1 f_C(h) + \tilde{k}_2^2 f_{RS}(h), \quad h \in (0, 1), \tag{6.17}$$

where the functions f_0*,* f_C^0*,* f_C*,* f_{RS} *and* f_{LS} *are the ones defined in* (6.11)*. Here the coefficients* k_i^j *and* $\tilde{k}_i^j$*, for* $i = 0, 1, 2$ *and* $j = 0, 1, 2, 3$*, depend on the parameters of system* (6.1)*.*

Theorem 17. *Suppose that system* $(6.1)|_{\epsilon=0}$ *is of the type CSS. Then, the first order Melnikov functions given by* (6.8)*,* (6.9) *and* (6.10) *associated with system* (6.1) *can be expressed as*

$$M_0(h) = k_0^0 f_0(h) + k_0^1 f_C^0(h) + k_0^2 f_{RS}(h) + k_0^3 f_{LC}(h), \quad h \in (1, \tau_{RS}), \tag{6.18}$$

$$M_1(h) = k_1^0 f_0(h) + k_1^1 f_C(h) + k_1^2 f_{RS}(h), \quad h \in (0, 1), \tag{6.19}$$

$$M_2(h) = k_2^0 f_0(h) + k_2^1 f_C(h) + k_2^2 f_{LC}(h), \quad h \in (0, 1), \tag{6.20}$$

where the functions f_0*,* f_C^0*,* f_C*,* f_{RS} *and* f_{LC} *are the ones defined in* (6.11)*, with* $\mu_2 = 0$ *(resp.* $\mu_2 = 1$*) if the left subsystem from* $(6.1)|_{\epsilon=0}$ *has a virtual (resp. real) center. Here the coefficients* k_i^j*, for* $i = 0, 1, 2$ *and* $j = 0, 1, 2, 3$*, depend on the parameters of system* (6.1)*.*

Theorem 18. *Suppose that system* $(6.1)|_{\epsilon=0}$ *is of the type CSC. Then, the first order Melnikov functions given by* (6.8)*,* (6.9) *and* (6.10) *associated with system* (6.1) *can be expressed as*

$$M_0(h) = k_0^0 f_0(h) + k_0^1 f_C^0(h) + k_0^2 f_{RC}(h) + k_0^3 f_{LC}(h), \quad h \in (1, \infty), \tag{6.21}$$

$$M_1(h) = k_1^0 f_0(h) + k_1^1 f_C(h) + k_1^2 f_{RC}(h), \quad h \in (0, 1), \tag{6.22}$$

$$M_2(h) = k_2^0 f_0(h) + k_2^1 f_C(h) + k_2^2 f_{LC}(h), \quad h \in (0, 1), \tag{6.23}$$

where the functions f_0*,* f_C^0*,* f_C*,* f_{RC} *and* f_{LC} *are the ones defined in* (6.11)*, with* $\mu_i = 0$ *(resp.* $\mu_i = 1$*),* $i = 1, 2$*, if the right or left subsystem from* $(6.1)|_{\epsilon=0}$ *has a virtual (resp. real) center.*

Here the coefficients k_i^j*, for* $i = 0, 1, 2$ *and* $j = 0, 1, 2, 3$*, depend on the parameters of system* (6.1)*.*

6.5 Proof of Theorem 15

In order to prove Theorem 15, we study the number of simultaneous zeros of the Melnikov functions M_0, M_1 and M_2 associated with system (6.1). Our study focuses on a neighborhood of the double homoclinic loop which separates the periodic annulus of orbits that pass through three zones from the two periodic annulus of orbits that pass through only two zones. Thus, to estimate the zeros of the Melnikov functions we consider its expansions at the point corresponding to this orbit. For this, we need of the following results.

Consider the real functions F, G, H given by

$$F(h) = \sum_{j=0}^{2} C_j^F(\delta)(h-\tau)^j + \mathcal{O}((h-\tau)^3), \text{ for } h \geq \tau, \tag{6.24}$$

$$G(h) = \sum_{j=0}^{2} C_j^G(\delta)(h-\tau)^j + \mathcal{O}((h-\tau)^3), \text{ for } h \leq \tau, \tag{6.25}$$

$$H(h) = \sum_{j=0}^{2} C_j^H(\delta)(h-\tau)^j + \mathcal{O}((h-\tau)^3), \text{ for } h \leq \tau, \tag{6.26}$$

such that $C_j^F = C_j^G + C_j^H$, $\tau \geq 0$ and the coefficients $C_j^i(\delta)$, $i = F, G, H$, $j = 0, \ldots, n$, are smooth functions depending on the parameters $\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. For each fixed δ, denote by $N_F(\delta)$, $N_G(\delta)$ and $N_H(\delta)$ the number of simple zeros of the function F, G and H, respectively. Then we have the following proposition.

Proposition 10. *Suppose that there exist* $\tilde{\delta} \in \mathbb{R}^m$*, with* $m \geq 5$*, such that*

$$\begin{gathered} C_j^G(\tilde{\delta}) = C_j^H(\tilde{\delta}) = 0, \quad j = 0, 1, 2, \\ \operatorname{rank} \frac{\partial(C_0^G, C_1^G, C_2^G, C_0^H, C_1^H)}{\partial(\delta_1, \ldots, \delta_5, \delta_6, \ldots, \delta_m)}(\tilde{\delta}) = 5, \end{gathered} \tag{6.27}$$

and

$$C_2^H = \alpha_0^G C_0^G + \alpha_1^G C_1^G + \alpha_2^G C_2^G + \alpha_0^H C_0^H + \alpha_1^H C_1^H, \tag{6.28}$$

where α_i^k*,* $i = 0, 1, 2$ *and* $k = G, H$*, are constants. If* $\alpha_2^G < 0$ *and* $\alpha_2^G \neq -1$ *(resp. either* $\alpha_2^G \geq 0$ *or* $\alpha_2^G = -1$*), then the* $\max\{N_F(\delta) + N_G(\delta) + N_H(\delta)\}$ *is at least six (resp. five) in a neighborhood of* $h = \tau$ *for all* δ *near* $\tilde{\delta}$*. Moreover, the following configurations* (N_F, N_G, N_H)*,* $N_F, N_G, N_H \in \{0, 1, 2\}$*, for the number of simple zeros, are achievable. In particular, when*

$\alpha_2^G < 0$ *and* $\alpha_2^G \neq -1$ *(resp. either* $\alpha_2^G = -1$ *or* $\alpha_2^G \geq 0$*), we can have* $(2,2,2)$ *(resp. either* $(1,2,2)$ *or* $(2,2,1)$ *or* $(2,1,2)$*).*

Proof. By the condition (6.27) we can assume that

$$\det \frac{\partial(C_0^G, C_1^G, C_2^G, C_0^H, C_1^H)}{\partial(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)}(\tilde{\delta}) \neq 0.$$

Then the change of parameters $\tilde{C}_j^G = C_j^G(\delta_1, \ldots, \delta_5, \tilde{\delta}_6, \ldots, \tilde{\delta}_m)$, $j = 0,1,2$, and $\tilde{C}_j^H = C_j^H(\delta_1, \ldots, \delta_5, \tilde{\delta}_6, \ldots, \tilde{\delta}_m)$, $j = 0,1$, has inverse $\delta_j(\tilde{C}_0^G, \tilde{C}_1^G, \tilde{C}_2^G, \tilde{C}_0^H, \tilde{C}_1^H)$, $j = 1, \ldots, 5$, defined in a neighborhood of the origin $0 \in \mathbb{R}^5$. Now to simplify the notation and becomes the proof more understandable, let us do the following change of parameters and variable $(C_0^G, C_1^G, C_2^G, C_0^H, C_1^H, h) \mapsto (\varepsilon^2 a, \varepsilon b, c, \varepsilon^2\alpha, \varepsilon\beta, \varepsilon x + \tau)$, for $0 < \varepsilon \ll 1$. Hence, by (6.28) and as $C_j^F = C_j^G + C_j^H$, for $j = 0,1,2$, in these new variables and parameters we can write (6.24), (6.25) and (6.26) as

$$\begin{aligned}
(\varepsilon^2)^{-1}F(\varepsilon x + \tau) &= (a+\alpha) + (b+\beta)x + \left(c(1+\alpha_2^G) + \mathcal{O}(\varepsilon)\right)x^2 + \mathcal{O}(\varepsilon),\\
(\varepsilon^2)^{-1}G(\varepsilon x + \tau) &= a + bx + cx^2 + \mathcal{O}(\varepsilon),\\
(\varepsilon^2)^{-1}H(\varepsilon x + \tau) &= \alpha + \beta x + \left(\alpha_2^G c + \mathcal{O}(\varepsilon)\right)x^2 + \mathcal{O}(\varepsilon).
\end{aligned}$$

Then, for $0 < \varepsilon \ll 1$, the right side of the above equations are small perturbations of the following polynomials

$$p(x) = (a+\alpha) + (b+\beta)x + (1+\sigma)\,cx^2, \tag{6.29}$$

$$g(x) = a + bx + cx^2, \tag{6.30}$$

$$q(x) = \alpha + \beta x + \sigma cx^2, \tag{6.31}$$

where $\sigma = \alpha_2^G$. Thus, to conclude the proof of the proposition, it is enough to study the simple roots of the above polynomials.

Denoting the real roots of (6.30) by r_1, r_2 and solving the system $\{g(r_1) = 0, g(r_2) = 0\}$, with respect to a and b we get

$$a = c\,r_1 r_2 \quad\text{and}\quad b = -c\,(r_1 + r_2).$$

Analogously, if s_1 and s_2 denote the real roots of (6.31), then the solution of the system $\{q(s_1) =$

$0, q(s_2) = 0\}$ with respect to α and β is

$$\alpha = c\sigma s_1 s_2 \text{ and } \beta = -c\sigma(s_1 + s_2).$$

Substituting these above relations into (6.29), we obtain the expression of the polynomial p, whose coefficients now depend on the new parameters c, σ, r_1, r_2, s_1 and s_2, given by

$$p(x) = c(1+\sigma)x^2 - c\left(r_1 + r_2 + \sigma(s_1 + s_2)\right)x + c(r_1 r_2 + \sigma s_1 s_2). \tag{6.32}$$

Observe that, for $c(1+\sigma) \neq 0$, the roots of (6.32) are

$$\frac{r_1 + r_2 + \sigma(s_1+s_2) \pm \sqrt{(r_1 + r_2 + \sigma(s_1+s_2))^2 - 4(1+\sigma)(r_1 r_2 + \sigma s_1 s_2)}}{2(1+\sigma)}.$$

We are looking for positive simple roots of (6.29) and negative simple roots of (6.30) and (6.31). Obviously, taking $c \neq 0$ and $\sigma = -1$, we can choose suitable values for the parameters a, b, c, α and β such that we have the configuration $(1, 2, 2)$ for the number of real simple roots. Now, for $c \neq 0$ and $\sigma \neq -1$, by the Routh-Hurwitz stability criterion (see [50, p. 58]), if we take $r_1 < 0$, $r_2 < 0$, $s_1 < 0$ and $s_2 < 0$, then (6.32) has roots with positive real part if and only if

$$\frac{r_1 + r_2 + \sigma(s_1 + s_2)}{1+\sigma} > 0 \text{ and } \frac{r_1 r_2 + \sigma s_1 s_2}{1+\sigma} > 0. \tag{6.33}$$

Therefore, for $\sigma < 0$ and $\sigma \neq -1$, we can choose suitable values for the parameters a, b, c, α and β such that we have the configurations $(2, 2, 2)$ for the number of real simple roots of polynomials (6.29), (6.30) and (6.31). When $\sigma \geq 0$, the conditions from (6.33) are not satisfied for negative roots r_1, r_2, s_1 and s_2, but we can choose suitable values for the parameters a, b, c, α and β such that polynomials (6.29), (6.30) and (6.31) have the configurations $(2, 2, 1)$ or $(2, 1, 2)$ for the number of real simple roots. This completes the proof of the proposition. $\square$

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^{k+1}, $k \geq 1$. Then*

$$f(x)\log(\tau - x) = \mathcal{P}(x)\log(\tau - x) + \mathcal{R}(x), \quad \textit{when} \quad x < \tau,$$

$$f(x)\log(x - \tau) = \mathcal{P}(x)\log(x - \tau) + \mathcal{R}(x), \quad \textit{when} \quad x > \tau,$$

where $\mathcal{P}(x) = \sum_{i=0}^{k} \frac{f^{(i)}(\tau)}{i!}(x-\tau)^i$ is the Taylor's polynomial of f at $x = \tau$ of degree k and $\mathcal{R}(x) = \log(\pm(\tau - x))(f(x) - \mathcal{P}(x))$. Moreover, $\lim_{x\to\tau} \mathcal{R}(x) = 0$.

Proof. To prove the lemma, it suffices to show that $\lim_{x\to\tau} \mathcal{R}(x) = 0$. By the Taylor's formula with Lagrange remainder, there is $c \in \mathbb{R}$ such that

$$\mathcal{R}(x) = \frac{f^{(k+1)}(c)}{(k+1)!} \log(\pm(\tau - x))(x - \tau)^{k+1}.$$

By the L'Hospital rule

$$\begin{aligned}
\lim_{x \to \tau} \mathcal{R}(x) &= \frac{f^{(k+1)}(c)}{(k+1)!} \lim_{x \to \tau} \log(\pm(\tau - x))(x - \tau)^{k+1} \\
&= \frac{f^{(k+1)}(c)}{(k+1)!} \lim_{x \to \tau} \frac{\log(\pm(\tau - x))}{(x - \tau)^{-(k+1)}} \\
&= \frac{f^{(k+1)}(c)}{(k+1)!} \lim_{x \to \tau} \frac{\pm(\tau - x)^{-1}}{(k+1)(x - \tau)^{-(k+2)}} \\
&= \mp \frac{f^{(k+1)}(c)}{(k+1)!(k+1)} \lim_{x \to \tau} (x - \tau)^{(k+1)} = 0.
\end{aligned}$$

□

Now, let is prove Theorem 15.

Proof of Theorem 15. In order to prove Theorem 15, let us consider the Melnikov functions M_0, M_1 and M_2 associated to system (6.1) given by Theorems 16, 17 and 18, respectively. We separate the proof in three cases. We begin with the case where system $(6.1)|_{\epsilon=0}$ is of the type SSS. For this case, we have two subcases. The first one is when $\tau_{LS} \neq \tau_{RS}$ (i.e., we have an homoclinic loop) and the second one is when $\tau_{LS} = \tau_{RS}$ (i.e., we have two heteroclinic orbits). The second and third cases are when system $(6.1)|_{\epsilon=0}$ is of the type CSS and CSC, respectively.

Case SSS. Consider the Melnikov functions given by Theorem 16. When we have homoclinic loop, i.e., $\tau_{LS} \neq \tau_{RS}$, by Lemma 2, we can expand functions (6.12)–(6.14) at $h = 1$ as

$$M_0(h) = \sum_{j=0}^{2} C_0^j (h-1)^j + \sum_{j=1}^{2} D_0^j \log(h-1)(h-1)^j + \mathcal{O}((h-1)^3), \tag{6.34}$$

$$M_1(h) = \sum_{j=0}^{2} C_1^j (h-1)^j + \sum_{j=1}^{2} D_1^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^3), \tag{6.35}$$

$$M_2(h) = \sum_{j=0}^{2} C_2^j (h-1)^j + \sum_{j=1}^{2} D_2^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^3), \tag{6.36}$$

where

$$
\begin{aligned}
C_0^0 &= \frac{2b_R}{b_L}(r_{10}-r_{00}) + 2(p_{00}+p_{10}-b_R u_{10}+b_R v_{01}) + \frac{b_R\tau_{LS}}{\omega_{LS}}(r_{10}+s_{01}) + \frac{b_R\tau_{RS}}{\omega_{RS}} \\
&\quad (p_{10}+q_{01}) - \frac{b_R}{2\omega_{LS}}(r_{10}+s_{01})(\tau_{LS}^2-1)\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right) - \frac{b_R}{2\omega_{RS}}(p_{10}+q_{01}) \\
&\quad (\tau_{RS}^2-1)\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right), \\
C_0^1 &= \frac{2b_R}{b_L}(r_{10}-r_{00}) + 2(p_{00}+p_{10}-b_R u_{10}+b_R v_{01}) + b_R(u_{10}+v_{01})\log(4) + \frac{b_R}{\omega_{LS}} \\
&\quad (r_{10}+s_{01})\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right) + \frac{b_R}{\omega_{RS}}(p_{10}+q_{01})\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right), \\
C_0^2 &= \frac{b_R}{2}\Bigg(2u_{10} + 2\bigg(v_{01} + \frac{\tau_{LS}}{\omega_{LS}(\tau_{LS}^2-1)}(r_{10}+s_{01}) + \frac{\tau_{RS}}{\omega_{RS}(\tau_{RS}^2-1)}(p_{10}+q_{01})\bigg) \\
&\quad + (u_{10}+v_{01})\log(4) + \frac{1}{\omega_{LS}}(r_{10}+s_{01})\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right) + \frac{1}{\omega_{RS}}(p_{10}+q_{01}) \\
&\quad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg), \\
C_1^0 &= 2p_{00} + 2p_{10} + b_R(v_{01}-2u_{00}-u_{10}) + \frac{b_R\tau_{RS}}{\omega_{RS}}(p_{10}+q_{01}) - \frac{b_R}{2\omega_{RS}}(p_{10}+q_{01}) \\
&\quad (\tau_{RS}^2-1)\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right), \\
C_1^1 &= 2p_{00} + 2p_{10} + b_R\Big(v_{01}-2u_{00}-u_{10}+(u_{10}+v_{01})\log(2)\Big) + \frac{b_R}{\omega_{RS}}(p_{10}+q_{01}) \\
&\quad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right), \\
C_1^2 &= \frac{1}{2}\Bigg(\frac{2\tau_{RS}}{\omega_{RS}(\tau_{RS}^2-1)}(p_{10}+q_{01}) + (u_{10}+v_{01})(1+\log(2)) + \frac{1}{\omega_{RS}}(p_{10}+q_{01}) \\
&\quad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg), \\
C_2^0 &= -\frac{2}{b_L}(r_{00}-r_{10}) + 2u_{00} - u_{10} + v_{01} + \frac{\tau_{LS}}{\omega_{LS}}(r_{10}+s_{01}) + \frac{1}{2\omega_{LS}}(r_{10}+s_{01}) \\
&\quad (\tau_{LS}^2-1)\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right), \\
C_2^1 &= -\frac{2}{b_L}(r_{00}-r_{10}) + 2u_{00} - u_{10} + v_{01} + (u_{10}+v_{01})\log(2) + \frac{1}{\omega_{LS}}(r_{10}+s_{01}) \\
&\quad \log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right), \\
C_2^2 &= \frac{1}{2}\Bigg(\frac{2\tau_{LS}}{\omega_{LS}(\tau_{LS}^2-1)}(r_{10}+s_{01}) + (u_{10}+v_{01})(1+\log(2)) + \frac{1}{\omega_{LS}}(r_{10}+s_{01}) \\
&\quad \log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right)\Bigg),
\end{aligned}
$$

and D_i^k, for $i = 0, 1, 2$ and $k = 1, 2$, depend of the parameters of system (6.1), whose expressions have been omitted for simplicity. Now, when we have two heteroclinic orbits, i.e., $\tau_{LS} = \tau_{RS}$, the expansions of Melnikov functions are the same as in (6.34)–(6.36) with $C_i^j = C_i^j \mid_{\tau_{LS}=\tau_{RS}}$ and $D_i^j = D_i^j \mid_{\tau_{LS}=\tau_{RS}}$, $i = 0,1,2$, $j = 0,1,2,3$. Moreover, as

$$\lim_{h\to 1} \log(h-1)(h-1)^j = 0, \quad j = 1,2, \quad \text{for} \quad 0 < h-1,$$

$$\lim_{h\to 1} \log(1-h)(h-1)^j = 0, \quad j = 1,2, \quad \text{for} \quad 0 < 1-h,$$

the factors $\log(\pm(h-1))$ of the expansions in (6.34)–(6.36) can be disregarded in the study of the number of zeros. Therefore, we consider only the coefficients C_i^k, for $i = 0,1,2$ and $k = 0,1,2,3$.

Furthermore, when $\tau_{LS} \neq \tau_{RS}$, we have that

$$\begin{aligned}
C_0^0 - (C_1^0 + C_2^0) = {} & \left(\frac{b_R - 1}{2b_L\omega_{LS}}\right)\Bigg(4\omega_{LS}(r_{10} - r_{00}) + 2b_L(2u_{00} - u_{10} + v_{01})\omega_{LS} \\
& + 2b_L(r_{10} + s_{01})\tau_{LS} - b_L(r_{10} + s_{01})(\tau_{LS}^2 - 1)\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right)\Bigg), \\
C_0^1 - (C_1^1 + C_2^1) = {} & \left(\frac{b_R - 1}{b_L\omega_{LS}}\right)\Bigg(\omega_{LS}\Big(2(r_{10} - r_{00}) + b_L\Big(2u_{00} - u_{10} + v_{01} \\
& + (u_{10} + v_{01})\log(2)\Big)\Big) + b_L(r_{10} + s_{01})\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right)\Bigg), \\
C_0^2 - (C_1^2 + C_2^2) = {} & \left(\frac{b_R - 1}{2\omega_{LS}(\tau_{LS}^2 - 1)}\right)\Bigg(2(r_{10} + s_{01})\tau_{LS} + (u_{10} + v_{01})\omega_{LS}(\tau_{LS}^2 - 1) \\
& (1 + \log(2)) + (r_{10} + s_{01})(\tau_{LS}^2 - 1)\log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right)\Bigg).
\end{aligned} \tag{6.37}$$

Thus, if $b_R = 1$, we have that $C_0^0 = C_1^0 + C_2^0$, $C_0^1 = C_1^1 + C_2^1$ and $C_0^2 = C_1^2 + C_2^2$. Moreover, when

$$\begin{aligned}
p_{00} &= \frac{1}{4\omega_{RS}}\Bigg(2(u_{10} - v_{01} + 2(u_{00} - p_{10}))\omega_{RS} - 2(p_{10} + q_{01})\tau_{RS} + (p_{10} + q_{01})(\tau_{RS}^2 - 1) \\
&\quad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg), \\
u_{10} &= -\frac{1}{\omega_{RS}\log(4)}\Bigg(-2(p_{10} + q_{01})\tau_{RS} + v_{01}\omega_{RS}\log(4) + (p_{10} + q_{01})(1 + \tau_{RS}^2) \\
&\quad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg), \\
p_{10} &= -q_{01}, \\
r_{00} &= \frac{1}{4\omega_{LS}}\Bigg(4(r_{10} + b_L(u_{00} + v_{01}))\omega_{RS} + 2b_L(r_{10} + s_{01})\tau_{LS} - b_L(r_{10} + s_{01})(\tau_{LS}^2 - 1) \\
&\quad \log\left(\frac{\tau_{LS}+1}{\tau_{LS}-1}\right)\Bigg), \\
r_{10} &= -s_{01},
\end{aligned}$$

we have that

$$C_1^0 = C_1^1 = C_1^2 = C_2^0 = C_2^1 = C_2^2 = 0, \quad \text{with} \quad \text{rank}\frac{\partial(C_1^0, C_1^1, C_1^2, C_2^0, C_2^1)}{\partial(p_{00}, u_{10}, p_{10}, r_{00}, r_{10})} = 5.$$

Moreover, $C_2^2 = \alpha_1^2 C_1^2 + \alpha_1^0 C_1^0 + \alpha_1^1 C_1^1 + \alpha_2^0 C_2^0 + \alpha_2^1 C_2^1$, where α_1^2 correspond to α_2^G from Proposition 10 and it is given by

$$\alpha_1^2 = \frac{\phi(\tau_{RS})}{\phi(\tau_{LS})}, \quad \text{with} \quad \phi(\tau) = \frac{\lambda_1(\tau)}{\lambda_2(\tau)},$$

where

$$\lambda_1(\tau) = (\tau^2 - 1)\Big((\tau^2 + 1)\log\left(\frac{\tau+1}{\tau-1}\right) - 2\tau\Big),$$
$$\lambda_2(\tau) = \tau\Big(2 - \log(4) - \tau^2(2 + \log(4))\Big) + \Big(\log(2) - 1 + \tau^4(1 + \log(2)) - \tau^2\log(4)\Big)$$
$$\log\left(\frac{\tau+1}{\tau-1}\right).$$

For $a_R = c_L = 0$, $b_R = c_R = a_L = 1$, $\beta_R = -3$, $b_L \in \{2/3, 1\}$ and $\beta_L \in \{2, 3\}$, the right and left subsystems from $(6.1)|_{\epsilon=0}$ have real saddles at the points $(3, 0)$ and $(-3, 3)$, respectively, with a homoclinic loop when $b_L = 2/3$ and $\beta_L = 3$. Now, for the same parameter values, except for $b_L = 1$ and $\beta_L = 2$, the right and left subsystems from $(6.1)|_{\epsilon=0}$ have real saddles at the points $(3, 0)$ and $(-3, 2)$, respectively, with two heteroclinic orbits. Moreover, in the homoclinic case, we have that $\alpha_1^2 \approx 0.80487$, and in the heteroclinic case, we have that $\alpha_1^2 = 1$. This implies, by Proposition 10, that C_1^0, C_1^1, C_1^2, C_2^0 and C_2^1 can be taken as free coefficients, such that $\max\{N_{M_0} + N_{M_1} + N_{M_2}\}$ is at least five in a neighborhood of $h = 1$. More precisely, the configuration of zeros $(2, 2, 1)$ is achievable (obviously we can obtain the configuration $(2, 1, 2)$ through a reflection with respect to the y-axis). Therefore, the number of limit cycles from system (6.1) that can bifurcate from the periodic annulus near $h = 1$, when system $(6.1)|_{\epsilon=0}$ is of the type SSS, is at least five.

Case CSS. Consider the Melnikov functions given by Theorem 17. By Lemma 2 we can expand these functions at $h = 1$ as

$$M_0(h) = \sum_{j=0}^{3} C_0^j (h-1)^j + \sum_{j=1}^{2} D_0^j \log(h-1)(h-1)^j + \mathcal{O}((h-1)^4), \tag{6.38}$$

$$M_1(h) = \sum_{j=0}^{2} C_1^j (h-1)^j + \sum_{j=1}^{2} D_1^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^3), \tag{6.39}$$

$$M_2(h) = \sum_{j=0}^{2} C_2^j (h-1)^j + \sum_{j=1}^{2} D_2^j \log(1-h)(h-1)^j + \mathcal{O}((h-1)^3), \tag{6.40}$$

where C_i^j and D_i^j, $i=0,1,2$ and $j=0,\ldots,3$, depending on the parameters of system (6.1). As in the previous case, we consider only the coefficients C_i^j, $i=0,1,2$ and $j=0,\ldots,3$, whose expressions are in Appendix 7.8.

Further, similarly as in (6.37), we have that if $b_R=1$ then $C_0^0=C_1^0+C_2^0$, $C_0^1=C_1^1+C_2^1$ and $C_0^2=C_1^2+C_2^2$. Moreover, when

$$
\begin{aligned}
p_{00} &= \frac{1}{4\omega_{RS}}\Bigg(2(u_{10}-v_{01}+2(u_{00}-p_{10}))\omega_{RS}-2(p_{10}+q_{01})\tau_{RS}+(p_{10}+q_{01})(\tau_{RS}^2-1)\\
&\qquad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg),\\
u_{10} &= -\frac{1}{\omega_{RS}\log(4)}\Bigg(-2(p_{10}+q_{01})\tau_{RS}+v_{01}\omega_{RS}\log(4)+(p_{10}+q_{01})(1+\tau_{RS}^2)\\
&\qquad \log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg),\\
p_{10} &= -q_{01},\\
r_{00} &= \frac{1}{4\omega_{LS}}\Bigg(4r_{10}\omega_{LC}+4b_L(u_{00}+v_{01})\omega_{LC}+2b_L(r_{10}+s_{01})(\pi\mu_2(1+\tau_{LC}^2)-\tau_{LC})\\
&\qquad +(-1)^{\mu_2}b_L(r_{10}+s_{01})(1+\tau_{LC}^2)\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right)\Bigg),\\
r_{10} &= -s_{01},
\end{aligned}
$$

we have that

$$
C_1^0=C_1^1=C_1^2=C_2^0=C_2^1=C_2^2=0, \quad \text{with} \quad \operatorname{rank}\frac{\partial(C_1^0,C_1^1,C_1^2,C_2^0,C_2^1)}{\partial(p_{00},u_{10},p_{10},r_{00},r_{10})}=5.
$$

Moreover, $C_2^2=\alpha_1^2C_1^2+\alpha_1^0C_1^0+\alpha_1^1C_1^1+\alpha_2^0C_2^0+\alpha_2^1C_2^1$, with α_1^2 given by

$$
\alpha_1^2=\phi_1(\tau_{RS})\phi_2(\tau_{LS}), \quad \text{with} \quad \phi_1(\tau)=\frac{\lambda_1(\tau)}{\lambda_2(\tau)} \quad \text{and} \quad \phi_2(\tau)=\frac{\lambda_3(\tau)}{\lambda_4(\tau)},
$$

where

$$\lambda_1(\tau) = (\tau^2-1)\Big((\tau^2+1)\log\left(\frac{\tau+1}{\tau-1}\right)-2\tau\Big),$$
$$\lambda_2(\tau) = \tau\Big(2-\log(4)-\tau^2(2+\log(4))\Big)+\Big(\log(2)-1+\tau^4(1+\log(2))-\tau^2\log(4)\Big)$$
$$\log\left(\frac{\tau+1}{\tau-1}\right),$$
$$\lambda_3(\tau) = (-1)^{\mu_2}\left(\arccos\left(\frac{\tau^2-1}{\tau^2+1}\right)\Big(-1+\log(2)+\tau^4(1+\log(2))+\tau^2\log(4)\Big)\right.$$
$$\left.+\tau\Big(\log(4)-2-\tau^2(2+\log(4))\Big)\right)+\pi\mu^2\Big(-2+\log(4)+\tau^4(2+\log(4))$$
$$+\tau^2\log(16)\Big),$$
$$\lambda_4(\tau) = (1+\tau^2)\left(2\pi\mu_2(-1+\tau^2)+(-1)^{\mu_2}\left(-2\tau+(-1+\tau^2)\arccos\left(\frac{\tau^2-1}{\tau^2+1}\right)\right)\right).$$

For this case, with $a_R=0$, $b_R=c_R=a_L=1$, $\beta_R=-3$, $b_L=2$, $c_L=-1$ and $\beta_L\in\{-4,-1/2\}$, the right subsystem from $(6.1)|_{\epsilon=0}$ has a real saddle at the point $(3,0)$ and the left subsystem has a virtual (resp. real) center at the point $(0,-1/2)$ (resp. $(-7,3)$) when $\beta_L=-1/2$ (resp. $\beta_L=-4$). Moreover, we have that $\alpha_1^2\approx-0.2170$ (resp. $\alpha_1^2\approx-3.3874$). This implies, by Proposition 10, that C_1^0, C_1^1, C_1^2, C_2^0 and C_2^1 can be taken as free coefficients, such that $\max\{N_{M_0}+N_{M_1}+N_{M_2}\}$ is at least six in a neighborhood of $h=1$. More precisely, the configuration of zeros $(2,2,2)$ is achievable. Therefore, the number of limit cycles from system (6.1) that can bifurcate from the periodic annulus near $h=1$, when system $(6.1)|_{\epsilon=0}$ is of the type CSS, is at least six.

Case CSC. Consider the Melnikov functions given by Theorem 18. By Lemma 2 we can expand these functions at $h=1$ as

$$M_0(h)=\sum_{j=0}^{3}C_0^j(h-1)^j+\sum_{j=1}^{2}D_0^j\log(h-1)(h-1)^j+\mathcal{O}((h-1)^4), \tag{6.41}$$
$$M_1(h)=\sum_{j=0}^{2}C_1^j(h-1)^j+\sum_{j=1}^{2}D_1^j\log(1-h)(h-1)^j+\mathcal{O}((h-1)^3), \tag{6.42}$$
$$M_2(h)=\sum_{j=0}^{2}C_2^j(h-1)^j+\sum_{j=1}^{2}D_2^j\log(1-h)(h-1)^j+\mathcal{O}((h-1)^3), \tag{6.43}$$

where C_i^j and D_i^j, $i=0,1,2$ and $j=0,\dots,3$, depending on the parameters of system (6.1). As in the previous cases, we consider only the coefficients C_i^j, $i=0,1,2$ and $j=0,\dots,3$, whose expressions are in Appendix 7.9.

Further, similarly as in (6.37), we have that if $b_R=1$ then $C_0^0=C_1^0+C_2^0$, $C_0^1=C_1^1+C_2^1$

and $C_0^2 = C_1^2 + C_2^2$. Moreover, when

$$
\begin{aligned}
p_{00} &= \frac{1}{4\omega_{RC}}\Big(-2\pi(p_{10}+q_{01})\mu_1 + 2\Big(2(u_{00}-p_{10}) + u_{10} - v_{01}\Big)\omega_{RC} + 2(p_{10}+q_{01})\tau_{RC} \\
&\quad - 2\pi(p_{10}+q_{01})\mu_1\tau_{RC}^2 + (-1)^{(1+\mu_1)}(p_{10}+q_{01})(1+\tau_{RC}^2)\arccos\left(\frac{\tau_{RC}^2-1}{\tau_{RC}^2+1}\right)\Big), \\
u_{10} &= -\frac{1}{\omega_{RC}\log(4)}\Big((p_{10}+q_{01})\Big(-2(-1)^{\mu_1} + (-1)^{\mu_1}(1+\tau_{RC}^2)\Big)\arccos\left(\frac{\tau_{RC}^2-1}{\tau_{RC}^2+1}\right)\Big) \\
&\quad + 2\Big((p_{10}+q_{01})\Big((-1)^{(1+\mu_1)}\tau_{RC} + \pi\mu_1(\tau_{RC}^2-1)\Big) - v_{01}\omega_{RC}\log(2)\Big)\Big), \\
p_{10} &= -q_{01}, \\
r_{00} &= \frac{1}{4\omega_{LS}}\Big(4r_{10}\omega_{LC} + 4b_L(u_{00}+v_{01})\omega_{LC} + 2b_L(r_{10}+s_{01})(\pi\mu_2(1+\tau_{LC}^2) - \tau_{LC}) \\
&\quad + (-1)^{\mu_2}b_L(r_{10}+s_{01})(1+\tau_{LC}^2)\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right)\Big), \\
r_{10} &= -s_{01},
\end{aligned}
$$

we have that

$$
C_1^0 = C_1^1 = C_1^2 = C_2^0 = C_2^1 = C_2^2 = 0, \quad \text{with} \quad \operatorname{rank}\frac{\partial(C_1^0, C_1^1, C_1^2, C_2^0, C_2^1)}{\partial(p_{00}, u_{10}, p_{10}, r_{00}, r_{10})} = 5.
$$

Furthermore, $C_2^2 = \alpha_1^2 C_1^2 + \alpha_1^0 C_1^0 + \alpha_1^1 C_1^1 + \alpha_2^0 C_2^0 + \alpha_2^1 C_2^1$, whose expression of parameter α_1^2 will be omitted due to its complexity. For this case, with $a_R = 0$, $b_R = a_L = 1$, $b_L = 2$, $c_R = c_L = -1$, $\beta_R \in \{0, 2\}$ and $\beta_L \in \{-3, -1/2\}$, the right subsystem from $(6.1)|_{\epsilon=0}$ has a virtual (resp. real) center at the point $(0,0)$ (resp. $(2,0)$) when $\beta_R = 0$ (resp. $\beta_R = 2$) and the left subsystem has a virtual (resp. real) center at the point $(0,-1/2)$ (resp. $(-5,2)$) when $\beta_L = -1/2$ (resp. $\beta_L = -3$). Moreover, we have that $\alpha_1^2 \approx -1.3811$ when $\beta_R = 0$ and $\beta_L = -1/2$, $\alpha_1^2 \approx -27.0327$ when $\beta_R = 0$ and $\beta_L = -3$, $\alpha_1^2 \approx -1.0591$ when $\beta_R = 2$ and $\beta_L = -3$. This implies, by Proposition 10, that C_1^0, C_1^1, C_1^2, C_2^0 and C_2^1 can be taken as free coefficients, such that $\max\{N_{M_0} + N_{M_1} + N_{M_2}\}$ is at least six in a neighborhood of $h = 1$. More precisely, the configuration of zeros $(2,2,2)$ is achievable. Therefore, the number of limit cycles from system (6.1) that can bifurcate from the periodic annulus near $h = 1$, when system $(6.1)|_{\epsilon=0}$ is of the type CSC, is at least six. □

Remark 3. *Through numerical simulations, apparently for the case SSS, we always have $\alpha_1^2 > 0$. Therefore, we cannot guarantee more than 5 limit cycles using our method proposed in Proposition 10.*

Chapter

7

Appendix

7.1 Proof of Theorem 11 from Chapter 4

Next, we present the proof of Theorem 11 from Chapter 4.

Proof. Firstly, let us obtain the expressions of Melnikov functions (4.32) and (4.33). For this, consider system $(4.1)|_{\epsilon=0}$ with $a_L = b_R = c_R = \beta_L = 1$, $b_L = 2$, $a_R = 0$, $c_L = -1$ and $\beta_R = -2$. The orbit $(x_R(t), y_R(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$\begin{aligned} x_R(t) &= \frac{e^t}{2}(h-1) - \frac{e^{-t}}{2}(h+1) + 2, \\ y_R(t) &= \frac{e^t}{2}(h-1) + \frac{e^{-t}}{2}(h+1). \end{aligned} \tag{7.1}$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$t_R = \log\left(\frac{1+h}{1-h}\right). \tag{7.2}$$

Now, for f_R and g_R defined in (4.3) and (4.4), respectively, we have that

$$\begin{aligned} I_R^n &= \int_{\widehat{AA_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \big(g_R(x_R(t), y_R(t))x_R'(t) - f_R(x_R(t), y_R(t))y_R'(t)\big)dt \\ &= \int_0^{t_R} \left(\sum_{i+j=0}^{n} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{n} p_{ij} x_R^i(t) y_R^j(t) y_R'(t)\right)dt, \end{aligned} \tag{7.3}$$

with $n = 2, 3$. Replacing (7.1) and (7.2) into (7.3), we obtain

$$\begin{aligned} I_R^2 &= (2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2q_{11}) f_0(h) \\ &\quad + \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01} + 2q_{11}\Big) f_0^S(h) + \frac{2}{3}\Big(p_{02} - 2p_{20} - q_{11}\Big) f_2(h), \end{aligned} \tag{7.4}$$

$$\begin{aligned} I_R^3 &= \frac{1}{3}\Big(8p_{00} + 12p_{10} - p_{12} + 24p_{20} + 59p_{30} + 4q_{01} - 3q_{03}\Big) f_0(h) \\ &\quad + \frac{1}{3}\Big(8q_{11} + 17q_{21}\Big) f_0(h) + \frac{1}{8}\Big(4p_{10} - p_{12} + 16p_{20} + 51p_{30}\Big) f_0^S(h) \\ &\quad + \frac{1}{8}\Big(4q_{01} - 3q_{03} + 8q_{11} + 17q_{21}\Big) f_0^S(h) + \frac{1}{12}\Big(8p_{02} + 13p_{12}\Big) f_2(h) \\ &\quad + \frac{1}{12}\Big(- 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21}\Big) f_2(h) \\ &\quad + \frac{1}{8}\Big(p_{12} - 3p_{30} + 3q_{03} - q_{21}\Big) f_2^S(h). \end{aligned} \tag{7.5}$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C1}(0),\, y_{C1}(0)) = (1, -h)$, is given by

$$\begin{aligned} x_{C1}(t) &= \cos(t) - h\sin(t), \\ y_{C1}(t) &= -h\cos(t) - \sin(t). \end{aligned} \tag{7.6}$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$t_{C1} = \arccos\left(\frac{h^2 - 1}{h^2 + 1}\right). \tag{7.7}$$

Now, for f_C and g_C defined in (4.3) and (4.4), respectively, we have that

$$\begin{aligned} \bar{I}_C^n &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ &= \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t)) x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t)) y'_{C1}(t)\big) dt \\ &= \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{n} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x'_{C1}(t) - \sum_{i+j=0}^{n} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y'_{C1}(t)\Bigg) dt, \end{aligned} \tag{7.8}$$

with $n = 2, 3$. Replacing (7.6) and (7.7) into (7.8), we obtain

$$\begin{aligned} \bar{I}_C^2 &= -\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01}) f_0(h) - 2v_{02}\, h^2 \\ &\quad + \frac{1}{2}\Big(u_{10} + v_{01}\Big) f_1(h), \end{aligned} \tag{7.9}$$

$$\begin{aligned}\bar{I}_C^3 = &-\frac{2}{3}\Big(u_{11}+3v_{00}+2v_{02}+v_{20}\Big)+\frac{1}{4}\Big(-4u_{10}+u_{12}-5u_{30}+4v_{01}\\ &+3v_{03}+v_{21}\Big)f_0(h)-2v_{02}h^2-\frac{1}{4}\Big(u_{12}+3u_{30}-5v_{03}+v_{21}\Big)f_2(h)\\ &+\frac{1}{8}\Big(4u_{10}+u_{12}+3u_{30}+4v_{01}+3v_{03}+v_{21}\Big)f_1(h)+\frac{1}{8}(u_{12}\\ &+3(u_{30}+v_{03})+v_{21})f_3(h).\end{aligned} \tag{7.10}$$

The orbit $(x_L(t), y_L(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_L(0),\, y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned} x_L(t) &= -4\cos(t) - 2h\sin(t) + 3,\\ y_L(t) &= (2-h)\cos(t) + (2+h)\sin(t) - 2. \end{aligned} \tag{7.11}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \arccos\left(\frac{4-h^2}{4+h^2}\right). \tag{7.12}$$

Now, for f_L and g_L defined in (4.3) and (4.4), respectively, we obtain

$$\begin{aligned} I_L^n &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy\\ &= \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\big)dt\\ &= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{n} s_{ij}x_L^i(t)y_L^j(t)x_L'(t) - \sum_{i+j=0}^{n} r_{ij}x_L^i(t)y_L^j(t)y_L'(t)\Bigg)dt \end{aligned} \tag{7.13}$$

with $n = 2, 3$. Replacing (7.11) and (7.12) into (7.13), we have that

$$\begin{aligned} I_L^2 = &-\frac{2}{3}\Big(3r_{00}+3(r_{10}-4r_{11}+13r_{20}+2s_{01}-8s_{02}+6s_{11})\Big)f_0(h)\\ &-\frac{2}{3}\Big(r_{02}-2r_{11}+8r_{20}-4s_{02}+4s_{11}\Big)f_2(h)+(r_{10}-2r_{11}+6r_{20}\\ &+s_{01}-4s_{02}+3s_{11})f_0^C(h), \quad \text{with} \quad \mu=0, \end{aligned} \tag{7.14}$$

$$\begin{aligned} I_L^3 = &-\frac{2}{3}\Big(3r_{00}+3(r_{10}-4r_{11}+12r_{12}+13r_{20}-32r_{21}+77r_{30}+2s_{01}\\ &-8s_{02}+36s_{03}+6s_{11}-32s_{12}+26s_{21})\Big)f_0(h)-\frac{2}{3}\Big(r_{02}-2r_{11}+8r_{12}\\ &+8r_{20}-22r_{21}+54r_{30}-4s_{02}+27s_{03}+4s_{11}-22s_{12}+18s_{21}\Big)f_2(h)\\ &+\frac{1}{2}\Big(2r_{10}-4r_{11}+12r_{12}+2(6r_{20}-16r_{21}+39r_{30}+s_{01}-4s_{02}\\ &+18s_{03}+3s_{11}-16s_{12}+13s_{21})\Big)f_0^C(h)+\frac{1}{2}\Big(r_{12}-2r_{21}+6r_{30}\\ &+3s_{03}-2s_{12}+2s_{21}\Big)f_1^C(h), \quad \text{with} \quad \mu=0. \end{aligned} \tag{7.15}$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C2}(0), y_{C2}(0)) = (-1, h)$, is given by

$$\begin{aligned} x_{C2}(t) &= -\cos(t) + h\sin(t), \\ y_{C2}(t) &= h\cos(t) + \sin(t). \end{aligned} \tag{7.16}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \arccos\left(\frac{h^2-1}{h^2+1}\right). \tag{7.17}$$

Now, for f_C and g_C defined in (4.3) and (4.4), respectively, we obtain

$$\begin{aligned} I_C^n &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \big(g_C(x_{C2}(t), y_{C2}(t))x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t))y'_{C2}(t)\big)dt \\ &= \int_0^{t_{C2}} \left(\sum_{i+j=0}^{1} v_{ij}x_{C2}^i(t)y_{C2}^j(t)x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij}x_{C2}^i(t)y_{C2}^j(t)y'_{C2}(t)\right)dt \end{aligned} \tag{7.18}$$

with $n = 2, 3$. Replacing (7.16) and (7.17) into (7.18), we have that

$$\begin{aligned} I_C^2 = {} & \frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01})f_0(h) + 2v_{02}\,h^2 \\ & + \frac{1}{2}\Big(u_{10} + v_{01}\Big)f_1(h), \end{aligned} \tag{7.19}$$

$$\begin{aligned} I_C^3 = {} & \frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) + \frac{1}{4}\Big(-4u_{10} + u_{12} - 5u_{30} + 4v_{01} + 3v_{03} \\ & + v_{21}\Big)f_0(h) + 2v_{02}h^2 - \frac{1}{4}\Big(u_{12} + 3u_{30} - 5v_{03} + v_{21}\Big)f_2(h) + \frac{1}{8}\Big(4u_{10} \\ & + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big)f_1(h) + \frac{1}{8}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big)f_3(h). \end{aligned} \tag{7.20}$$

Moreover, by Corollary 9, the first order Melnikov function associated to system (4.1) is given by

$$M(h) = I_C^n + \frac{1}{2}I_L^n + \bar{I}_C^n + I_R^n, \quad n = 2, 3. \tag{7.21}$$

Therefore, if $n = 2$, replacing (7.4), (7.9), (7.14) and (7.19) into (7.21) we obtain (4.32), with

$$\begin{aligned}
k_{12}^0 &= 2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2(q_{11} - r_{00} - r_{10} + 4r_{11} - 13r_{20} - 2s_{01} + 8s_{02} - 6s_{11} \\
&\quad - u_{10} + v_{01}), \\
k_{12}^1 &= u_{10} + v_{01}, \\
k_{12}^2 &= -\frac{2}{3}\Big(p_{02} - 2p_{20} - q_{11} - r_{02} + 2r_{11} - 8r_{20} + 4s_{02} - 4s_{11}\Big), \\
l_{12}^0 &= \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01}\Big) + q_{11}, \\
m_{12}^0 &= r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}.
\end{aligned}$$

Finally, if $n = 3$, replacing (7.5), (7.10), (7.15) and (7.20) into (7.21) we obtain (4.33), with

$$\begin{aligned}
k_{13}^0 &= 2p_{00} + 3p_{10} + \frac{1}{4}\Big(24p_{20} - p_{12} + 59p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21} - 2(2r_{00} + 2r_{10} \\
&\quad - 8r_{11} + 24r_{12} + 26r_{20} - 64r_{21} + 154r_{30} + 4s_{01} - 16s_{02} + 72s_{03} + 12s_{11} - 64s_{12} \\
&\quad + 52s_{21} + 4u_{10} - u_{12} + 5u_{30} - 4v_{01} - 3v_{03} - v_{21})\Big), \\
k_{13}^1 &= \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\
k_{13}^2 &= \frac{1}{12}\Big(8p_{02} + 13p_{12} - 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21} - 2(2r_{02} - 4r_{11} + 16r_{12} \\
&\quad + 16r_{20} - 44r_{21} + 108r_{30} - 8s_{02} + 54s_{03} + 8s_{11} - 44s_{12} + 3(12s_{21} + u_{12} + 3u_{30} \\
&\quad - 5v_{03} + v_{21}))\Big), \\
k_{13}^3 &= \frac{1}{4}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big), \\
l_{13}^0 &= \frac{1}{8}\Big(4p_{10} - p_{12} + 16p_{20} + 51p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21}\Big), \\
l_{13}^1 &= \frac{1}{8}\Big(p_{12} - 3p_{30} + 3q_{03} - q_{21}\Big), \\
m_{13}^0 &= \frac{1}{4}\Big(2r_{10} - 4r_{11} + 12r_{12} + 2(6r_{20} - 16r_{21} + 39r_{30} + s_{01} - 4s_{02} + 18s_{03} + 3s_{11} \\
&\quad - 16s_{12} + 13s_{21})\Big), \\
m_{13}^1 &= \frac{1}{4}\Big(r_{12} - 2r_{21} + 6r_{30} + 3s_{03} - 2s_{12} + 2s_{21}\Big).
\end{aligned}$$

With a similar proof, that will be omitted to simplifying the text, we can obtain the expressions of the Melnikov functions (4.34) and (4.35), but it is important to emphasize that for this case the flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$t_L = 2\pi - \arccos\left(\frac{1 - h^2}{1 + h^2}\right).$$

Moreover, the coefficients $k^m_{i,j}$ and $l^m_{i,j}$ of Melnikov function (4.34) are given by

$$\begin{aligned}
k^0_{22} =& \, 2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2(q_{11} - r_{00} + 2r_{10} + r_{11} - 7r_{20} + s_{01} + 2s_{02} \\
& - 3s_{11} - u_{10} + v_{01}), \\
k^1_{22} =& \, u_{10} + v_{01}, \\
k^2_{22} =& \, \frac{2}{3}\Big(p_{02} - 2p_{20} - q_{11} - r_{02} + 2r_{11} - 8r_{20} + 4s_{02} - 4s_{11}\Big), \\
l^0_{22} =& \, \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01}\Big) + q_{11}, \\
m^2_{22} =& \, r_{10} + r_{11} - 6r_{20} + s_{01} + 2s_{02} - 3s_{11}.
\end{aligned}$$

For Melnikov function (4.35), we have that

$$\begin{aligned}
k^0_{23} =& \, \frac{1}{4}\Big(8p_{00} + 12p_{10} - p_{12} + 24p_{20} + 59p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21} + 2(-2r_{00} \\
& + 4r_{10} + 2r_{11} + 3r_{12} - 14r_{20} - 14r_{21} + 62r_{30} + 2s_{01} + 4s_{02} + 9s_{03} - 6s_{11} - 14s_{12} \\
& + 20s_{21} - 4u_{10} + u_{12} - 5u_{30} + 4v_{01} + 3v_{03} + v_{21})\Big), \\
k^1_{23} =& \, \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\
k^2_{23} =& \, \frac{1}{12}\Big(8p_{02} + 13p_{12} - 16p_{20} - 87p_{30} + 15q_{03} - 8q_{11} - 29q_{21} - 4r_{02} + 8r_{11} + 22r_{12} \\
& - 32r_{20} - 68r_{21} + 252r_{30} + 16s_{02} + 54s_{03} - 16s_{11} - 68s_{12} - 6(-14s_{21} + u_{12} \\
& + 3u_{30} - 5v_{03} + v_{21})\Big), \\
k^3_{23} =& \, \frac{1}{3}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big), \\
l^0_{23} =& \, \frac{1}{8}\Big(4p_{10} - p_{12} + 16p_{20} + 51p_{30} + 4q_{01} - 3q_{03} + 8q_{11} + 17q_{21}\Big), \\
l^1_{23} =& \, \frac{1}{8}\Big(p_{12} - 3p_{30} + 3q_{03} - q_{21}\Big), \\
m^2_{23} =& \, \frac{1}{4}\Big(2r_{10} + 2r_{11} + 3r_{12} - 12r_{20} - 14r_{21} + 60r_{30} + 2s_{01} + 4s_{02} + 9s_{03} - 6s_{11} \\
& - 14s_{12} + 20s_{21}\Big), \\
m^3_{23} =& \, \frac{1}{4}\Big(r_{12} + -2r_{21} + 6r_{30} + 3s_{03} - 2s_{12} + 2s_{21}\Big).
\end{aligned}$$

□

7.2 Proof of Theorem 12 from Chapter 4

Next, we present the proof of Theorem 12 from Chapter 4.

Proof. Firstly, let us obtain the expressions of Melnikov functions (4.36) and (4.37). For this, consider system $(4.1)|_{\epsilon=0}$ with $a_L = b_R = \beta_L = 1$, $b_L = 2$, $a_R = \beta_R = 0$ and $c_L = c_R = -1$. The orbit $(x_R(t), y_R(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$\begin{aligned} x_R(t) &= \cos(t) + h\sin(t), \\ y_R(t) &= h\cos(t) - \sin(t). \end{aligned} \tag{7.22}$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$t_R = \arccos\left(\frac{1-h^2}{1+h^2}\right). \tag{7.23}$$

Now, for f_R and g_R defined in (4.3) and (4.4), respectively, we obtain

$$\begin{aligned} I_R^n &= \int_{\widehat{AA_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \big(g_R(x_R(t), y_R(t))x_R'(t) - f_R(x_R(t), y_R(t))y_R'(t)\big)dt \\ &= \int_0^{t_R} \left(\sum_{i+j=0}^{n} q_{ij}x_R^i(t)y_R^j(t)x_R'(t) - \sum_{i+j=0}^{n} p_{ij}x_R^i(t)y_R^j(t)y_R'(t)\right)dt, \end{aligned} \tag{7.24}$$

with $n = 2, 3$. Replacing (7.22) and (7.23) into (7.24), we have that

$$\begin{aligned} I_R^2 &= \Big(2p_{00} + p_{10} + 2p_{20} - q_{01}\Big) f_0(h) + \frac{2}{3}\Big(p_{02} + 2p_{20} + q_{11}\Big) f_2(h) \\ &\quad + \frac{1}{2}\Big(p_{10} + q_{01}\Big) f_2^C(h), \quad \text{with} \quad \lambda = 0, \end{aligned} \tag{7.25}$$

$$\begin{aligned} I_R^3 &= \frac{1}{24}\Big(48p_{00} + 6(4p_{10} - p_{12} + 8p_{20} + 5p_{30} - 4q_{01} - 3q_{03} - q_{21})\Big) f_0(h) \\ &\quad + \frac{1}{12}\Big(8p_{02} + 3p_{12} + 16p_{20} + 9p_{30} - 15q_{03} + 8q_{11} + 3q_{21}\Big) f_2(h) \\ &\quad + \frac{1}{8}\Big(4p_{10} + p_{12} + 3p_{30} + 4q_{01} + 3q_{03} + q_{21}\Big) f_2^C(h) + \frac{1}{8}\Big(3(p_{30} + q_{03}) \\ &\quad + q_{21} + p_{12}\Big) f_3^C(h), \quad \text{with} \quad \lambda = 0. \end{aligned} \tag{7.26}$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C1}(0),\, y_{C1}(0)) = (1, -h)$, is given by

$$\begin{aligned} x_{C1}(t) &= \cos(t) - h\sin(t), \\ y_{C1}(t) &= -h\cos(t) - \sin(t). \end{aligned} \tag{7.27}$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$t_{C1} = \arccos\left(\frac{h^2-1}{h^2+1}\right). \tag{7.28}$$

Now, for f_C and g_C defined in (4.3) and (4.4), respectively, we obtain

$$\begin{aligned}\bar{I}_C^n &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ &= \int_0^{t_{C1}} \big(g_C(x_{C1}(t), y_{C1}(t))x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t))y'_{C1}(t)\big)dt \\ &= \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{n} v_{ij}x_{C1}^i(t)y_{C1}^j(t)x'_{C1}(t) - \sum_{i+j=0}^{n} u_{ij}x_{C1}^i(t)y_{C1}^j(t)y'_{C1}(t)\Bigg)dt,\end{aligned} \tag{7.29}$$

with $n = 2, 3$. Replacing (7.27) and (7.28) into (7.29), we have that

$$\begin{aligned}\bar{I}_C^2 = &-\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01})f_0(h) - 2v_{02}\,h^2 \\ &+\frac{1}{2}\Big(u_{10} + v_{01}\Big)f_1(h),\end{aligned} \tag{7.30}$$

$$\begin{aligned}\bar{I}_C^3 = &-\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) + \frac{1}{4}\Big(- 4u_{10} + u_{12} - 5u_{30} + 4v_{01} \\ &+3v_{03} + v_{21}\Big)f_0(h) - 2v_{02}h^2 - \frac{1}{4}\Big(u_{12} + 3u_{30} - 5v_{03} + v_{21}\Big)f_2(h) \\ &+\frac{1}{8}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big)f_1(h) + \frac{1}{8}(u_{12} \\ &+3(u_{30} + v_{03}) + v_{21})f_3(h).\end{aligned} \tag{7.31}$$

The orbit $(x_L(t), y_L(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_L(0),\, y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned}x_L(t) &= -4\cos(t) - 2h\sin(t) + 3, \\ y_L(t) &= (2 - h)\cos(t) + (2 + h)\sin(t) - 2.\end{aligned} \tag{7.32}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \arccos\left(\frac{4 - h^2}{4 + h^2}\right). \tag{7.33}$$

Now, for f_L and g_L defined in (4.3) and (4.4), respectively, we obtain

$$\begin{aligned}I_L^n &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\ &= \int_0^{t_L} \big(g_L(x_L(t), y_L(t))x'_L(t) - f_L(x_L(t), y_L(t))y'_L(t)\big)dt \\ &= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{n} s_{ij}x_L^i(t)y_L^j(t)x'_L(t) - \sum_{i+j=0}^{n} r_{ij}x_L^i(t)y_L^j(t)y'_L(t)\Bigg)dt\end{aligned} \tag{7.34}$$

with $n=2,3$. Replacing (7.32) and (7.33) into (7.34), we have that

$$
\begin{aligned}
I_L^2 =& -\frac{2}{3}\Big(3r_{00}+3(r_{10}-4r_{11}+13r_{20}+2s_{01}-8s_{02}+6s_{11})\Big)f_0(h) \\
&-\frac{2}{3}\Big(r_{02}-2r_{11}+8r_{20}-4s_{02}+4s_{11}\Big)f_2(h)+(r_{10}-2r_{11}+6r_{20} \\
&+s_{01}-4s_{02}+3s_{11})f_0^C(h), \quad \text{with} \quad \mu=0,
\end{aligned} \tag{7.35}
$$

$$
\begin{aligned}
I_L^3 =& -\frac{2}{3}\Big(3r_{00}+3(r_{10}-4r_{11}+12r_{12}+13r_{20}-32r_{21}+77r_{30}+2s_{01} \\
&-8s_{02}+36s_{03}+6s_{11}-32s_{12}+26s_{21})\Big)f_0(h)-\frac{2}{3}\Big(r_{02}-2r_{11}+8r_{12} \\
&+8r_{20}-22r_{21}+54r_{30}-4s_{02}+27s_{03}+4s_{11}-22s_{12}+18s_{21}\Big)f_2(h) \\
&+\frac{1}{2}\Big(2r_{10}-4r_{11}+12r_{12}+2(6r_{20}-16r_{21}+39r_{30}+s_{01}-4s_{02} \\
&+18s_{03}+3s_{11}-16s_{12}+13s_{21})\Big)f_0^C(h)+\frac{1}{2}\Big(r_{12}-2r_{21}+6r_{30} \\
&+3s_{03}-2s_{12}+2s_{21}\Big)f_1^C(h), \quad \text{with} \quad \mu=0.
\end{aligned} \tag{7.36}
$$

The orbit $(x_{C2}(t),y_{C2}(t))$ of system $(4.1)|_{\epsilon=0}$, such that $(x_{C2}(0),y_{C2}(0))=(-1,h)$, is given by

$$
\begin{aligned}
x_{C2}(t) &= -\cos(t)+h\sin(t), \\
y_{C2}(t) &= h\cos(t)+\sin(t).
\end{aligned} \tag{7.37}
$$

The flight time of the orbit $(x_{C2}(t),y_{C2}(t))$, from $A_3(h)=(-1,h)$ to $A(h)=(1,h)$, is

$$
t_{C2}=\arccos\left(\frac{h^2-1}{h^2+1}\right). \tag{7.38}
$$

Now, for f_C and g_C defined in (4.3) and (4.4), respectively, we obtain

$$
\begin{aligned}
I_C^n &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\
&= \int_0^{t_{C2}} \Big(g_C(x_{C2}(t),y_{C2}(t))x'_{C2}(t)-f_C(x_{C2}(t),y_{C2}(t))y'_{C2}(t)\Big)dt \\
&= \int_0^{t_{C2}} \Bigg(\sum_{i+j=0}^{1} v_{ij}x_{C2}^i(t)y_{C2}^j(t)x'_{C2}(t)-\sum_{i+j=0}^{1} u_{ij}x_{C2}^i(t)y_{C2}^j(t)y'_{C2}(t)\Bigg)dt,
\end{aligned} \tag{7.39}
$$

with $n = 2, 3$. Replacing (7.37) and (7.38) into (7.39), we have that

$$\begin{aligned} I_C^2 = &\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) - (u_{10} - v_{01})f_0(h) + 2v_{02}\,h^2 \\ &+\frac{1}{2}\Big(u_{10} + v_{01}\Big)f_1(h), \end{aligned} \tag{7.40}$$

$$\begin{aligned} I_C^3 = &\frac{2}{3}\Big(u_{11} + 3v_{00} + 2v_{02} + v_{20}\Big) + \frac{1}{4}\Big(- 4u_{10} + u_{12} - 5u_{30} + 4v_{01} \\ &+3v_{03} + v_{21}\Big)f_0(h) + 2v_{02}h^2 - \frac{1}{4}\Big(u_{12} + 3u_{30} - 5v_{03} + v_{21}\Big)f_2(h) \\ &+\frac{1}{8}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big)f_1(h) + \frac{1}{8}\Big(u_{12} \\ &+3(u_{30} + v_{03}) + v_{21}\Big)f_3(h). \end{aligned} \tag{7.41}$$

Moreover, by Corollary 9, the first order Melnikov function associated to system (4.1) is given by

$$M(h) = I_C^n + \frac{1}{2}I_L^n + \bar{I}_C^n + I_R^n, \quad n = 2, 3. \tag{7.42}$$

Therefore, if $n = 2$, replacing (7.25), (7.30), (7.35) and (7.40) into (7.42) we obtain (4.36), with

$$\begin{aligned} k_{12}^0 &= 2p_{00} + p_{10} + 2p_{20} - q_{01} - r_{00} - r_{10} + 4r_{11} - 13r_{20} - 2(s_{01} - 4s_{02} + 3s_{11} + u_{10} - v_{01}) \\ k_{12}^1 &= u_{10} + v_{01}, \\ k_{12}^2 &= \frac{1}{3}\Big(2p_{02} + 4p_{20} + 2q_{11} - r_{02} + 2r_{11} - 8r_{20} + 4s_{02} - 4s_{11}\Big), \\ l_{12}^0 &= \frac{1}{2}\Big(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}\Big), \\ l_2 &= \frac{1}{2}\Big(p_{10} + q_{01}\Big). \end{aligned}$$

Finally, if $n = 3$, replacing (7.26), (7.31), (7.36) and (7.41) into (7.42) we obtain (4.37), with

$$\begin{aligned} k_{13}^0 = &\frac{1}{4}\Big(8p_{00} + 4p_{10} - p_{12} + 8p_{20} + 5p_{30} - 4q_{01} - 3q_{03} - q_{21} - 2(2r_{00} + 2r_{10} - 8r_{11} \\ &+ 24r_{12} + 26r_{20} - 64r_{21} + 154r_{30} + 4s_{01} - 16s_{02} + 72s_{03} + 12s_{11} - 64s_{12} + 52s_{21} \\ &+ 4u_{10} - u_{12} + 5u_{30} - 4v_{01} - 3v_{03} - v_{21})\Big), \\ k_{13}^1 = &\frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\ k_{13}^2 = &\frac{1}{12}\Big(8p_{02} + 3p_{12} + 16p_{20} + 9p_{30} - 15q_{03} + 8q_{11} + 3q_{21} - 2(2r_{02} - 4r_{11} + 16r_{12} \\ &+ 16r_{20} - 44r_{21} + 108r_{30} - 8s_{02} + 54s_{03} + 8s_{11} - 44s_{12} + 3(12s_{21} + u_{12} + 3u_{30} \\ &- 5v_{03} + v_{21}))\Big), \end{aligned}$$

$$
\begin{aligned}
k_{13}^3 &= \frac{1}{4}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big),\\
l_{13}^0 &= \frac{1}{4}\Big(2r_{10} - 4r_{11} + 12r_{12} + 2(6r_{20} - 16r_{21} + 39r_{30} + s_{01} - 4s_{02} + 18s_{03} + 3s_{11}\\
&\quad - 16s_{12} + 13s_{21})\Big),\\
l_{13}^1 &= \frac{1}{4}\Big(r_{12} - 2r_{21} + 6r_{30} + 3s_{03} - 2s_{12} + 2s_{21}\Big),\\
l_{13}^2 &= \frac{1}{8}\Big(4p_{10} + p_{12} + 3p_{30} + 4q_{01} + 3q_{03} + q_{21}\Big),\\
l_{13}^3 &= \frac{1}{8}\Big(p_{12} + 3(p_{30} + q_{03}) + q_{21}\Big).
\end{aligned}
$$

With a similar proof, that will be omitted to simplifying the text, we can obtain the expressions of the Melnikov functions (4.38), (4.39), (4.40) and (4.41), but it is important to emphasize that for these cases the flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$t_L = 2\pi - \arccos\left(\frac{4 - h^2}{4 + h^2}\right),$$

and the flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, such that $(x_R(0), y_R(0)) = (1, h)$, is given by

$$t_R = 2\pi\lambda + (-1)^\lambda \arccos\left(\frac{1 - h^2}{1 + h^2}\right),$$

with $\lambda = 0$ (resp. $\lambda = 1$) if the right subsystem from $(4.1)|_{\epsilon=0}$ has a virtual (resp. real) center. Moreover, the coefficients $k_{i,j}^m$ and $l_{i,j}^m$ of Melnikov function (4.38) are given by

$$
\begin{aligned}
k_{22}^0 &= 2p_{00} + p_{10} + 2p_{20} - q_{01} - r_{00} + 3r_{10} + 4r_{11} - 21r_{20} + 2(s_{01} + 4s_{02} - 5s_{11}\\
&\quad - u_{10} + v_{01}),\\
k_{22}^1 &= u_{10} + v_{01},\\
k_{22}^2 &= \frac{1}{3}\Big(2p_{02} + 4p_{20} + 2q_{11} - r_{02} + 2r_{11} - 8r_{20} + 4s_{02} - 4s_{11}\Big),\\
l_{22}^0 &= \frac{1}{2}\Big(r_{10} + 2r_{11} - 10r_{20} + s_{01} + 4s_{02} - 5s_{11}\Big),\\
l_{22}^2 &= \frac{1}{2}\Big(p_{10} + q_{01}\Big).
\end{aligned}
$$

For Melnikov function (4.39), we have that

$$
\begin{aligned}
k_{23}^0 =\ & \frac{1}{4}\Big(8p_{00} + 4p_{10} - p_{12} + 8p_{20} + 5p_{30} - 4q_{01} - 3q_{03} - q_{21} + 2(-2r_{00} + 6r_{10} + 8r_{11} \\
& + 24r_{12} - 42r_{20} - 96r_{21} + 350r_{30} + 4s_{01} + 16s_{02} + 72s_{03} - 20s_{11} - 96s_{12} + 116s_{21} \\
& - 4u_{10} + u_{12} - 5u_{30} + 4v_{01} + 3v_{03} + v_{21})\Big) f_0(h), \\
k_{23}^1 =\ & \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\
k_{23}^2 =\ & \frac{1}{12}\Big(8p_{02} + 3p_{12} + 16p_{20} + 9p_{30} - 15q_{03} + 8q_{11} + 3q_{21} - 2(2r_{02} - 4r_{11} - 20r_{12} \\
& + 16r_{20} + 60r_{21} - 204r_{30} - 8s_{02} - 54s_{03} + 8s_{11} + 60s_{12} - 68s_{21} + 3(u_{12} + 3u_{30} \\
& - 5v_{03} + v_{21}))\Big), \\
k_{23}^3 =\ & \frac{1}{4}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big), \\
l_{23}^0 =\ & \frac{1}{4}\Big(2r_{10} + 4r_{11} + 12r_{12} + 2(-10r_{20} - 24r_{21} + 87r_{30} + s_{01} + 4s_{02} + 18s_{03} - 5s_{11} \\
& - 24s_{12} + 29s_{21})\Big), \\
l_{23}^1 =\ & \frac{1}{4}\Big(r_{12} + (-2r_{21} + 6r_{30} + 3s_{03} - 2s_{12} + 2s_{21})\Big), \\
l_{23}^2 =\ & \frac{1}{8}\Big(4p_{10} + p_{12} + 3p_{30} + 4q_{01} + 3q_{03} + q_{21}\Big), \\
l_{23}^2 =\ & \frac{1}{8}\Big(p_{12} + 3(p_{30} + q_{03}) + q_{21}\Big).
\end{aligned}
$$

For Melnikov function (4.40), we have that

$$
\begin{aligned}
k_{32}^0 =\ & 2p_{00} + 3p_{10} + 6p_{20} + q_{01} + 2q_{11} - r_{00} + 3r_{10} + 4r_{11} - 21r_{20} + 2(s_{01} + 4s_{02} \\
& - 5s_{11} - u_{10} + v_{01}), \\
k_{32}^1 =\ & u_{10} + v_{01}, \\
k_{32}^2 =\ & \frac{1}{3}\Big(2p_{02} + 4p_{20} + 2q_{11} - r_{02} + 2r_{11} - 8r_{20} + 4s_{02} - 4s_{11}\Big), \\
l_{32}^0 =\ & \frac{1}{2}\Big(r_{10} + 2r_{11} - 10r_{20} + s_{01} + 4s_{02} - 5s_{11}\Big), \\
l_{32}^2 =\ & \frac{1}{2}\Big(p_{10} + 4p_{20} + q_{01} + 2q_{11}\Big).
\end{aligned}
$$

Finally, for Melnikov function (4.41), we have that

$$
\begin{aligned}
k_{33}^0 &= \frac{1}{4}\Big(8p_{00} + 12p_{10} + p_{12} + 24p_{20} + 59p_{30} + 4q_{01} + 3q_{03} + 8q_{11} + 17q_{21} + 2(-2r_{00} \\
&\quad + 6r_{10} + 8r_{11} + 24r_{12} - 42r_{20} - 96r_{21} + 350r_{30} + 4s_{01} + 16s_{02} + 72s_{03} - 20s_{11} \\
&\quad - 96s_{12} + 116s_{21} - 4u_{10} + u_{12} - 5u_{30} + 4v_{01} + 3v_{03} + v_{21})\Big), \\
k_{33}^1 &= \frac{1}{4}\Big(4u_{10} + u_{12} + 3u_{30} + 4v_{01} + 3v_{03} + v_{21}\Big), \\
k_{33}^2 &= \frac{1}{12}\Big(8p_{02} + 13p_{12} + 16p_{20} + 87p_{30} + 15q_{03} + 8q_{11} + 29q_{21} - 2(2r_{02} - 4r_{11} - 20r_{12} \\
&\quad + 16r_{20} + 60r_{21} - 204r_{30} - 8s_{02} - 54s_{03} + 8s_{11} + 60s_{12} - 68s_{21} + 3(u_{12} + 3u_{30} \\
&\quad - 5v_{03} + v_{21}))\Big),
\end{aligned}
$$

$$
\begin{aligned}
k_{33}^3 &= \frac{1}{4}\Big(u_{12} + 3(u_{30} + v_{03}) + v_{21}\Big), \\
l_{33}^0 &= \frac{1}{4}\Big(2r_{10} + 4r_{11} + 12r_{12} + 2(-10r_{20} - 24r_{21} + 87r_{30} + s_{01} + 4s_{02} + 18s_{03} - 5s_{11} \\
&\quad - 24s_{12} + 29s_{21})\Big), \\
l_{33}^1 &= \frac{1}{4}\Big(r_{12} + (-2r_{21} + 6r_{30} + 3s_{03} - 2s_{12} + 2s_{21})\Big), \\
l_{33}^2 &= \frac{1}{8}\Big(4p_{10} + p_{12} + 16p_{20} + 51p_{30} + 4q_{01} + 3q_{03} + 8q_{11} + 17q_{21}\Big), \\
l_{33}^3 &= \frac{1}{8}\Big(p_{12} + 3(p_{30} + q_{03}) + q_{21}\Big).
\end{aligned}
$$

□

7.3 Proof of Theorem 16 from Chapter 6

Next, we present the proof of Theorem 16 from Chapter 6.

Proof. Firstly, let us simplify the Melnikov function M_0 given by (6.8) for the case $\tau_{LS} \neq \tau_{RS}$. For this purpose, consider the orbit $(x_R(t), y_R(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, given by

$$
\begin{aligned}
x_R(t) &= -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(b_R h - b_R e^{2t\omega_{RS}} h - 2e^{t\omega_{RS}}\omega_{RS} + b_R\tau_{RS} - 2b_R e^{t\omega_{RS}}\tau_{RS} + b_R e^{2t\omega_{RS}}\tau_{RS}\Big), \\
y_R(t) &= -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(-a_R h + a_R e^{2t\omega_{RS}} h - \omega_{RS} h - e^{2t\omega_{RS}}\omega_{RS} h - a_R\tau_{RS} + 2a_R e^{t\omega_{RS}}\tau_{RS} \\
&\quad - a_R e^{2t\omega_{RS}}\tau_{RS} - \omega_{RS}\tau_{RS} + e^{2t\omega_{RS}}\omega_{RS}\tau_{RS}\Big).
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$
t_R = \frac{1}{\omega_{RS}} \log\left(\frac{h + \tau_{RS}}{\tau_{RS} - h}\right).
$$

Now, for f_R and g_R defined in (6.2) and (6.3), respectively, we have

$$\begin{aligned} I_R^0 &= \int_{\widehat{AA_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t)) x_R'(t) - f_R(x_R(t), y_R(t)) y_R'(t) \Big) dt \\ &= \int_0^{t_R} \Big(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y_R'(t) \Big) dt \\ &= \alpha_1 f_0(h) + \alpha_2 f_{RS}(h) \end{aligned} \tag{7.43}$$

with

$$\alpha_1 = \frac{1}{\omega_{RS}} \Big(2(p_{00} + p_{10})\omega_{RS} + b_R(p_{10} + q_{01})\tau_{RS} \Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RS}} \Big(p_{10} + q_{01} \Big).$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$\begin{aligned} x_{C1}(t) &= \frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1), \\ y_{C1}(t) &= -\frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1). \end{aligned}$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$t_{C1} = \log\left(\frac{h+1}{h-1} \right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned} \bar{I}_C^0 &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\ &= \int_0^{t_{C1}} \Big(g_C(x_{C1}(t), y_{C1}(t)) x_{C1}'(t) - f_C(x_{C1}(t), y_{C1}(t)) y_{C1}'(t) \Big) dt \\ &= \int_0^{t_{C1}} \Big(\sum_{i+j=0}^{1} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x_{C1}'(t) - \sum_{i+j=0}^{1} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y_{C1}'(t) \Big) dt \\ &= -2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h) \end{aligned} \tag{7.44}$$

with

$$\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{1}{2}\Big(u_{10} + v_{01} \Big).$$

The orbit $(x_L(t), y_L(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$\begin{aligned} x_L(t) &= \frac{e^{-t\omega_{LS}}}{2\omega_{LS}}\Big(b_L h - b_L e^{2t\omega_{LS}}h - 2e^{t\omega_{LS}}\omega_{LS} + b_L\tau_{LS} - 2b_L e^{t\omega_{LS}}\tau_{LS} + b_L e^{2t\omega_{LS}}\tau_{LS}\Big),\\ y_L(t) &= \frac{e^{-t\omega_{LS}}}{2\omega_{LS}}\Big(- a_L h + a_L e^{2t\omega_{LS}}h - \omega_{LS}h - e^{2t\omega_{LS}}\omega_{LS}h - a_L\tau_{LS} + 2a_L e^{t\omega_{LS}}\tau_{LS}\\ &\quad - a_L e^{2t\omega_{LS}}\tau_{LS} - \omega_{LS}\tau_{LS} + e^{2t\omega_{LS}}\omega_{LS}\tau_{LS}\Big). \end{aligned}$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \frac{1}{\omega_{LS}}\log\left(\frac{h+\tau_{LS}}{\tau_{LS}-h}\right).$$

Now, for f_L and g_L defined in (6.2) and (6.3), respectively, we have

$$\begin{aligned} I_L^0 &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy\\ &= \int_0^{t_L}\Big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\Big)dt\\ &= \int_0^{t_L}\left(\sum_{i+j=0}^{1} s_{ij}x_L^i(t)y_L^j(t)x_L'(t) - \sum_{i+j=0}^{1} r_{ij}x_L^i(t)y_L^j(t)y_L'(t)\right)dt\\ &= \alpha_5 f_0(h) + \alpha_6 f_{LS}(h) \end{aligned} \tag{7.45}$$

with

$$\alpha_5 = \frac{1}{\omega_{LS}}\Big(2(r_{10}-r_{00})\omega_{LS} + b_L(r_{10}+s_{01})\tau_{LS}\Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LS}}\Big(r_{10}+s_{01}\Big).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$\begin{aligned} x_{C2}(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1),\\ y_{C2}(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1). \end{aligned}$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \log\left(\frac{h+1}{h-1}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned} I_C^0 &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \Big(g_C(x_{C2}(t), y_{C2}(t)) x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t)) y'_{C2}(t) \Big) dt \\ &= \int_0^{t_{C2}} \left(\sum_{i+j=0}^{1} v_{ij} x_{C2}^i(t) y_{C2}^j(t) x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C2}^i(t) y_{C2}^j(t) y'_{C2}(t) \right) dt \\ &= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h). \end{aligned} \tag{7.46}$$

Hence, by Corollary 11, the first order Melnikov function M_0 associated with system (6.1) is given by

$$M_0(h) = b_R I_C^0 + \frac{b_R}{b_L} I_L^0 + b_R \bar{I}_C^0 + I_R^0. \tag{7.47}$$

Replacing (7.43), (7.44), (7.45) and (7.46) into (7.47) we obtain (6.12), with

$$k_0^0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k_0^1 = 2b_R\alpha_4, \quad k_0^2 = \alpha_2 \quad \text{and} \quad k_0^3 = \frac{b_R}{b_L}\alpha_6.$$

Now, let us simplify the Melnikov function M_1 given by (6.9) for the case $\tau_{LS} \neq \tau_{RS}$. Similarly as in (7.43), we have that

$$\begin{aligned} I_R^1 &= \int_{\widehat{BB_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t)) x'_R(t) - f_R(x_R(t), y_R(t)) y'_R(t) \Big) dt \\ &= \int_0^{t_R} \left(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x'_R(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y'_R(t) \right) dt \\ &= \alpha_1 f_0(h) + \alpha_2 f_{RS}(h). \end{aligned} \tag{7.48}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (1, -h)$, is given by

$$\begin{aligned} x_C(t) &= \frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1), \\ y_C(t) &= -\frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1). \end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $B_1(h) = (1, -h)$ to $B(h) = (1, h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}
\bar{I}_C^1 &= \int_{\widehat{B_1B}} g_C dx - f_C dy \\
&= \int_0^{t_C} (g_C(x_C(t), y_C(t))x_C'(t) - f_C(x_C(t), y_C(t))y_C'(t))dt \\
&= \int_0^{t_C} \left(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \right) dt \\
&= \alpha_7 f_0(h) + \alpha_4 f_C(h)
\end{aligned} \tag{7.49}$$

with

$$\alpha_7 = -2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_1 associated with system (6.1) is given by

$$M_1(h) = b_R I_C^1 + I_R^1. \tag{7.50}$$

Replacing (7.48) and (7.49) into (7.50) we obtain (6.13), with

$$k_1^0 = \alpha_1 + b_R\alpha_7, \quad k_1^1 = b_R\alpha_4, \quad \text{and} \quad k_1^2 = \alpha_2.$$

Now, let us simplify the Melnikov function M_2 given by (6.10) for the case $\tau_{LS} \neq \tau_{RS}$. Similarly as in (7.45), we have that

$$\begin{aligned}
I_L^2 &= \int_{\widehat{C_1C}} g_L dx - f_L dy \\
&= \int_0^{t_L} \Big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\Big)dt \\
&= \int_0^{t_L} \left(\sum_{i+j=0}^{1} s_{ij} x_L^i(t) y_L^j(t) x_L'(t) - \sum_{i+j=0}^{1} r_{ij} x_L^i(t) y_L^j(t) y_L'(t) \right) dt \\
&= \alpha_5 f_0(h) + \alpha_6 f_{LS}(h),
\end{aligned} \tag{7.51}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (-1, h)$, is given by

$$\begin{aligned}
x_C(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1), \\
y_C(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1).
\end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $C(h) = (-1, h)$ to $C_1(h) = (-1, -h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned} \bar{I}_C^2 &= \int_{\widehat{CC1}} g_C dx - f_C dy \\ &= \int_0^{t_C} \Big(g_C(x_C(t), y_C(t)) x_C'(t) - f_C(x_C(t), y_C(t)) y_C'(t) \Big) dt \\ &= \int_0^{t_C} \left(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \right) dt \\ &= \alpha_8 f_0(h) + \alpha_4 f_C(h) \end{aligned} \tag{7.52}$$

with

$$\alpha_8 = 2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_2 associated with system (6.1) is given by

$$M_2(h) = \frac{1}{b_L} I_L^2 + I_C^2. \tag{7.53}$$

Replacing (7.51) and (7.52) into (7.53) we obtain (6.14), with

$$k_2^0 = \frac{1}{b_L}\alpha_5 + \alpha_8, \quad k_2^1 = \alpha_4, \quad \text{and} \quad k_2^2 = \frac{1}{b_L}\alpha_6.$$

Finally, replacing $\tau_{LS} = \tau_{RS}$ in (6.12), (6.13) and (6.14) we obtain the expression (6.15), (6.16) and (6.17), with

$$\tilde{k}_0^0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L\omega_{LS}}\Big(2(r_{10} - r_{00})\omega_{LS} + b_L(r_{10} + s_{01})\tau_{RS}\Big), \ \tilde{k}_0^1 = 2b_R\alpha_4,$$

$$\tilde{k}_0^2 = \alpha_2 + \frac{b_R}{b_L}\alpha_6, \quad \text{and} \quad \tilde{k}_i^j = k_i^j|_{\tau_{LS}=\tau_{RS}}, \quad \text{for } i = 1, 2, \ j = 0, 1, 2.$$

□

7.4 Proof of Theorem 17 from Chapter 6

Next, we present the proof of Theorem 17 from Chapter 6.

Proof. Firstly, let us simplify the Melnikov function M_0 given by (6.8). For this purpose, consider the orbit $(x_R(t), y_R(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, given by

$$
\begin{aligned}
x_R(t) =& -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(b_R h - b_R e^{2t\omega_{RS}} h - 2e^{t\omega_{RS}}\omega_{RS} + b_R\tau_{RS} - 2b_R e^{t\omega_{RS}}\tau_{RS} + b_R e^{2t\omega_{RS}}\tau_{RS}\Big),\\
y_R(t) =& -\frac{e^{-t\omega_{RS}}}{2\omega_{RS}}\Big(- a_R h + a_R e^{2t\omega_{RS}} h - \omega_{RS} h - e^{2t\omega_{RS}}\omega_{RS} h - a_R\tau_{RS} + 2a_R e^{t\omega_{RS}}\tau_{RS}\\
& - a_R e^{2t\omega_{RS}}\tau_{RS} - \omega_{RS}\tau_{RS} + e^{2t\omega_{RS}}\omega_{RS}\tau_{RS}\Big).
\end{aligned}
$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$
t_R = \frac{1}{\omega_{RS}} \log\left(\frac{h + \tau_{RS}}{\tau_{RS} - h}\right).
$$

Now, for f_R and g_R defined in (6.2) and (6.3), respectively, we have

$$
\begin{aligned}
I_R^0 &= \int_{\widehat{AA_1}} g_R dx - f_R dy\\
&= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t))x_R'(t) - f_R(x_R(t), y_R(t))y_R'(t)\Big)dt\\
&= \int_0^{t_R} \Bigg(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y_R'(t)\Bigg)dt\\
&= \alpha_1 f_0(h) + \alpha_2 f_{RS}(h),
\end{aligned}
\tag{7.54}
$$

with

$$
\alpha_1 = \frac{1}{\omega_{RS}}\Big(2(p_{00} + p_{10})\omega_{RS} + b_R(p_{10} + q_{01})\tau_{RS}\Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RS}}(p_{10} + q_{01}).
$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \frac{e^{-t}}{2}(h + 1) - \frac{e^t}{2}(h - 1),\\
y_{C1}(t) &= -\frac{e^{-t}}{2}(h + 1) - \frac{e^t}{2}(h - 1).
\end{aligned}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$
t_{C1} = \log\left(\frac{h + 1}{h - 1}\right).
$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$
\begin{aligned}
\bar{I}_C^0 &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\
&= \int_0^{t_{C1}} \Big(g_C(x_{C1}(t), y_{C1}(t)) x'_{C1}(t) - f_C(x_{C1}(t), y_{C1}(t)) y'_{C1}(t) \Big) dt \\
&= \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x'_{C1}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y'_{C1}(t) \Bigg) dt \\
&= -\, 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h)
\end{aligned} \tag{7.55}
$$

with

$$\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{u_{10} + v_{01}}{2}.$$

The orbit $(x_L(t), y_L(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$
\begin{aligned}
x_L(t) &= -\frac{1}{\omega_{LC}} \Big(\omega_{LC} - b_L \tau_{LC} + b_L \tau_{LC} \cos(t\omega_{LC}) + b_L h \sin(t\omega_{LC}) \Big), \\
y_L(t) &= \frac{1}{\omega_{LC}} \Big(- a_L \tau_{LC} + (-\omega_{LC} h + a_L \tau_{LC}) \cos(t\omega_{LC}) + (a_L h + \omega_{LC} \tau_{LC}) \sin(t\omega_{LC}) \Big).
\end{aligned}
$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$t_L = \frac{1}{\omega_{LC}} \left(2\pi\mu_2 + (-1)^{\mu_2} \arccos\left(\frac{\tau_{LC}^2 - h^2}{\tau_{LC}^2 + h^2} \right) \right),$$

with $\mu_2 = 0$ (resp. $\mu_2 = 1$) if the center of the left subsystem from $(6.1)|_{\epsilon=0}$ is virtual (resp. real). Now, for f_L and g_L defined in (6.2) and (6.3), respectively, we have

$$
\begin{aligned}
I_L^0 &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy \\
&= \int_0^{t_L} \Big(g_L(x_L(t), y_L(t)) x'_L(t) - f_L(x_L(t), y_L(t)) y'_L(t) \Big) dt \\
&= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{1} s_{ij} x_L^i(t) y_L^j(t) x'_L(t) - \sum_{i+j=0}^{1} r_{ij} x_L^i(t) y_L^j(t) y'_L(t) \Bigg) dt \\
&= \alpha_5 f_0(h) + \alpha_6 f_{LC}(h),
\end{aligned} \tag{7.56}
$$

with

$$\alpha_5 = \frac{1}{\omega_{LC}} \Big(2(r_{10} - r_{00})\omega_{LC} - b_L(r_{10} + s_{01})\tau_{LC} \Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LC}}(r_{10} + s_{01}).$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$
\begin{aligned}
x_{C2}(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^{t}}{2}(h-1), \\
y_{C2}(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^{t}}{2}(h-1).
\end{aligned}
$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$t_{C2} = \log\left(\frac{h+1}{h-1}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}
I_C^0 &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\
&= \int_0^{t_{C2}} \Big(g_C(x_{C2}(t), y_{C2}(t)) x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t)) y'_{C2}(t) \Big) dt \\
&= \int_0^{t_{C2}} \left(\sum_{i+j=0}^{1} v_{ij} x_{C2}^i(t) y_{C2}^j(t) x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C2}^i(t) y_{C2}^j(t) y'_{C2}(t) \right) dt \\
&= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h).
\end{aligned} \tag{7.57}$$

Hence, by Corollary 11, the first order Melnikov function M_0 associated to system (6.1) is given by

$$M_0(h) = b_R I_C^0 + \frac{b_R}{b_L} I_L^0 + b_R \bar{I}_C^0 + I_R^0. \tag{7.58}$$

Replacing (7.54), (7.55), (7.56) and (7.57) into (7.58) we obtain (6.18), with

$$k_0^0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k_0^1 = 2b_R\alpha_4, \quad k_0^2 = \alpha_2 \quad \text{and} \quad k_0^3 = \frac{b_R}{b_L}\alpha_6.$$

Now, let us simplify the Melnikov function M_1 given by (6.9). Similarly as in (7.54), we have that

$$\begin{aligned}
I_R^1 &= \int_{\widehat{BB_1}} g_R dx - f_R dy \\
&= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t)) x'_R(t) - f_R(x_R(t), y_R(t)) y'_R(t) \Big) dt \\
&= \int_0^{t_R} \left(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x'_R(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y'_R(t) \right) dt \\
&= \alpha_1 f_0(h) + \alpha_2 f_{RS}(h).
\end{aligned} \tag{7.59}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (1, -h)$, is given by

$$\begin{aligned}
x_C(t) &= \frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1), \\
y_C(t) &= -\frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1).
\end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $B_1(h) = (1, -h)$ to $B(h) = (1, h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}\bar{I}_C^1 &= \int_{\widehat{B_1B}} g_C dx - f_C dy \\ &= \int_0^{t_C} \Big(g_C(x_C(t), y_C(t)) x_C'(t) - f_C(x_C(t), y_C(t)) y_C'(t) \Big) dt \\ &= \int_0^{t_C} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \Bigg) dt \\ &= \alpha_7 f_0(h) + \alpha_4 f_C(h)\end{aligned} \tag{7.60}$$

with

$$\alpha_7 = -2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_1 associated to system (6.1) is given by

$$M_1(h) = b_R I_C^1 + I_R^1. \tag{7.61}$$

Replacing (7.59) and (7.60) into (7.61) we obtain (6.19), with

$$k_1^0 = \alpha_1 + b_R \alpha_7, \quad k_1^1 = b_R \alpha_4, \quad \text{and} \quad k_1^2 = \alpha_2.$$

Finally, let us simplify the Melnikov function M_2 given by (6.10). Similarly as in (7.56), we have that

$$\begin{aligned}I_L^2 &= \int_{\widehat{C_1C}} g_L dx - f_L dy \\ &= \int_0^{t_L} \Big(g_L(x_L(t), y_L(t)) x_L'(t) - f_L(x_L(t), y_L(t)) y_L'(t) \Big) dt \\ &= \int_0^{t_L} \Bigg(\sum_{i+j=0}^{1} s_{ij} x_L^i(t) y_L^j(t) x_L'(t) - \sum_{i+j=0}^{1} r_{ij} x_L^i(t) y_L^j(t) y_L'(t) \Bigg) dt \\ &= \alpha_5 f_0(h) + \alpha_6 f_{LC}(h).\end{aligned} \tag{7.62}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (-1, h)$, is given by

$$\begin{aligned}x_C(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1), \\ y_C(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1).\end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $C(h) = (-1, h)$ to $C_1(h) = (-1, -h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}\bar{I}_C^2 &= \int_{\widehat{CC1}} g_C dx - f_C dy \\ &= \int_0^{t_C} \Big(g_C(x_C(t), y_C(t)) x_C'(t) - f_C(x_C(t), y_C(t)) y_C'(t) \Big) dt \\ &= \int_0^{t_C} \bigg(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \bigg) dt \\ &= \alpha_8 f_0(h) + \alpha_4 f_C(h)\end{aligned} \tag{7.63}$$

with

$$\alpha_8 = 2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_2 associated to system (6.1) is given by

$$M_2(h) = \frac{1}{b_L} I_L^2 + I_C^2. \tag{7.64}$$

Replacing (7.62) and (7.63) into (7.64) we obtain (6.20), with

$$k_2^0 = \frac{1}{b_L}\alpha_5 + \alpha_8, \quad k_2^1 = \alpha_4, \quad \text{and} \quad k_2^2 = \frac{1}{b_L}\alpha_6.$$

□

7.5 Proof of Theorem 18 from Chapter 6

Next, we present the proof of Theorem 18 from Chapter 6.

Proof. Firstly, let us simplify the Melnikov function M_0 given by (6.8). For this propose, consider the orbit $(x_R(t), y_R(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_R(0), y_R(0)) = (1, h)$, given by

$$\begin{aligned} x_R(t) &= \frac{1}{\omega_{RC}} \Big(\omega_{RC} - b_R \tau_{RC} + b_R \tau_{RC} \cos(t\omega_{RC}) + b_R h \sin(t\omega_{RC}) \Big), \\ y_R(t) &= \frac{1}{\omega_{RC}} \Big(a_R \tau_{RC} + (\omega_{RC} h - a_R \tau_{RC}) \cos(t\omega_{RC}) - (a_R h + \omega_{RC}\tau_{RC}) \sin(t\omega_{RC}) \Big). \end{aligned}$$

The flight time of the orbit $(x_R(t), y_R(t))$, from $A(h) = (1, h)$ to $A_1(h) = (1, -h)$, is

$$t_R = \frac{1}{\omega_{RC}} \left(2\pi\mu_1 + (-1)^{\mu_1} \arccos\left(\frac{\tau_{RC}^2 - h^2}{\tau_{RC}^2 + h^2} \right) \right),$$

with $\mu_1 = 0$ (resp. $\mu_1 = 1$) if the center of the right subsystem from $(6.1)|_{\epsilon=0}$ is virtual (resp.

real). Now, for f_R and g_R defined in (6.2) and (6.3), respectively, we have

$$
\begin{aligned}
I_R^0 &= \int_{\widehat{AA_1}} g_R dx - f_R dy \\
&= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t)) x_R'(t) - f_R(x_R(t), y_R(t)) y_R'(t) \Big) dt \\
&= \int_0^{t_R} \Bigg(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x_R'(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y_R'(t) \Bigg) dt \\
&= \alpha_1 f_0(h) + \alpha_2 f_{RC}(h)
\end{aligned}
\tag{7.65}
$$

with

$$
\alpha_1 = \frac{1}{\omega_{RC}} \Big(2(p_{00} + p_{10}) \omega_{RC} - b_R (p_{10} + q_{01}) \tau_{RC} \Big) \quad \text{and} \quad \alpha_2 = \frac{b_R}{2\omega_{RC}} (p_{10} + q_{01}).
$$

The orbit $(x_{C1}(t), y_{C1}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C1}(0), y_{C1}(0)) = (1, -h)$, is given by

$$
\begin{aligned}
x_{C1}(t) &= \frac{e^{-t}}{2}(h+1) - \frac{e^{t}}{2}(h-1), \\
y_{C1}(t) &= -\frac{e^{-t}}{2}(h+1) - \frac{e^{t}}{2}(h-1).
\end{aligned}
$$

The flight time of the orbit $(x_{C1}(t), y_{C1}(t))$, from $A_1(h) = (1, -h)$ to $A_2(h) = (-1, -h)$, is

$$
t_{C1} = \log\left(\frac{h+1}{h-1}\right).
$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$
\begin{aligned}
\bar{I}_C^0 &= \int_{\widehat{A_1A_2}} g_C dx - f_C dy \\
&= \int_0^{t_{C1}} \Big(g_C(x_{C1}(t), y_{C1}(t)) x_{C1}'(t) - f_C(x_{C1}(t), y_{C1}(t)) y_{C1}'(t) \Big) dt \\
&= \int_0^{t_{C1}} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_{C1}^i(t) y_{C1}^j(t) x_{C1}'(t) - \sum_{i+j=0}^{1} u_{ij} x_{C1}^i(t) y_{C1}^j(t) y_{C1}'(t) \Bigg) dt \\
&= -\, 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h)
\end{aligned}
\tag{7.66}
$$

with

$$
\alpha_3 = v_{01} - u_{10} \quad \text{and} \quad \alpha_4 = \frac{u_{10} + v_{01}}{2}.
$$

The orbit $(x_L(t), y_L(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_L(0), y_L(0)) = (-1, -h)$, is given by

$$
\begin{aligned}
x_L(t) &= -\frac{1}{\omega_{LC}}\Big(\omega_{LC} - b_L\tau_{LC} + b_L\tau_{LC}\cos(t\omega_{LC}) + b_L h\sin(t\omega_{LC})\Big),\\
y_L(t) &= \frac{1}{\omega_{LC}}\Big(- a_L\tau_{LC} + (-\omega_{LC}h + a_L\tau_{LC})\cos(t\omega_{LC}) + (a_L h + \omega_{LC}\tau_{LC})\sin(t\omega_{LC})\Big).
\end{aligned}
$$

The flight time of the orbit $(x_L(t), y_L(t))$, from $A_2(h) = (-1, -h)$ to $A_3(h) = (-1, h)$, is

$$
t_L = \frac{1}{\omega_{LC}}\left(2\pi\mu_2 + (-1)^{\mu_2}\arccos\left(\frac{\tau_{LC}^2 - h^2}{\tau_{LC}^2 + h^2}\right)\right),
$$

with $\mu_2 = 0$ (resp. $\mu_2 = 1$) if the center of the left subsystem from $(6.1)|_{\epsilon=0}$ is virtual (resp. real). Now, for f_L and g_L defined in (6.2) and (6.3), respectively, we have

$$
\begin{aligned}
I_L^0 &= \int_{\widehat{A_2A_3}} g_L dx - f_L dy\\
&= \int_0^{t_L}\Big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t)\Big)dt\\
&= \int_0^{t_L}\left(\sum_{i+j=0}^{1} s_{ij}x_L^i(t)y_L^j(t)x_L'(t) - \sum_{i+j=0}^{1} r_{ij}x_L^i(t)y_L^j(t)y_L'(t)\right)dt\\
&= \alpha_5 f_0(h) + \alpha_6 f_{LC}(h),
\end{aligned}
\tag{7.67}
$$

with

$$
\alpha_5 = \frac{1}{\omega_{LC}}\Big(2(r_{10} - r_{00})\omega_{LC} - b_L(r_{10} + s_{01})\tau_{LC}\Big) \quad \text{and} \quad \alpha_6 = \frac{b_L}{2\omega_{LC}}(r_{10} + s_{01}).
$$

The orbit $(x_{C2}(t), y_{C2}(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_{C2}(0),\, y_{C2}(0)) = (-1, h)$, is given by

$$
\begin{aligned}
x_{C2}(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1),\\
y_{C2}(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1).
\end{aligned}
$$

The flight time of the orbit $(x_{C2}(t), y_{C2}(t))$, from $A_3(h) = (-1, h)$ to $A(h) = (1, h)$, is

$$
t_{C2} = \log\left(\frac{h+1}{h-1}\right).
$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned} I_C^0 &= \int_{\widehat{A_3A}} g_C dx - f_C dy \\ &= \int_0^{t_{C2}} \Big(g_C(x_{C2}(t), y_{C2}(t)) x'_{C2}(t) - f_C(x_{C2}(t), y_{C2}(t)) y'_{C2}(t) \Big) dt \\ &= \int_0^{t_{C2}} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_{C2}^i(t) y_{C2}^j(t) x'_{C2}(t) - \sum_{i+j=0}^{1} u_{ij} x_{C2}^i(t) y_{C2}^j(t) y'_{C2}(t) \Bigg) dt \\ &= 2v_{00} + \alpha_3 f_0(h) + \alpha_4 f_C^0(h). \end{aligned} \tag{7.68}$$

Hence, by Corollary 11, the first order Melnikov function M_0 associated to system (6.1) is given by

$$M_0(h) = b_R I_C^0 + \frac{b_R}{b_L} I_L^0 + b_R \bar{I}_C^0 + I_R^0. \tag{7.69}$$

Replacing (7.65), (7.66), (7.67) and (7.68) into (7.69) we obtain (6.21), with

$$k_0^0 = \alpha_1 + 2b_R\alpha_3 + \frac{b_R}{b_L}\alpha_5, \quad k_0^1 = 2b_R\alpha_4, \quad k_0^2 = \alpha_2 \quad \text{and} \quad k_0^3 = \frac{b_R}{b_L}\alpha_6.$$

Let us simplify the Melnikov function M_1 given by (6.9). Similarly as in (7.65), we have that

$$\begin{aligned} I_R^1 &= \int_{\widehat{BB_1}} g_R dx - f_R dy \\ &= \int_0^{t_R} \Big(g_R(x_R(t), y_R(t)) x'_R(t) - f_R(x_R(t), y_R(t)) y'_R(t) \Big) dt \\ &= \int_0^{t_R} \Bigg(\sum_{i+j=0}^{1} q_{ij} x_R^i(t) y_R^j(t) x'_R(t) - \sum_{i+j=0}^{1} p_{ij} x_R^i(t) y_R^j(t) y'_R(t) \Bigg) dt \\ &= \alpha_1 f_0(h) + \alpha_2 f_{RC}(h). \end{aligned} \tag{7.70}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (1, -h)$, is given by

$$\begin{aligned} x_C(t) &= \frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1), \\ y_C(t) &= -\frac{e^{-t}}{2}(h+1) - \frac{e^t}{2}(h-1). \end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $B_1(h) = (1, -h)$ to $B(h) = (1, h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}\bar{I}_C^1 &= \int_{\widehat{B_1B}} g_C dx - f_C dy \\ &= \int_0^{t_C} \Big(g_C(x_C(t), y_C(t))x_C'(t) - f_C(x_C(t), y_C(t))y_C'(t) \Big) dt \\ &= \int_0^{t_C} \left(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \right) dt \\ &= \alpha_7 f_0(h) + \alpha_4 f_C(h),\end{aligned} \tag{7.71}$$

with

$$\alpha_7 = -2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_1 associated to system (6.1) is given by

$$M_1(h) = b_R I_C^1 + I_R^1. \tag{7.72}$$

Replacing (7.70) and (7.71) into (7.72) we obtain (6.22), with

$$k_1^0 = \alpha_1 + b_R\alpha_7, \quad k_1^1 = b_R\alpha_4, \quad \text{and} \quad k_1^2 = \alpha_2.$$

Finally, let us simplify the Melnikov function M_2 given by (6.10). Similarly as in (7.67), we have that

$$\begin{aligned}I_L^2 &= \int_{\widehat{C_1C}} g_L dx - f_L dy \\ &= \int_0^{t_L} \Big(g_L(x_L(t), y_L(t))x_L'(t) - f_L(x_L(t), y_L(t))y_L'(t) \Big) dt \\ &= \int_0^{t_L} \left(\sum_{i+j=0}^{1} s_{ij} x_L^i(t) y_L^j(t) x_L'(t) - \sum_{i+j=0}^{1} r_{ij} x_L^i(t) y_L^j(t) y_L'(t) \right) dt \\ &= \alpha_5 f_0(h) + \alpha_6 f_{LC}(h).\end{aligned} \tag{7.73}$$

The orbit $(x_C(t), y_C(t))$ of system $(6.1)|_{\epsilon=0}$, such that $(x_C(0), y_C(0)) = (-1, h)$, is given by

$$\begin{aligned}x_C(t) &= -\frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1), \\ y_C(t) &= \frac{e^{-t}}{2}(h+1) + \frac{e^t}{2}(h-1).\end{aligned}$$

The flight time of the orbit $(x_C(t), y_C(t))$, from $C(h) = (-1, h)$ to $C_1(h) = (-1, -h)$, is

$$t_C = \log\left(\frac{h+1}{1-h}\right).$$

Now, for f_C and g_C defined in (6.2) and (6.3), respectively, we obtain

$$\begin{aligned}
\bar{I}_C^2 &= \int_{\widehat{CC1}} g_C dx - f_C dy \\
&= \int_0^{t_C} \Big(g_C(x_C(t), y_C(t)) x_C'(t) - f_C(x_C(t), y_C(t)) y_C'(t) \Big) dt \\
&= \int_0^{t_C} \Bigg(\sum_{i+j=0}^{1} v_{ij} x_C^i(t) y_C^j(t) x_C'(t) - \sum_{i+j=0}^{1} u_{ij} x_C^i(t) y_C^j(t) y_C'(t) \Bigg) dt \\
&= \alpha_8 f_0(h) + \alpha_4 f_C(h),
\end{aligned} \tag{7.74}$$

with

$$\alpha_8 = 2u_{00} - u_{10} + v_{01}.$$

Hence, by Corollary 11, the first order Melnikov function M_2 associated to system (6.1) is given by

$$M_2(h) = \frac{1}{b_L} I_L^2 + I_C^2. \tag{7.75}$$

Replacing (7.73) and (7.74) into (7.75) we obtain (6.23), with

$$k_2^0 = \frac{1}{b_L}\alpha_5 + \alpha_8, \quad k_2^1 = \alpha_4, \quad \text{and} \quad k_2^2 = \frac{1}{b_L}\alpha_6.$$

□

7.6 Coefficients of functions (4.44) and (4.45) from Chapter 4

The coefficients list of functions (4.44) and (4.45) from Chapter 4 are the following:

$$\begin{aligned}
C_{12}^0 &= 2p_{00} + \frac{2}{3}p_{02} + 3p_{10} + \frac{14}{3}p_{20} + q_{01} + \frac{2}{3}\Big(2q_{11} - 3r_{00} - r_{02} + 6r_{10} + 5r_{11} - 19r_{20} \\
&\quad + 3s_{01} + 10s_{02} - 8s_{11}\Big) + (\pi - 2)u_{10} + (\pi + 2)v_{01} - \frac{3}{2}\Big(r_{10} + 2r_{11} - 6r_{20} + s_{01} \\
&\quad + 4s_{02} - 3s_{11}\Big) \log(3), \\
C_{12}^1 &= 2p_{00} + 2p_{02} + 3p_{10} + q_{01} - 2r_{00} - 2r_{02} + 2r_{10} + 2r_{20} + 2s_{11} + (\pi - 4)u_{10} + \pi v_{01} \\
&\quad + (p_{10} + q_{01}) \log(2) + (r_{10} - 6r_{20} + s_{01} - 3s_{11}) \log(3) + q_{11} \log(4) + r_{11}(\log(9) \\
&\quad - 2) + p_{20}(2 + \log(16)) + s_{02}(\log(81) - 4),
\end{aligned}$$

$$C_{12}^{2} = \frac{1}{6}\Big(12p_{02} + 3p_{10} + 3q_{01} - 12r_{02} + 4r_{10} + 4s_{01} - 8s_{02} + 3(\pi - 2)(u_{10} + v_{01}) + 12p_{20}$$
$$(\log(2) - 1) - 9(2r_{20} + s_{11})\log(3) + (p_{10} + q_{01})\log(8) + (r_{10} + s_{01} + 4s_{02})\log(27)$$
$$+ q_{11}(\log(64) - 6) + r_{11}(\log(729) - 4)\Big),$$
$$C_{12}^{3} = \frac{1}{226}\Big(144p_{02} + 27p_{10} - 180p_{20} + 27q_{01} - 90q_{11} - 8\Big(18r_{02} - 16r_{10} - 14r_{11} + 60r_{20}$$
$$- 16s_{01} - 28s_{02} + 30s_{11} + 9(u_{10} + v_{01})\Big)\Big),$$
$$C_{12}^{4} = \frac{1}{1296}\Big(- 27p_{10} - 108p_{20} - 27q_{01} - 54q_{11} + 8\Big(32r_{10} + 64r_{11} - 192r_{20} + 32s_{01}$$
$$+ 128s_{02} - 96s_{11} + 27(u_{10} + v_{01})\Big)\Big),$$
$$C_{13}^{0} = 2p_{00} + \frac{1}{6}\Big(4p_{02} + 18p_{10} + 5p_{12} + 28p_{20} + 45p_{30} + 6q_{01} + 3q_{03} + 8q_{11} + 11q_{21} - 12r_{00}$$
$$- 4r_{02} + 24r_{10} + 20r_{11} + 44r_{12} - 76r_{20} - 134r_{21} + 309r_{30} + 12s_{01} + 40s_{02} + 120s_{03}$$
$$- 32s_{11} - 134s_{12} + 3\big(33s_{21} + 2(\pi - 2)u_{10} - 8u_{30} + 4v_{01} + 8v_{03} + \pi\big(u_{12} + 3u_{30}$$
$$+ 2v_{01} + 3v_{03} + v_{21}\big)\big)\Big) - \frac{3}{8}\Big(4r_{10} + 8r_{11} + 16r_{12} - 24r_{20} - 54r_{21} + 117r_{30} + 4s_{01}$$
$$+ 16s_{02} + 48s_{03} - 12s_{11} - 54s_{12} + 39s_{21}\Big)\log(3),$$
$$C_{13}^{1} = 2p_{00} + 2p_{02} + 3p_{10} + 3p_{12} + 2p_{20} - 7p_{30} + q_{01} + 3q_{03} - 3q_{21} - 2r_{00} - 2r_{02} + 2r_{10}$$
$$- 2r_{12} + 2r_{20} + 16r_{21} - 28r_{30} - 4s_{02} - 12s_{03} + 2s_{11} + 16s_{12} - 10s_{21} + (\pi - 4)u_{10}$$
$$- 2(u_{12} + 5u_{30} - 3v_{03} + v_{21}) + \pi(u_{12} + 3u_{30} + v_{01} + 3v_{03} + v_{21}) + (p_{10} + 12p_{30}$$
$$+ q_{01})\log(2) + \frac{1}{2}\Big(2r_{10} - 12r_{20} - 30r_{21} + 63r_{30} + 2s_{01} + 24s_{03} - 6(s_{11} + 5s_{12})$$
$$+ 21s_{21}\Big)\log(3) + q_{11}\log(4) + r_{11}(\log(9) - 2) + (p_{20} + q_{21})\log(16) + (r_{12} + s_{02})$$
$$\log(81),$$
$$C_{13}^{2} = \frac{1}{12}\Big(24p_{02} + 6p_{10} + 39p_{12} + 6q_{01} - 63q_{21} - 24r_{02} + 8r_{10} - 8r_{11} + 8r_{12} + 72r_{21}$$
$$- 108r_{30} + 8s_{01} - 16s_{02} + 6(12s_{12} - 8s_{03} - 6s_{21} + (\pi - 2)u_{10} + \pi(2u_{12} + 6u_{30}$$
$$+ v_{01} + 6v_{03} + 2v_{21}) - 2(3u_{12} + 9u_{30} + v_{01} - 3v_{03} + 3v_{21})) + 24p_{20}(\log(2) - 1)$$
$$+ 27p_{30}(\log(4) - 7) + 6q_{11}(\log(4) - 2) + 9q_{03}(5 + \log(4)) + (2r_{10} + 4r_{11} + 8r_{12}$$
$$- 12r_{20} - 26r_{21} + 57r_{30} + 2s_{01} + 8s_{02} + 24s_{03} - 6s_{11} - 26s_{12} + 19s_{21})\log(27)$$
$$+ (p_{10} + p_{12} + q_{01} + 3q_{21})\log(64)\Big),$$

$$
\begin{aligned}
C_{13}^{3} = \frac{1}{216}\Big(&144p_{02} + 27p_{10} - 180p_{20} + 27q_{01} - 90q_{11} + 108q_{03}(4 + \log(8)) + 36p_{12}(8 \\
&+ \log(8)) - 108p_{30}(13 + \log(8)) + 4(32r_{10} - 36r_{02} + 28r_{11} + 92r_{12} - 120r_{20} \\
&+ 396r_{30} + 32s_{01} + 56s_{02} + 3(56s_{03} - 20s_{11} - 48s_{12} + 44s_{21} - 6u_{10} + 3(3\pi - 10) \\
&u_{12} + 9\pi(3(u_{30} + v_{03}) + v_{21}) - 6(15u_{30} + v_{01} + 3v_{03} + 5v_{21})) - 27(3r_{30} - 2s_{12} + s_{21}) \\
&\log(3) - 9q_{21}(13 + \log(8)) + 18r_{21}(\log(27) - 8))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{13}^{4} = \frac{1}{1296}\Big(&- 27p_{10} - 108p_{20} - 27q_{01} - 54q_{11} + 2(128r_{10} + 256r_{11} + 512r_{12} - 768r_{20} \\
&- 1128r_{21} + 2844r_{30} + 128s_{01} + 512s_{02} + 3(512s_{03} - 128s_{11} - 376s_{12} + 316s_{21} \\
&+ 9(4u_{10} + (3\pi - 8)u_{12} + 3(3\pi - 8)u_{30} + 4v_{01} + (3\pi - 8)(3v_{03} + v_{21}))) + 81(2r_{21} \\
&- 3r_{30} + 2s_{12} - s_{21})\log(3)) + 81p_{12}(3 + \log(4)) + 81q_{03}(9 + \log(64)) - 81p_{30}(13 \\
&+ \log(64)) - 27q_{21}(13 + \log(64))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{13}^{5} = \frac{1}{25920}\Big(&135p_{10} + 540p_{12} + 540p_{20} + 135q_{01} + 1620q_{03} + 270q_{11} + 32(96r_{10} + 192r_{11} \\
&+ 384r_{12} - 576r_{20} - 896r_{21} + 2208r_{30} + 96s_{01} + 384s_{02} + 1152s_{03} - 288s_{11} - 896s_{12} \\
&+ 736s_{21} - 27(2u_{10} + u_{12} + 3u_{30} + 2v_{01} + 3v_{03} + v_{21}))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{13}^{6} = \frac{1}{466560}\Big(&- 729p_{10} - 1215p_{12} - 2916p_{20} - 5103p_{30} - 729q_{01} - 3645q_{03} - 1458q_{11} \\
&- 1701q_{21} + 32(1088r_{10} + 2176r_{11} + 4352r_{12} - 6528r_{20} - 11520r_{21} + 27072r_{30} \\
&+ 1088s_{01} + 4352s_{02} + 3(4352s_{03} - 1088s_{11} - 3840s_{12} + 3008s_{21} + 81(u_{10} + u_{12} \\
&+ 3u_{30} + v_{01} + 3v_{03} + v_{21}))))\Big),
\end{aligned}
$$

$$
\begin{aligned}
C_{13}^{7} = \frac{1}{9797760}\Big(&5103p_{10} + 5103p_{12} + 20412p_{20} + 45927p_{30} + 5103q_{01} + 15309q_{03} \\
&+ 10206q_{11} + 15309q_{21} + 32(16064r_{10} + 32128r_{11} + 64256r_{12} - 96384r_{20} \\
&- 175872r_{21} + 408384r_{30} + 16064s_{01} + 64256s_{02} + 3(64256s_{03} - 16064s_{11} \\
&- 58624s_{12} + 45376s_{21} + 243(u_{10} - 3u_{12} - 9u_{30} + v_{01} - 9v_{03} - 3v_{21})))\Big).
\end{aligned}
$$

7.7 Coefficients of functions (4.46)–(4.49) from Chapter 4

The coefficients list of functions (4.46)–(4.49) from Chapter 4 are the following:

$$C_{12}^{0} = 2p_{00} + \frac{1}{3}\Big(2p_{02} + 9p_{10} + 14p_{20} + 3q_{01} + 4q_{11} - 2(3r_{00} + r_{02} + 3r_{10} - 14r_{11} + 47r_{20} + 6s_{01} - 28s_{02} + 22s_{11})\Big) + (\pi - 2)u_{10} + (2+\pi)v_{01} + 5(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}) \arccos\left(\frac{3}{5}\right),$$

$$C_{12}^{1} = 2p_{00} + 2p_{02} + 3p_{10} + q_{01} - 2r_{00} - 2r_{02} + 4r_{11} - 18r_{20} + 8s_{02} - 8s_{11} + (\pi - 4)u_{10} + \pi v_{01} + 2\left(r_{10} + (r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}) \arccos\left(\frac{3}{5}\right)\right) + (p_{10} + q_{01}) \log(2) + q_{11}\log(4) + p_{20}(2 + \log(16)),$$

$$C_{12}^{2} = \frac{1}{10}\Big(20p_{02} + 5p_{10} + 5q_{01} - 20r_{02} + 8r_{10} + 24r_{11} - 112r_{20} + 8s_{01} + 48s_{02} - 56s_{11} + 5(\pi - 2)(u_{10} + v_{01}) + 10(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}) \arccos\left(\frac{3}{5}\right) + 20p_{20}(\log(2) - 1) + 10q_{11}(\log(2) - 1) + (p_{10} + q_{01}) \log(32)\Big),$$

$$C_{12}^{3} = \frac{1}{24}\Big(16p_{02} + 3p_{10} - 20p_{20} + 3q_{01} - 10q_{11}\Big) + \frac{1}{75}\Big(32r_{10} - 50r_{02} + 36r_{11} - 208r_{20} + 32s_{01} + 72s_{02} - 104s_{11} - 25(u_{10} + v_{01})\Big),$$

$$C_{12}^{4} = \frac{1}{6000}\Big(- 125p_{10} - 500p_{20} - 125q_{01} - 250q_{11} - 8\big(64r_{10} - 128r_{11} + 384r_{20} + 64s_{01} - 256s_{02} + 192s_{11} - 125(u_{10} + v_{01})\big)\Big),$$

$$C_{13}^{0} = 2p_{00} + \frac{1}{6}\Big(4p_{02} + 18p_{10} + 5p_{12} + 28p_{20} + 45p_{30} + 6q_{01} + 3q_{03} + 8q_{11} + 11q_{21} - 6r_{00} - 2r_{02} - 6r_{10} + 28r_{11} - 88r_{12} - 94r_{20} + 236r_{21} - 570r_{30} - 12s_{01} + 56s_{02} - 270s_{03} - 44s_{11} + 236s_{12} + 3(-64s_{21} + 2(\pi - 2)u_{10} - 8u_{30} + 4v_{01} + 8v_{03} + \pi(u_{12} + 3u_{30} + 2v_{01} + 3v_{03} + v_{21}))\Big) + \frac{5}{4}(2r_{10} - 4r_{11} + 13r_{12} + 12r_{20} - 34r_{21} + 84r_{30} + 2s_{01} - 8s_{02} + 39s_{03} + 6s_{11} - 34s_{12} + 28s_{21}) \arccos\left(\frac{3}{5}\right),$$

$$C_{13}^{1} = 2p_{00} + 2p_{02} + 3p_{10} + 3p_{12} + 2p_{20} - 7p_{30} + q_{01} + 3q_{03} - 3q_{21} - r_{00} - r_{02} + r_{10} + 2r_{11} - 7r_{12} - 9r_{20} + 20r_{21} - 47r_{30} + 4s_{02} - 24s_{03} - 4s_{11} + 20s_{12} - 16s_{21} + (\pi - 4)u_{10} - 2(u_{12} + 5u_{30} - 3v_{03} + v_{21}) + \pi(u_{12} + 3u_{30} + v_{01} + 3v_{03} + v_{21}) + (r_{10} - 2r_{11} + 9r_{12} + 6r_{20} - 22r_{21} + 57r_{30} + s_{01} - 4s_{02} + 27s_{03} + 3s_{11} - 22s_{12} + 19s_{21}) \arccos\left(\frac{3}{5}\right) + (p_{10} + 12p_{30} + q_{01}) \log(2) + q_{11}\log(4) + (p_{20} + q_{21}) \log(16),$$

$$C_{13}^2 = \frac{1}{20}\Big(40p_{02} + 10p_{10} + 65p_{12} - 40p_{20} - 315p_{30} + 10q_{01} + 75q_{03} - 20q_{11} - 105q_{21}$$
$$+ 2(4r_{10} - 10r_{02} + 12r_{11} - 34r_{12} - 56r_{20} + 112r_{21} - 252r_{30} + 4s_{01} + 24s_{02} - 132s_{03}$$
$$- 28s_{11} + 112s_{12} - 84s_{21} + 5((\pi - 2)u_{10} + \pi(2u_{12} + 6u_{30} + v_{01} + 6v_{03} + 2v_{21})$$
$$- 2(3u_{12} + 9u_{30} + v_{01} - 3v_{03} + 3v_{21}))) + \big(10r_{10} - 20r_{11} + 110r_{12} + 60r_{20} - 260r_{21}$$
$$+ 690r_{30} + 10s_{01} - 40s_{02} + 330s_{03} + 30s_{11} - 260s_{12} + 230s_{21}\big) \arccos\left(\frac{3}{5}\right) + \big(10p_{10}$$
$$+ 10p_{12} + 40p_{20} + 90p_{30} + 10q_{01} + 30q_{03} + 20q_{11} + 30q_{21}\big) \log(2)\Big),$$

$$C_{13}^3 = \frac{1}{600}\Big(400p_{02} + 75p_{10} + 800p_{12} - 500p_{20} - 3900p_{30} + 75q_{01} + 1200q_{03} - 250q_{11}$$
$$- 1300q_{21} - 200r_{02} + 128r_{10} + 144r_{11} + 72r_{12} - 832r_{20} + 544r_{21} - 384r_{30} + 128s_{01}$$
$$+ 288s_{02} - 384s_{03} - 416s_{11} + 544s_{12} - 128s_{21} - 200u_{10} - 1000u_{12} + 300\pi u_{12}$$
$$- 3000u_{30} + 900\pi u_{30} - 200v_{01} - 600v_{03} + 900\pi v03 - 1000v_{21} + 300\pi v_{21} + (600r_{12}$$
$$- 1200r_{21} + 3600r_{30} + 1800s_{03} - 1200s_{12} + 1200s_{21}) \arccos\left(\frac{3}{5}\right) + (300p_{12} - 900p_{30}$$
$$+ 900q_{03} - 300q_{21}) \log(2)\Big),$$

$$C_{13}^4 = \frac{1}{6000}\Big(3375q_{03} - 125p_{10} - 500p_{20} - 4875p_{30} - 125q_{01} - 250q_{11} - 1625q_{21} + 2\big(256r_{11}$$
$$- 128r_{10} + 408r_{12} - 768r_{20} - 304r_{21} + 2064r_{30} - 128s_{01} + 512s_{02} + 1224s_{03}$$
$$- 384s_{11} - 304s_{12} + 688s_{21} + 125(4u_{10} + 4v_{01} + (3\pi - 8)(u_{12} + 3u_{30} + 3v_{03} + v_{21}))\big)$$
$$+ (1500r_{12} - 3000r_{21} + 9000r_{30} + 4500s_{03} - 3000s_{12} + 3000s_{21}) \arccos\left(\frac{3}{5}\right)$$
$$+ (2250q_{03} - 2250p_{30} - 750q_{21}) \log(2) + 375p_{12}(3 + \log(4))\Big),$$

$$C_{13}^5 = \frac{1}{192}\Big(p_{10} + 4p_{12} + 4p_{20} + q_{01} + 12q_{03} + 2q_{11}\Big) + \frac{1}{18750}\Big(32r_{10} - 64r_{11} + 1408r_{12}$$
$$+ 192r_{20} - 2944r_{21} + 8544r_{30} + 32s_{01} - 128s_{02} + 4224s_{03} + 96s_{11} - 2944s_{12}$$
$$+ 2848s_{21} - 625(2u_{10} + u_{12} + 3u_{30} + 2v_{01} + 3v_{03} + v_{21})\Big),$$

$$C_{13}^6 = \frac{1}{6000000}\Big(- 9375p_{10} - 15625p_{12} - 37500p_{20} - 65625p_{30} - 9375q_{01} - 46875q_{03}$$
$$- 18750q_{11} - 21875q_{21} + 14336r_{10} - 28672r_{11} - 24576r_{12} + 86016r_{20} - 8192r_{21}$$
$$- 104448r_{30} + 14336s_{01} - 57344s_{02} - 73728s_{03} + 43008s_{11} - 8192s_{12} - 34816s_{21}$$
$$+ 100000u_{10} + 100000u_{12} + 300000u_{30} + 100000v_{01} + 300000v_{03} + 100000v_{21}\Big),$$

$$C_{13}^{7} = \frac{1}{42000000}\Big(21875p_{10} + 21875p_{12} + 87500p_{20} + 196875p_{30} + 21875q_{01} + 65625q_{03} + 43750q_{11} + 65625q_{21} - 34816r_{10} + 69632r_{11} - 90112r_{12} - 208896r_{20} + 319488r_{21} - 645120r_{30} - 34816s_{01} + 139264s_{02} - 270336s_{03} - 104448s_{11} + 319488s_{12} - 215040s_{21} + 100000u_{10} - 300000u_{12} - 900000u_{30} + 100000v_{01} - 900000v_{03} - 300000v_{21}\Big),$$

$$C_{22}^{0} = 2p_{00} + \frac{1}{3}\Big(2p_{02} + 9p_{10} + 14p_{20} + 3q_{01} + 4q_{11} - 6r_{00} - 2r_{02} + 3(4+3\pi)r_{10} + 10r_{11} - 58r_{20} + 6s_{01} + 20s_{02} + 9\pi(r_{11} - 6r_{20} + s_{01} + 2s_{02} - 3s_{11}) - 26s_{11}\Big) + (\pi - 2)u_{10} + (2+\pi)v_{01},$$

$$C_{22}^{1} = 2p_{00} + 2p_{02} + 3p_{10} + q_{01} - 2r_{00} - 2r_{02} + (2+3\pi)r_{10} + 4r_{11} - 18r_{20} + 8s_{02} - 8s_{11} - 4u_{10} + \pi(3r_{11} - 18r_{20} + 3s_{01} + 6s_{02} - 9s_{11} + u_{10} + v_{01}) + (p_{10} + q_{01})\log(2) + q_{11}\log(4) + p_{20}(2 + \log(16)),$$

$$C_{22}^{2} = \frac{1}{2}\Big(4p_{02} + p_{10} + q_{01} - 4r_{02} + (3\pi - 2)r_{10} + 3(2+\pi)r_{11} + \pi(3s_{01} - 18r_{20} + 6s_{02} - 9s_{11} + u_{10} + v_{01}) - 2(10r_{20} + s_{01} - 6s_{02} + 5s_{11} + u_{10} + v_{01}) + (p_{10} + q_{01})\log(2) + q_{11}(\log(4) - 2) + p_{20}(\log(16) - 4)\Big),$$

$$C_{22}^{3} = \frac{1}{24}\Big(16p_{02} + 3p_{10} - 20p_{20} + 3q_{01} - 10q_{11} - 8(2r_{02} + r_{10} - 3r_{11} + 10r_{20} + s_{01} - 6s_{02} + 5s_{11} + u_{10} + v_{01})\Big),$$

$$C_{22}^{4} = \frac{1}{48}\Big(-p_{10} - 4p_{20} - q_{01} - 2q_{11} + 8(r_{10} + r_{11} - 6r_{20} + s_{01} + 2s_{02} - 3s_{11} + u_{10} + v_{01})\Big),$$

$$C_{23}^{0} = \frac{1}{6}\Big(12p_{00} + 4p_{02} + 18p_{10} + 5p_{12} + 28p_{20} + 45p_{30} + 6q_{01} + 3q_{03} + 8q_{11} + 11q_{21} - 6r_{00} - 2r_{02} + 3(4+3\pi)r_{10} + 2\big(5r_{11} + 10r_{12} - 29r_{20} - 38r_{21} + 156r_{30} + 3s_{01} + 10s_{02} + 27s_{03} - 13s_{11} - 38s_{12} + 51s_{21} - 6u_{10} - 12u_{30} + 6v_{01} + 12v_{03}\big) + 3\pi\big(3r_{11} + 6r_{12} - 18r_{20} - 24r_{21} + 99r_{30} + 3s_{01} + 6s_{02} + 18s_{03} - 9s_{11} - 24s_{12} + 33s_{21} + 2u_{10} + u_{12} + 3u_{30} + 2v_{01} + 3v_{03} + v_{21}\big)\Big),$$

$$C_{23}^{1} = 2p_{00} + 2p_{02} + 3p_{10} + 3p_{12} + 2p_{20} - 7p_{30} + q_{01} + 3q_{03} - 3q_{21} - r_{00} - r_{02} + r_{10} + \frac{3\pi r_{10}}{2} + 2r_{11} + 5r_{12} - 9r_{20} - 16r_{21} + 61r_{30} + 4s_{02} + 12s_{03} - 2(2s_{11} + 8s_{12} - 10s_{21} + 2u_{10} + u_{12} + 5u_{30} - 3v_{03} + v_{21}) + \frac{\pi}{2}\Big(3r_{11} + 9r_{12} - 18r_{20} - 30r_{21} + 117r_{30} + 3s_{01} + 6s_{02} + 27s_{03} - 9s_{11} - 30s_{12} + 39s_{21} + 2\big(u_{10} + u_{12} + 3u_{30} + v_{01} + 3v_{03} + v_{21}\big)\Big) + (p_{10} + 12p_{30} + q_{01})\log(2) + q_{11}\log(4) + (p_{20} + q_{21})\log(16),$$

$$C_{23}^2 = \frac{1}{4}\Big(8p_{02} + 2p_{10} + 13p_{12} - 63p_{30} + 2q_{01} + 15q_{03} - 21q_{21} - 4r_{02} + (3\pi - 2)r10 +$$
$$\pi\big(3r_{11} + 15r_{12} - 18r_{20} - 42r_{21} + 153r_{30} + 3s_{01} + 6s_{02} + 45s_{03} - 9s_{11} - 42s_{12}$$
$$+ 51s_{21} + 2(u_{10} + 2u_{12} + 6u_{30} + v_{01} + 6v_{03} + 2v_{21})\big) + 2\big(3r_{11} + 7r_{12} - 10r_{20}$$
$$- 22r_{21} + 81r_{30} - s_{01} + 6s_{02} + 15s_{03} - 5s_{11} - 22s_{12} + 27s_{21} - 2(u_{10} + 3u_{12}$$
$$+ 9u_{30} + v_{01} - 3v_{03} + 3v_{21})\big) + 8p_{20}(\log(2) - 1) + (p_{10} + p_{12} + 9p_{30} + q_{01})\log(4)$$
$$+ q11(\log(16) - 4) + (q_{03} + q_{21})\log(64)\Big),$$

$$C_{23}^3 = \frac{1}{24}\Big(16p_{02} + 3p_{10} - 20p_{20} + 3q_{01} - 10q_{11} + 12q_{03}(4 + \log(8)) + 4p_{12}(8 + \log(8))$$
$$- 12p_{30}(13 + \log(8)) - 4\big(2r_{02} + r_{10} - 3r_{11} + 10r_{20} + 14r_{21} - 57r_{30} + s_{01} - 6s_{02}$$
$$- 3s_{03} + 5s_{11} + 14s_{12} - 19s_{21} + 2(u_{10} + 5u_{12} + 15u_{30} + v_{01} + 3v_{03} + 5v_{21}) - 3(r_{12}$$
$$+ 3\pi r_{12} + \pi(18r_{30} - 6r_{21} + 9s_{03} - 6s_{12} + 6s_{21} + u_{12} + 3(u_{30} + v_{03}) + v_{21})) + q_{21}$$
$$(13 + \log(18)))\Big),$$

$$C_{23}^4 = \frac{1}{48}\Big(27q_{03} - p_{10} - 4p_{20} - 39p_{30} - q_{01} - 2q_{11} - 13q_{21} + 4r_{10} + 4r_{11} + 4\big(2r_{21} - 6r_{20}$$
$$+ 3r_{30} + s_{01} + 2s_{02} - 9s_{03} - 3s_{11} + 2s_{12} + s_{21} + 2(u_{10} - 2u_{12} - 6u_{30} + v_{01} - 6v_{03}$$
$$- 2v_{21})\big) + 6\big((3\pi - 2)r_{12} + \pi(18r_{30} - 6r_{21} + 9s_{03} - 6s_{12} + 6s_{21} + u_{12} + 3(u_{30}$$
$$+ v_{03}) + v_{21})\big) - 6(3p_{30} - 3q_{03} + q_{21})\log(2) + p_{12}(9 + \log(64))\Big),$$

$$C_{23}^5 = \frac{1}{960}\Big(5p_{10} + 20p_{12} + 20p_{20} + 5q_{01} + 60q_{03} + 10q_{11} - 32\big(r_{10} + r_{11} + 2r_{12} - 6r_{20} - 8r_{21}$$
$$+ 33r_{30} + s_{01} + 2s_{02} + 6s_{03} - 3s_{11} - 8s_{12} + 11s_{21} + 2u_{10} + u_{12} + 3u_{30} + 2v_{01} + 3v_{03}$$
$$+ v_{21}\big)\Big),$$

$$C_{23}^6 = \frac{1}{1920}\Big(-3p_{10} - 5p_{12} - 12p_{20} - 21p_{30} - 3q_{01} - 15q_{03} - 6q_{11} - 7q_{21} + 16\big(r_{10} + r_{11}$$
$$+ 3r_{12} - 6r_{20} - 10r_{21} + 39r_{30} + s_{01} + 2s_{02} + 9s_{03} - 3s_{11} - 10s_{12} + 13s_{21} + 2(u_{10}$$
$$+ u_{12} + 3u_{30} + v_{01} + 3v_{03} + v_{21})\big)\Big),$$

$$C_{23}^7 = \frac{1}{13440}\Big(7p_{10} + 7p_{12} + 28p_{20} + 63p_{30} + 7q_{01} + 21q_{03} + 14q_{11} + 21q_{21} + 16\big(r_{10} + r_{11}$$
$$- 5r_{12} - 6r_{20} + 6r_{21} - 9r_{30} + s_{01} + 2s_{02} - 15s_{03} - 3s_{11} + 6s_{12} - 3s_{21} + 2\big(u_{10}$$
$$- 3u_{12} - 9u_{30} + v_{01} - 9v_{03} - 3v_{21}\big)\big)\Big).$$

7.8 Coefficients of functions (6.38)–(6.40) from Chapter 6

The coefficients list of functions (6.38)–(6.40) from Chapter 6 are the following:

$$C_0^0 = 2(p_{00}+p_{10}) + \frac{2b_R}{b_L}(r_{10}-r_{00}) + b_R\Big(2v_{01} - 2u_{10} + \frac{1}{\omega_{LC}}(r_{10}+s_{01})(\pi\mu_2(1+\tau_{LC}^2)$$
$$- \tau_{LC}) + \frac{\tau_{RS}}{\omega_{RS}}(p_{10}+q_{01})\Big) + \frac{(-1)^{\mu_2} b_R}{2\omega_{LC}}(r_{10}+s_{01})(\tau_{LC}^2+1)\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right)$$
$$- \frac{b_R}{2\omega_{RS}}(p_{10}+q_{01})(\tau_{RS}^2-1)\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right),$$

$$C_0^1 = 2(p_{00}+p_{10}) + \frac{(-1)^{\mu_2} b_R}{\omega_{LC}}(r_{10}+s_{01})\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right) + b_R\Big(\frac{2}{b_L}(r_{10}-r_{00}) + \frac{1}{\omega_{LC}}$$
$$(r_{10}+s_{01})(2\pi\mu_2 + ((-1)^{\mu_2}-1)\tau_{LC}) + u_{10}(\log(4)-2) + v_{01}(2+\log(4))\Big)$$
$$+ \frac{b_R}{\omega_{RS}}(p_{10}+q_{01})\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right),$$

$$C_0^2 = \frac{b_R}{2\omega_{LC}\omega_{RS}}\Bigg(2\Big(\pi(r_{10}+s_{01})\mu_2\omega_{RS} + (u_{10}+v_{01})\omega_{LC}\omega_{RS} + \frac{(-1)^{\mu_2}\tau_{LC}\omega_{RS}}{\tau_{LC}^2+1}(r_{10}+s_{01})$$
$$\frac{\tau_{RS}\omega_{LC}}{\tau_{RS}^2-1}(p_{10}+q_{01})\Big) + (-1)^{\mu_2}(r_{10}+s_{01})\omega_{RS}\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right) + (u_{10}+v_{01})\omega_{LC}$$
$$\omega_{RS}\log(4) + (p_{10}+q_{01})\omega_{LC}\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg),$$

$$C_0^3 = \frac{b_R}{12}\Big(3(u_{10}+v_{01}) + \frac{8(-1)^{\mu_2}\tau_{LC}^3}{\omega_{LC}(\tau_{LC}^2+1)^2}(r_{10}+s_{01}) + \frac{8\tau_{RS}^3}{\omega_{RS}(\tau_{RS}^2-1)^2}(p_{10}+q_{01}),\Big)$$

$$C_1^0 = 2(p_{00}+p_{10}) + b_R(v_{01} - 2u_{00} - u_{10}) + \frac{b_R\tau_{RS}}{\omega_{LC}}(p_{10}+q_{01}) - \frac{b_R}{2\omega_{RS}}(p_{10}+q_{01})$$
$$(\tau_{RS}^2-1)\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right),$$

$$C_1^1 = 2(p_{00}+p_{10}) + b_R\Big(v_{01} - 2u_{00} - u_{10} + (u_{10}+v_{01})\log(2)\Big) + \frac{b_R}{\omega_{RS}}(p_{10}+q_{01})$$
$$\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right),$$

$$C_1^2 = \frac{b_R}{2}\Bigg(\frac{2\tau_{RS}}{\omega_{RS}(\tau_{RS}^2-1)}(p_{10}+q_{01}) + (u_{10}+v_{01})(1+\log(2)) + \frac{1}{\omega_{RS}}(p_{10}+q_{01})$$
$$\log\left(\frac{\tau_{RS}+1}{\tau_{RS}-1}\right)\Bigg),$$

$$C_2^0 = \frac{2}{b_L}(r_{10}-r_{00}) + 2u_{00} - u_{10} + v_{01} + \frac{1}{\omega_{LC}}(r_{10}+s_{01})(\pi\mu_2(1+\tau_{LC}^2) - \tau_{LC})$$
$$\frac{(-1)^{\mu_2}(\tau_{LC}^2+1)}{2\omega_{LC}}(r_{10}+s_{01})\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right),$$

$$C_2^1 = \frac{2}{b_L}(r_{10} - r_{00}) + 2u_{00} - u_{10} + v_{01} + \frac{1}{\omega_{LC}}(r_{10} + s_{01})(2\pi\mu_2 + ((-1)^{\mu_2} - 1)$$
$$\tau_{LC}) + (u_{10} + v_{01})\log(2) + \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10} + s_{01})\arccos\left(\frac{\tau_{LC}^2 - 1}{\tau_{LC}^2 + 1}\right),$$
$$C_2^2 = \frac{1}{2}\left(\frac{2}{\omega_{LC}}\left(\pi\mu_2 + \frac{(-1)^{\mu_2}\tau_{LC}}{\tau_{LC}^2 + 1}\right)(r_{10} + s_{01}) + \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10} + s_{01})\right.$$
$$\left.\arccos\left(\frac{\tau_{LC}^2 - 1}{\tau_{LC}^2 + 1}\right) + (u_{10} + v_{01})(1 + \log(2))\right).$$

7.9 Coefficients of functions (6.41)–(6.43) from Chapter 6

The coefficients list of functions (6.41)–(6.43) from Chapter 6 are the following:

$$C_0^0 = \frac{1}{2}\left(4(p_{00} + p_{10}) + 2b_R\left(\frac{2}{b_L}(r_{10} - r_{00}) - 2(u_{10} + v_{01}) - \frac{1}{\omega_{LC}}(r_{10} + s_{01})(\pi\mu_2(1 + \tau_{LC}^2)\right.\right.$$
$$\left. - \tau_{LC})\right) + \frac{2b_R}{\omega_{RC}}(p_{10} + q_{01})(\pi\mu_1(1 + \tau_{RC}^2) - \tau_{RC}) + \frac{(-1)^{\mu_2}b_R}{\omega_{LC}}(r_{10} + s_{01})(\tau_{LC}^2 + 1)$$
$$\left.\arccos\left(\frac{\tau_{LC}^2 - 1}{\tau_{LC}^2 + 1}\right) + \frac{(-1)^{\mu_1}b_R}{\omega_{RC}}(p_{10} + q_{01})(\tau_{RC}^2 + 1)\arccos\left(\frac{\tau_{RC}^2 - 1}{\tau_{RC}^2 + 1}\right)\right),$$

$$C_0^1 = \frac{1}{b_L\omega_{LC}\omega_{RC}}\Bigg(\omega_{LC}\omega_{RC}(2b_L(p_{00} + p_{10}) + 2b_R(r_{10} - r_{00}) + 2b_Lb_R(v_{01} - u_{10})) + b_Lb_R$$
$$\Bigg(2\pi(r_{10} + s_{01})\mu_2\omega_{RC} + (-r_{10} - s_{01} + (-1)^{\mu_2}(r_{10} + s_{01}))\tau_{LC}\omega_{RC} + p_{10}\omega_{LC}(2\pi\mu_1$$
$$+ (-1 + (-1)^{\mu_1})\tau_{LC}) + q_{01}(2\pi\mu_1\omega_{LC} - \omega_{LC}\tau_{RC} + (-1)^{\mu_1}\omega_{LC}\tau_{RC})\Bigg) + b_Lb_R\Bigg((-1)^{\mu_2}$$
$$(r_{10} + s_{01})\omega_{RC}\arccos\left(\frac{\tau_{LC}^2 - 1}{\tau_{LC}^2 + 1}\right) + (-1)^{\mu_1}(p_{10} + q_{01})\omega_{LC}\arccos\left(\frac{\tau_{RC}^2 - 1}{\tau_{RC}^2 + 1}\right)$$
$$+ (u_{10} + v_{01})\omega_{LC}\omega_{RC}\log(4)\Bigg)\Bigg),$$

$$C_0^2 = \frac{b_L}{2}\Bigg(2u_{10} + 2\Bigg(v_{01} + \frac{1}{\omega_{RC}}\Bigg(\frac{1}{\omega_{LC}}\Bigg(\pi(p_{10} + q_{01})\mu_1\omega_{LC} + \pi(r_{10} + s_{01})\mu_2\omega_{RC} + \frac{(-1)^{\mu_2}}{\tau_{LC}^2 + 1}$$
$$\tau_{LC}\omega_{LC}(r_{10} + s_{01})\Bigg) + \frac{(-1)^{\mu_1}\tau_{RC}}{\tau_{RC}^2 + 1}(p_{10} + q_{01})\Bigg)\Bigg) + \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10} + s_{01})$$
$$\arccos\left(\frac{\tau_{LC}^2 - 1}{\tau_{LC}^2 + 1}\right) + \frac{(-1)^{\mu_1}}{\omega_{RC}}(p_{10} + q_{01})\arccos\left(\frac{\tau_{RC}^2 - 1}{\tau_{RC}^2 + 1}\right) + (u_{10} + v_{01})\log(4)\Bigg),$$
$$C_0^3 = \frac{b_R}{12}\left(3(u_{10} + v_{01}) + \frac{8(-1)^{\mu_2}\tau_{LC}^3}{\omega_{LC}(\tau_{LC}^2 + 1)^2}(r_{10} + s_{01}) + \frac{8(-1)^{\mu_1}\tau_{RC}^3}{\omega_{RC}(\tau_{RC}^2 - 1)^2}(p_{10} + q_{01})\right),$$
$$C_1^0 = \frac{1}{2\omega_{RC}}\Bigg(4(p_{00} + p_{10})\omega_{RC} - 2b_R(2u_{00} + u_{10} - v_{01})\omega_{RC} + 2b_R(p_{10} + q_{01})(\pi\mu_1(1 + \tau_{RC}^2)$$
$$- \tau_{RC}) + (-1)^{\mu_1}b_R(p_{10} + q_{01})(1 + \tau_{RC}^2)\arccos\left(\frac{\tau_{RC}^2 - 1}{\tau_{RC}^2 + 1}\right)\Bigg),$$

$$C_1^1 = \frac{1}{\omega_{RC}}\Bigg(2(p_{00}+p_{10})\omega_{RC} + b_R(p_{10}+q_{01})(2\pi\mu_1 + ((-1)^{\mu_1}-1)\tau_{RC}) + (-1)^{\mu_1} b_R$$
$$(p_{10}+q_{01})\arccos\left(\frac{\tau_{RC}^2-1}{\tau_{RC}^2+1}\right) + b_R\omega_{RC}(v_{01} - 2u_{00} - u_{10} + (u_{10}+v_{01})\log(2))\Bigg),$$
$$C_1^2 = \frac{b_R}{2\omega_{RC}(\tau_{RC}^2+1)}\Bigg((-1)^{\mu_1}(p_{10}+q_{01})\Bigg(2\tau_{RC} + (1+\tau_{RC}^2)\arccos\left(\frac{\tau_{RC}^2-1}{\tau_{RC}^2+1}\right)\Bigg)$$
$$+ (1+\tau_{RC}^2)\Big(2\pi(p_{10}+q_{01})\mu_1 + (u_{10}+v_{01})\omega_{RC}(1+\log(2))\Big)\Bigg),$$

$$C_2^0 = \frac{2}{b_L}(r_{10}-r_{00}) + 2u_{00} - u_{10} + v_{01} + \frac{1}{\omega_{LC}}(r_{10}+s_{01})(\pi\mu_2(1+\tau_{LC}^2) - \tau_{LC}) + \frac{(-1)^{\mu_2}}{2\omega_{LC}}$$
$$(r_{10}+s_{01})(\tau_{LC}^2+1)\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right),$$
$$C_2^1 = \frac{2}{b_L}(r_{10}-r_{00}) + 2u_{00} - u_{10} + v_{01} + \frac{1}{\omega_{LC}}(r_{10}+s_{01})(2\pi\mu_2 + ((-1)^{\mu_2}-1)\tau_{LC})$$
$$+ \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10}+s_{01})\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right) + (u_{10}+v_{01})\log(2),$$
$$C_2^2 = \frac{1}{2}\Bigg(\frac{2}{\omega_{LC}}(r_{10}+s_{01})\left(\pi\mu_2 + \frac{(-1)^{\mu_2}\tau_{LC}}{\tau_{LC}^2+1}\right) + \frac{(-1)^{\mu_2}}{\omega_{LC}}(r_{10}+s_{01})\arccos\left(\frac{\tau_{LC}^2-1}{\tau_{LC}^2+1}\right)$$
$$+ (u_{10}+v_{01})(1+\log(2))\Bigg).$$

7.10 Parameter values to function (4.44) from Chapter 4

Next, we provide the perturbation parameter values of system (4.1) such that the coefficients of function (4.44) from Chapter 4 satisfies $C_{12}^i = 0$, for $i = 0, \ldots, 3$, and $C_{12}^4 \neq 0$.

$$p_{00} = \frac{1}{6}\Big(6r_{00} - 2p_{02} - 9p_{10} - 14p_{20} - 3q_{01} - 4q_{11} + 2r_{02} - 12r_{10} - 10r_{11} + 38r_{20} - 6s_{01}$$
$$- 20s_{02} + 16s_{11} + 6u_{10} - 3\pi u_{10} - 3(2+\pi)v_{01}\Big) + \frac{3\log(3)}{4}\Big(r_{10} + 2r_{11} - 6r_{20} + s_{01}$$
$$+ 4s_{02} - 3s_{11}\Big),$$
$$p_{10} = \frac{1}{\log(60)}\Big(16p_{20} - 8p_{02} + 8q_{11} + 8r_{02} + 12r_{10} + 4(8r_{11} - 22r_{20} + 3s_{01} + 16s_{02} - 11s_{11}$$
$$+ 3(u_{10}+v_{01})) - 15(r_{10} + 2r_{11} - 6r_{20} + s_{01} + 4s_{02} - 3s_{11})\log(3) - 4(2p_{20} + q_{11})$$
$$\log(8) - q_{01}\log(64)\Big),$$
$$p_{20} = \frac{1}{32\log(2) - 16}\Big(8r_{02} + 12r_{10} + 32r_{11} - 88r_{20} + 12s_{01} + 64s_{02} - 44s_{11} + 24r_{11}\log(2)$$
$$+ 20(r_{10}+s_{01})\log(2) - 4(4r_{02} + 22r_{20} - 12s_{02} + 11s_{11})\log(2) + 6(u_{10}+v_{01})(2$$
$$+ \pi\log(2)) + 8p_{02}(\log(4)-1) - 8q_{11}(\log(4)-1) - 3(r_{10} + 2r_{11} - 6r_{20} + s_{01} + 4s_{02}$$
$$- 3s_{11})\log(3)(5+\log(8))\Big),$$

$$
\begin{aligned}
s_{01} = \frac{1}{\lambda}\Big(&2(876r_{20} - 584s_{02} + 438s_{11} + 9(3\pi - 4)(u_{10} + v_{01}) - 4(104s_{02} - 156r_{20} - 78s_{11}\\
&+ 9(4 + 3\pi)(u_{10} + v_{01}))\log(2)) + r_{10}(189\log(3) - 292 + 4\log(2)(81\log(3) - 52))\\
&- 27(6r_{20} - 4s_{02} + 3s_{11})\log(3)(7 + \log(4096)) + r_{11}(-584 - 416\log(2) + 54\log(3)\\
&(7 + \log(4096)))\Big),\\
u_{10} \neq -v_{01},&
\end{aligned}
$$

with $\lambda = 292 + 4\log(2)(52 - 81\log(3)) - 189\log(3)$.

7.11 Parameter values to function (4.45) from Chapter 4

Next, we provide the perturbation parameter values of system (4.1) such that the coefficients of function (4.45) from Chapter 4 satisfies $C^i_{12} = 0$, for $i = 0, \ldots, 7$.

$$
\begin{aligned}
q_{11} &= -\frac{p_{10}}{2} - 2p_{12} - 2p_{20} - \frac{q_{01}}{2} - 6q_{03},\\
q_{21} &= p_{12} - 3p_{30} + 3q_{03},\\
r_{00} &= p_{00} + p_{10} + p_{20} + p_{30} + r_{10} - r_{20} + r_{30} - 2(u_{10} + u_{30}),\\
r_{02} &= \frac{1}{6}\Big(6p_{02} + 3p_{10} + 3q_{01} - 18q_{03} + 2r_{10} - 2r_{11} + 2r_{12} + 2s_{01} - 4s_{02} - 12s_{03} + 36v_{03}\Big),\\
s_{11} &= \frac{1}{3}\Big(r_{10} + 2r_{11} + 4r_{12} - 6r_{20} + 6r_{21} + s_{01} + 4s_{02} + 12s_{03} + 6s_{12}\Big),\\
s_{21} &= 2r_{21} - 3r_{30} + 2s_{12},\\
v_{01} &= -u_{10},\\
v_{21} &= -u_{12} - 3(u_{30} + v_{03}).
\end{aligned}
$$

7.12 Parameter values to functions (4.46) – (4.49) from Chapter 4

Next, we provide the perturbation parameter values of system (4.1) such that the coefficients of functions (4.46) – (4.49) from Chapter 4 satisfies $C^j_{i2} = 0$ and $C^4_{i2} \neq 0$, for $i = 1, 2$ and $j = 0, \ldots, 3$. For function (4.46), we have that

$$
\begin{aligned}
p_{00} = \frac{1}{6}\Big(&6r_{00} - 2p_{02} - 9p_{10} - 14p_{20} - 3q_{01} - 4q_{11} + 2r_{02} + 6r_{10} - 28r_{11} + 94r_{20} + 12s_{01} \\
&- 56s_{02} + 44s_{11} + 6u10 - 3\pi u_{10} - 3(2+\pi)v_{01} - 15(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} \\
&+ 3s_{11}) \arccos\left(\frac{3}{5}\right)\Big), \\
p_{02} = {}&2p_{20} + q_{11} + r_{02} - 3r_{10} + 4r_{11} - 10r_{20} - 3s_{01} + 8s_{02} - 5s_{11} + \frac{3(u_{10}+v_{01})}{2} \\
&+ \frac{9}{4}\Big(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}\Big) \arccos\left(\frac{3}{5}\right) - \frac{3\log(2)}{4}\Big(p_{10} + q_{01}\Big) \\
&- \frac{\log(8)}{2}\Big(2p_{20} + q_{11}\Big), \\
q_{11} = {}&\frac{1}{10(\log(4)-1)}\Big(5p_{10} + 5q_{01} - 52r_{10} + 104r_{11} - 312r_{20} - 52s_{01} + 208s_{02} - 156s_{11} \\
&+ 5(4+\pi)(u_{10}+v_{01}) + 55(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}) \arccos\left(\frac{3}{5}\right) \\
&- 20p_{20}(\log(4)-1) - (p_{10}+q_{01})\log(1024)\Big),
\end{aligned}
$$

$$
\begin{aligned}
v_{01} = \frac{1}{\lambda_1}\Big(&984r_{20} + 164s_{01} - 656s_{02} + 492s_{11} - 100u_{10} + 75\pi u_{10} + 18720r_{20}\log(2) \\
&+ 3120s_{01}\log(2) - 12480s_{02}\log(2) + 9360s_{11}\log(2) - 1200u_{10}\log(2) - 300\pi \\
&u_{10}\log(2) + 4r_{10}(41 + 780\log(2) - 236\log(4)) - 8r_{11}(41 + 780\log(2) - 236 \\
&\log(4)) + 75p_{10}\log(4) + 75q_{01}\log(4) - 5664r_{20}\log(4) - 944s_{01}\log(4) + 3776 \\
&s_{02}\log(4) - 2832s_{11}\log(4) + 400u_{10}\log(4) - 300p_{10}\log(2)\log(4) - 300q_{01}\log(2) \\
&\log(4) - 15p_{10}\log(1024) - 15q_{01}\log(1024) + 60p_{10}\log(2)\log(1024) + 60q_{01}\log(2) \\
&\log(1024) - 75(r_{10} - 2r_{11} + 6r_{20} + s_{01} - 4s_{02} + 3s_{11}) \arccos\left(\frac{3}{5}\right)(1 + \log(1048576))\Big), \\
s_{11} \neq {}&\frac{1}{3}\Big(2r_{11} - r_{10} - 6r_{20} - s_{01} + 4s_{02}\Big),
\end{aligned}
$$

with $\lambda_1 = 25\big(4 + \pi(\log(4096) - 3) + \log(65536)\big)$.

For function (4.48), we have that

$$
\begin{aligned}
p_{00} = \frac{1}{6}\Big(&6r_{00} - 2p_{02} - 9p_{10} - 14p_{20} - 3q_{01} - 4q_{11} + 2r_{02} - 10r_{11} + 58r_{20} - 6s_{01} - 20s_{02} \\
&+ 26s_{11} + 6u_{10} - 6v_{01} - 3((4+3\pi)r10 + \pi(3r_{11} - 18r_{20} + 3s_{01} + 6s_{02} - 9s_{11} + u_{10} \\
&+ v_{01}))\Big), \\
p_{02} = \frac{1}{4}\Big(&8p_{20} + 4q_{11} + 4r_{02} + 6r_{10} - 2(r_{11} + 2r_{20} - 3s_{01} + 2s_{02} + s_{11} - 3(u_{10} + v_{01})) - (p_{10} \\
&+ 4p_{20} + q_{01} + 2q_{11})\log(8)\Big),
\end{aligned}
$$

$$q_{11} = \frac{1}{(\log(16)-2)}\Big(p_{10} + q_{01} + (4+3\pi)r_{10} + \pi(3r_{11} - 18r_{20} + 3s_{01} + 6s_{02} - 9s_{11} + u_{10}$$
$$+ v_{01}) + 4(r_{11} - 6r_{20} + s_{01} + 2s_{02} - 3s_{11} + u_{10} + v_{01}) + p20(4 - 8\log(2)) - (p_{10}$$
$$+ q_{01})\log(4)\Big),$$
$$v_{01} = -\frac{1}{\lambda_2}\Big(108\pi r_{20} - 48r_{20} + 8s_{01} - 18\pi s_{01} + 16s_{02} - 36\pi s_{02} - 24s_{11} + 54\pi s_{11} + 8u_{10}$$
$$- 6\pi u_{10} + 48p_{20}\log(2) + 6p_{10}\log(4) + 6q_{01}\log(4) - 192r_{20}\log(8) - 144\pi r_{20}\log(8)$$
$$+ 32s_{01}\log(8) + 24\pi s_{01}\log(8) + 64s_{02}\log(8) + 48\pi s_{02}\log(8) - 96s_{11}\log(8) - 72\pi s_{11}$$
$$\log(8) + 32u_{10}\log(8) + 8\pi u_{10}\log(8) - 3p_{10}\log(16) - 12p_{20}\log(16) - 3q_{01}\log(16)$$
$$+ 96r_{20}\log(16) - 16s_{01}\log(16) - 32s_{02}\log(16) + 48s_{11}\log(16) - 16u_{10}\log(16)$$
$$+ 2r_{11}(4 - 9\pi + 12\pi\log(8) + \log(65536)) + 2r_{10}(4 + 3\pi(4\log(8)) + \log(65536) - 3)\Big),$$
$$s_{11} \neq \frac{1}{3}\Big(r_{10} + r_{11} - 6r_{20} + s_{01} + 2s_{02}\Big),$$

with $\lambda_2 = 8 - 6\pi + 8\pi\log(8) + 8\log(16)$.

For function (4.47), we have that

$$q_{11} = -\frac{p_{10}}{2} - 2p_{12} - 2p_{20} - \frac{q_{01}}{2} - 6q_{03},$$
$$q_{21} = p_{12} - 3p_{30} + 3q_{03},$$
$$r_{00} = 2p_{00} + 2p_{10} + 2p_{20} + 2p_{30} + r_{10} - r_{20} + r_{30} - 4u_{10} - 4u_{30},$$
$$r_{02} = 2p_{02} + p_{10} + q_{01} - 6q_{03} + 2r_{11} + r_{12} - 8r_{20} + 4r_{21} + 4s_{02} - 4s_{11} + 4s_{12} + 12v_{03},$$
$$s_{01} = \frac{1}{2}\Big(4r_{11} - 2r_{10} + r_{12} - 12r_{20} + 6r_{21} + 8s_{02} + 3s_{03} - 6s_{11} + 6s_{12}\Big),$$
$$s_{21} = -\frac{r_{12}}{2} + r_{21} - 3r_{30} - \frac{3s_{03}}{2} + s_{12},$$
$$v_{01} = -u_{10},$$
$$v_{21} = -u_{12} - 3(u_{30} + v_{03}).$$

For function (4.49), we have that

$$q_{11} = -\frac{p_{10}}{2} - 2p_{12} - 2p_{20} - \frac{q_{01}}{2} - 6q_{03},$$
$$q_{21} = p_{12} - 3p_{30} + 3q_{03},$$
$$r_{00} = 2p_{00} + 2p_{10} + 2p_{20} + 2p_{30} + r_{10} - r_{20} + r_{30} - 4(u_{10} + u_{30}),$$
$$r_{02} = 2p_{02} + p_{10} + q_{01} - 6q_{03} + 2r_{11} - 5r_{12} - 8r_{20} + 4r_{21} + 4s_{02} - 18s_{03} - 4s_{11} + 4s_{12}$$
$$+ 12v_{03},$$

$$
\begin{aligned}
s_{01} &= r_{10} - r_{11} + \frac{7r_{12}}{2} + 6r_{20} - 3r_{21} - 2s_{02} + \frac{21s_{03}}{2} + 3s_{11} - 3s_{12}, \\
s_{21} &= -\frac{r_{12}}{2} + r_{21} - 3r_{30} - \frac{3s_{03}}{2} + s_{12}, \\
v_{01} &= -u_{10}, \\
v_{21} &= -u_{12} - 3(u_{30} + v_{03}).
\end{aligned}
$$

CHAPTER

8

Conclusions

In this work, we estimate the number of crossing limit cycles in planar piecewise linear Hamiltonian differential systems with two or three zones separated by straight lines, and parallel if there are more than one, such that the linear systems that define the piecewise one have isolated singular points, i.e. centers or saddles. More precisely, we started with the study of the number of limit cycles for continuous or discontinuous piecewise linear Hamiltonian differential systems with two or three zones. In this case, we showed that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones or continuous with three zones, then it has no limit cycles, but if it is discontinuous with three zones then it has at most one limit cycle, and we provide examples with one limit cycle.

Next, we study the number of crossing limit cycles that can bifurcate from periodic annulis in a discontinuous piecewise linear Hamiltonian differential system with three zones, after polynomials functions perturbations. For theses cases, we prove that if the linear differential system defined in the region between the two parallel lines has a center at the origin and the others subsystems have centers or saddles then we have at least three, four or seven limit cycles bifurcating from a periodic annulus after linear, quadratic or cubic polynomials perturbations, respectively. Now, when the central subsystem has a real or boundary center and the others subsystems have only real saddles then at least six limit cycles can bifurcate simultaneously from two periodic annuli, after linear perturbations. Finally, if the central subsystem has a real saddle and the others subsystems have centers or saddles then at least six limit cycles can bifurcate simultaneously from three periodic annuli, after linear perturbations.

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