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Open-closed superstring amplitudes using vertex operators in $\mathbf{AdS_5 \times S^5}$

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Resumo

Nesta tese, uma dada amplitude de espalhamento de supercordas envolvendo um operador de vértice da corda fechada e N operadores de vértice da corda aberta em $\mathrm{AdS}_5 \times \mathrm{S}^5$ é detalhadamente estudada, utilizando-se o formalismo de espinores puros. Após rever o material de apoio e realizar alguns cálculos preliminares, mostramos que a amplitude nível-de-árvore contendo um estado de supergravidade e N estados de super-Yang–Mills localizados em uma D3-brana próxima à fronteira de AdS_5 pode ser expressa como uma integral no superespaço harmônico $\mathcal{N} = 4$, $\mathrm{d} = 4$ em termos dos supercampos de supergravidade e super-Yang–Mills, demonstrando assim uma conjectura feita recentemente.

Palavras Chaves: Supersimetria; Supergravidade; Correspondência AdS/CFT; Superespaço Harmônico; Amplitudes de espalhamento.

Áreas do conhecimento: Ciências Exatas e da Terra; Física de Partículas e Campos; Teoria de Supercordas.

Abstract

In this thesis, a particular superstring scattering amplitude involving one closed string and N open string vertex operators in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is studied in detail, using the pure-spinor formalism. After reviewing some background material and performing a few preliminary calculations, we show that the tree-level amplitude containing one supergravity state and N super-Yang–Mills states located on a D3-brane near the AdS_5 boundary can be expressed as an $\mathcal{N} = 4$, $\mathrm{d} = 4$ harmonic superspace integral in terms of the supergravity and super-Yang–Mills superfields, thus proving a conjecture recently made.

Contents

Chapter 1

Introduction

Presently, superstring theory is the only consistent quantum model that can describe gravitational interactions between the elementary particles. All the other attempts are believed to suffer from quantum divergences in the scattering amplitudes which cannot be removed through a renormalization procedure. Even if superstring theory turns out not to be the final theory of gravitation, it is likely to contain some of the essential ingredients of such theory.

Different formulations of the theory have been developed in the last decades. In 1971, Neveu, Schwarz and Ramond (NSR) constructed a formalism in which Lorentz covariance is manifest, but spacetime supersymmetry is hidden. In 1981, Green and Schwarz constructed another formalism in which supersymmetry is manifest, but Lorentz covariance is hidden. An excelent introduction to both formulations can be found in the book by Green, Schwarz and Witten [1]. More recently, a third formalism was developed, involving pure spinors, in which all the (super)symmetries of spacetime are manifest [2].

The pure-spinor formalism of the superstring has proven to be very well suited for scattering-amplitude computations involving massless states in flat space [3, 4]. For example, scattering amplitudes using the NSR formalism have been computed up to 6 fermions at tree level, whereas the manifest super-Poincaré covariance of the pure-spinor formalism makes it possible to compute tree-level amplitudes involving any number of fermions [5].

Another good feature of the pure-spinor formalism is that it can be used to describe the propagation of strings in curved spaces, and in particular in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, which is relevant for AdS/CFT. Given the success in the computation of scattering amplitudes in flat space, the computation of amplitudes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ using pure spinors also seems promising.

The interest in string theory in an (asymptotically) AdS background has considerably increased since Maldacena's AdS/CFT correspondence was conjectured in

1997 [6] — see [7, 8] for reviews. The original paper has literally opened a new field of research, having been cited over 10^4 times since its publication. Besides the studies of the duality within "pure" string theory, there are now many phenomenological applications, ranging from QCD-like models (AdS/QCD) to Condensed Matter Theory (AdS/CMT), as can be seen in [9, 10], for example.

In its original formulation, AdS/CFT relates type IIB superstring theory in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with N_c units of five-form flux on the S^5 to its dual gauge theory, namely the maximally supersymmetric ($\mathcal{N} = 4$) Yang–Mills theory in four dimensions with $\mathrm{U}(N_\mathrm{c})$ gauge group — roughly a supersymmetric version of QCD. In the weakest form of the conjecture, the correspondence is said to be valid only in the limit in which both N_c and the 't Hooft parameter $\lambda_\mathrm{t} := g_\mathrm{YM}^2 N_\mathrm{c}$ go to infinity, which corresponds to the supergravity limit of string theory on the AdS side. However, it is believed that AdS/CFT remains valid beyond the limits of supergravity, and it is important to test the duality in regimes in which the full string theory plays a role.

In order to study superstrings in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, it is possible to use either the Green–Schwarz [11, 12] or the pure-spinor [2, 13] formalism. Although the superstring action is known in both formalisms, the explicit superfield construction of the vertex operators of the theory is still an open problem. Vertex operators correspond to physical states in string theory, and knowing their expressions is necessary to compute scattering amplitudes.

Even though the NSR formalism cannot be used to describe $\mathrm{AdS}_5 \times \mathrm{S}^5$, since it is a Ramond-Ramond background, flat-space NSR vertex operators have been used to compute scattering amplitudes in that background in certain limits [14, 15, 16].

In the pure-spinor formalism, the first work on vertex operators in $\mathrm{AdS}_5 \times \mathrm{S}^5$ was [17]. There the authors have constructed massless (supergravity) vertex operators corresponding to on-shell fluctuations around that background, but the expansion in powers of θ was not computed and the connection between the vertex operators and the duals of the half-BPS operators was not found. More recently, a step in that direction was taken. In [18], vertex operator expressions have been found for a particular case, namely the massless states in the limit $z \to 0$, i.e. close to the AdS boundary (z is the radial coordinate of AdS_5). The vertex operators are in the ghost-number +2 cohomology of a BRST operator, and come in a family $\{V_{(N)}\}$ ($N = 1, 2, \ldots$) such that $V_{(N)}$ is dual to a half-BPS operator involving N super-Yang–Mills (SYM) fields.

Half-BPS operators can be described in an elegant manner in harmonic superspace as [19, 20, 21]

$$W^{(N)}(x,\theta,u) := (uu)^{i_1 j_1} \ldots (uu)^{i_N j_N} \, \mathrm{tr} \left[W_{i_1 j_1}(x,\theta) \ldots W_{i_N j_N}(x,\theta) \right], \tag{1.1}$$

where W_{ij} is the $\mathcal{N}=4$, $d=4$ Sohnius superfield strength [22], whose expansion in component fields is

$$W_{ij}(x,\theta,\bar{\theta}) = \phi_{ij}(x) - \varepsilon_{ijk\ell}\theta^k\xi^\ell(x) + 2\bar{\theta}_{[i}\bar{\xi}_{j]}(x) + \frac{1}{4}\varepsilon_{ijk\ell}\theta^k\sigma^{ab}\theta^\ell f_{ab}(x) - \frac{1}{2}\bar{\theta}_i\tilde{\sigma}^{ab}\bar{\theta}_j f_{ab}(x) + \cdots, \tag{1.2}$$

where ϕ_{ij}, ξ, $\bar{\xi}$ and f_{ab} are, respectively, the $\mathcal{N}=4$ SYM scalars, chiral and anti-chiral gluinos and gluon field-strength, $i,j,\ldots=1$ to 4 are SU(4) indices and $a,b=0$ to 3 are SO(3,1) indices. In addition, the supergravity vertex operators $V_{(N)}$ in [18] were written in terms of harmonic superfields $T^{(4-N)}$ which couple naturally to $W^{(N)}$ via [20]

$$\int \mathrm{d}^4x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(N)}(x,\theta,u)\, T^{(4-N)}(x,\theta,u,\overline{u})\right]. \tag{1.3}$$

This led to the following conjecture: the tree-level (disk) scattering amplitude with one closed string supergravity vertex operator $V_{(N)} \propto T^{(4-N)}$ in the bulk and N open string SYM vertex operators located on a D3-brane near the AdS_5 boundary would be proportional to the coupling (1.3).

In [23], this conjecture was proven to be indeed true. Using (super)symmetry and BRST arguments, i.e. using the fact that amplitudes in the pure-spinor formalism are supersymmetric and BRST-invariant, we have shown that

$$\mathcal{M}_N := \int \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{N-1} \left\langle V_{\mathrm{SYM}}\, V_{(N)}\, U_{\mathrm{SYM}}(\xi_1) \ldots U_{\mathrm{SYM}}(\xi_{N-1}) \right\rangle_{\mathrm{D3-brane}} \tag{1.4}$$

can be written as (1.3), up to an overall numerical factor. Here V_{SYM} is the unintegrated vertex operator of SYM, U_{SYM} is the integrated one and the "D3-brane" subscript indicates that these vertex operators are located on a D3-brane parallel and close to the AdS_5 boundary. This was the first application of the vertex operators found in [18].

It is important to note that, although the AdS/CFT correspondence motivates the main computation done in this thesis, the latter is not of the usual bulk-boundary type. In other words, the goal is not to compare the result on one side with that on the other. Instead, the purpose is to show that the coupling (1.3) — which is nothing but the usual coupling between the dilaton and the $\mathcal{N}=4$, d = 4 Lagrangian, the metric and the energy-momentum tensor, and so on, written in a supersymmetric way — can be obtained as a scattering amplitude with closed and open string states. It should also be clear that the D3-brane on which the open string vertex operators are located has no relation with the D3-branes that are responsible for the geometry of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space, which are located at $z \to \infty$ [6].

In this thesis, the work done in [23] will be reviewed in detail. In chapter 2, we will give a brief review of the pure-spinor formalism, focusing on the aspects that will be useful in the following chapters. In chapter 3, we will review the computation of the zero-mode cohomology of the $AdS_5 \times S^5$ BRST operator done in [18], obtaining explicit superfield expressions for the behavior of supergravity vertex operators near the boundary of AdS_5. In chapter 4, after performing some preliminary calculations, we will use the pure-spinor prescription and supersymmetry to compute the above-mentioned tree-level amplitude, first in the case $N = 1$ and then extending the result to $N > 1$. Finally, we summarize our results and present some perspectives in chapter 5. The appendices contain notations, conventions and further information useful for the reader, and should be considered as important as the main chapters of the thesis.

Chapter 2

Brief review of the pure-spinor formalism

Until the beginning of the year 2000, there were basically two ways of studying superstring theory. One could use either the Neveu–Schwarz–Ramond (NSR) or the Green–Schwarz (GS) formalism [1].

The NSR formalism has the advantage of being quantizable in a manifestly Lorentz-covariant manner, but lacks manifest spacetime supersymmetry, which can only be recognized as a symmetry of the spectrum of the theory after one performs the GSO projection. This drawback makes it hard to compute scattering amplitudes involving fermions within this formalism. Another issue with this formalism is that it cannot be used to study the propagation of strings in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, since it includes a Ramond-Ramond 5-form flux.

In the GS formalism, the situation is inverted. It is manifestly spacetime-supersymmetric, but seems to be impossible to covariantly quantize. Indeed, it has only been quantized in the light-cone gauge. Classically, however, it is possible to write a worldsheet action for the propagation of strings in any curved background using this formalism.

In the year 2000, Nathan Berkovits developed a new formalism for the superstring which had the good features of the previous ones, without sharing any of their disadvantages [2]. It was shown for the first time how the superstring could be quantized in a manifestly super-Poincaré-covariant manner. His formalism could also be used to describe the propagation of strings in curved backgrounds. It became known as the pure-spinor formalism, for reasons to be shortly clarified, and will be briefly reviewed in this chapter, since it is the formalism we will work in throughout this thesis. Most of the details omitted here can be found in the thesis by Mafra [3], except for the curved background part, whose details can be found in the thesis by Guttenberg [24].

2.1 The pure-spinor formalism in flat space

In 1985, Warren Siegel proposed the following world-sheet action for the flat-space superstring [25]:

$$S_{\text{Siegel}} = \frac{1}{2\pi}\int \mathrm{d}^2\zeta \left[\frac{1}{2}\partial x_\mu \bar{\partial} x^\mu + p_{\hat{\alpha}} \bar{\partial}\theta^{\hat{\alpha}}\right], \tag{2.1}$$

where x^μ ($\mu = 0$ to 9) and $\theta^{\hat{\alpha}}$ ($\hat{\alpha} = 1$ to 16) are the usual ten-dimensional superspace coordinates and $p_{\hat{\alpha}}$ is the conjugate momentum to $\theta^{\hat{\alpha}}$. For simplicity, we have written only the left-moving sector. Note that we have set α', the square of the string length scale, to $\alpha' = 2$, and this will be the case in the rest of this thesis. If desired, one can easily recover factors of α' in any expression by dimensional analysis, since x (resp. θ) has (mass-)dimension -1 (resp. $-\frac{1}{2}$) (in units such that $c = \hbar = 1$).

From (2.1), it is straightforward to derive the OPE's

$$p_{\hat{\alpha}}(\zeta)\,\theta^{\hat{\beta}}(\xi) \sim \frac{\delta_{\hat{\alpha}}^{\hat{\beta}}}{\zeta - \xi} \qquad \text{and} \qquad x^\mu(\zeta,\bar{\zeta})\,x^\nu(\xi,\bar{\xi}) \sim -\eta^{\mu\nu}\log|\zeta - \xi|^2 \,. \tag{2.2}$$

Then, one can show the action is invariant under the transformations generated by the supersymmetry generator

$$q_{\hat{\alpha}} = \frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}\zeta \left[p_{\hat{\alpha}} + \partial x^\mu(\theta\gamma_\mu)_{\hat{\alpha}} + \frac{1}{6}(\theta\gamma^\mu\partial\theta)(\theta\gamma_\mu)_{\hat{\alpha}}\right], \tag{2.3}$$

since, with a constant fermionic (anticommuting) parameter $\epsilon^{\hat{\alpha}}$, (2.3) and (2.2) imply $\delta_\epsilon x^\mu = -(\epsilon\gamma^\mu\theta)$, $\delta_\epsilon\theta^{\hat{\alpha}} = \epsilon^{\hat{\alpha}}$ and $\delta_\epsilon p_{\hat{\alpha}} = \partial x^\mu(\gamma_\mu\epsilon)_{\hat{\alpha}} - \frac{1}{2}(\epsilon\gamma^\mu\theta)(\gamma_\mu\partial\theta)_{\hat{\alpha}}$.

Siegel's idea was to avoid the difficulties in covariantly quantizing the GS superstring by letting $p_{\hat{\alpha}}$ be an unconstrained variable. In the GS formalism, $p_{\hat{\alpha}}$ has to satisfy the supersymmetric constraint

$$d_{\hat{\alpha}} := p_{\hat{\alpha}} - \partial x^\mu(\theta\gamma_\mu)_{\hat{\alpha}} - \frac{1}{2}(\theta\gamma^\mu\partial\theta)(\theta\gamma_\mu)_{\hat{\alpha}} = 0\,. \tag{2.4}$$

Using (2.2), one obtains

$$d_{\hat{\alpha}}(\zeta)\,d_{\hat{\beta}}(\xi) \sim -\frac{2(\gamma^\mu)_{\hat{\alpha}\hat{\beta}}}{\zeta - \xi}\Pi_\mu\,, \tag{2.5}$$

where $\Pi_\mu := \partial x_\mu + \theta\gamma_\mu\partial\theta$ is the supersymmetric momentum. This OPE implies that half the constraints in $d_{\hat{\alpha}}$ are first-class, while the other half are second-class constraints. To apply Dirac's method for quantizing constrained systems, one has to separate the two kinds of constraints. This can be done in the light-cone gauge, but apparently is impossible to be done in a Lorentz-covariant manner.

Although Siegel's approach was successful in the quantization of the superparticle, it does not work so well in the superstring case. The energy-momentum tensor obtained from (2.1) yields a central charge of -22. A non-vanishing central charge leads to conformal anomalies at the quantum level, and thus is not a desirable feature of a conformal field theory. Moreover, the Lorentz currents $M^{(\text{Siegel})}_{\mu\nu}$ constructed from the fields of the theory have an OPE that differs from the NSR one, implying that amplitude computations in the two formulations would not be equivalent.

The pure-spinor formalism can be seen as an extension of Siegel's approach in which the two above-mentioned issues are solved in a clever way. The first step is to introduce a ghost variable $\lambda^{\hat{\alpha}}$ and its conjugate momentum $w_{\hat{\alpha}}$ in (2.1), thus obtaining the (left-moving sector of the) pure-spinor action:

$$S_{\text{PS}} = \frac{1}{2\pi} \int \mathrm{d}^2\zeta \left[\frac{1}{2} \partial x_\mu \bar{\partial} x^\mu + p_{\hat{\alpha}} \bar{\partial} \theta^{\hat{\alpha}} + w_{\hat{\alpha}} \bar{\partial} \lambda^{\hat{\alpha}} \right] . \tag{2.6}$$

As will be seen in the following, the simple addition of this term has far-reaching consequences, provided that $\lambda^{\hat{\alpha}}$ satisfies

$$\lambda \gamma^\mu \lambda = 0 \,, \tag{2.7}$$

which is the defining condition for a ten-dimensional pure-spinor, whence the name of the formalism.

Note that, because of (2.7), $w_{\hat{\alpha}}$ is defined up to the gauge transformation $\delta w_{\hat{\alpha}} = \Lambda^\mu (\gamma_\mu \lambda)_{\hat{\alpha}}$, for arbitrary Λ^μ, since the pure-spinor action is invariant under this transformation. This implies that $w_{\hat{\alpha}}$ can only appear in combinations of the gauge-invariant quantities

$$N_{\mu\nu} = \frac{1}{2} w \gamma_{\mu\nu} \lambda \,, \qquad J_{\text{g}} = w \lambda \,, \tag{2.8}$$

which are respectively the SO(9, 1) Lorentz currents in the ghost sector and the ghost-number current.

The total Lorentz currents $M_{\mu\nu} = M^{(\text{Siegel})}_{\mu\nu} + N_{\mu\nu}$ can be shown to have the same OPE as the NSR ones. Thus, one of the issues with Siegel's approach is already solved in the pure-spinor formalism.

Furthermore, the total central charge vanishes in this formalism. One can convince oneself this is plausible by counting the degrees of freedom of the fields. In the Siegel action alone, there are 10 bosonic (x^μ) and 32 fermionic ($\theta^{\hat{\alpha}}$ and $p_{\hat{\alpha}}$) degrees of freedom. Since the fermionic ones enter the computation with a negative sign, one is left with the central charge of -22. Now, the pure-spinor condition (2.7) together with the gauge transformation of $w_{\hat{\alpha}}$ imply that $\lambda^{\hat{\alpha}}$ and $w_{\hat{\alpha}}$ have 11 degrees of freedom each, and since they are bosonic they contribute with a positive sign. So the total central charge is given by $-22 + 2 \times 11 = 0$. Of course, this is just a plausibility argument, not a rigorous proof.

In addition to those generated by (2.3), the pure-spinor action (2.6) is also invariant under the transformations generated by

$$Q = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}\zeta\, \lambda^{\hat{\alpha}} d_{\hat{\alpha}}\,. \tag{2.9}$$

Consisting of a ghost multiplying a constraint, this looks like a BRST charge. In fact, it also has an important property of BRST charges, namely it is nilpotent:

$$Q^2 = \frac{1}{2}\{Q, Q\} \propto \int \mathrm{d}\zeta\, (\lambda\gamma^{\mu}\lambda)\Pi_{\mu} = 0\,, \tag{2.10}$$

where we used (2.5) and (2.7).

In the pure-spinor formalism, Q is indeed regarded as the BRST operator of the theory. Its origin is a bit mysterious, since the world-sheet action is introduced already in a gauge-fixed form. However, a derivation of the BRST operator from gauge-fixing a reparametrization-invariant worldsheet action has been shown to be possible [26, 27].

Given a BRST operator, the physical vertex operators of the theory, corresponding to the physical states, are given by its cohomology. An operator Ψ satisfying $[Q, \Psi\} = 0$ is called BRST-invariant or BRST-closed, while an operator Υ satisfying $\Upsilon = [Q, \omega\}$ for some ω is called BRST-trivial or BRST-exact.* The cohomology of Q is defined as the set of BRST-invariant operators that are not BRST-trivial. In other words, the cohomology is the set of BRST-invariant operators identified via the equivalence relation

$$\Psi_1 \sim_Q \Psi_2 \iff \Psi_1 = \Psi_2 + [Q, \omega\} \tag{2.11}$$

for some ω.

Since Q does not mix the different λ-levels, we can consider the cohomology on each subspace of a specific ghost-number separately. For open superstrings, physical vertex operators are defined as those in the ghost-number $+1$ cohomology of Q. For example, massless states are described by the (unintegrated) vertex operator

$$V = \lambda^{\hat{\alpha}} A_{\hat{\alpha}}(x, \theta)\,, \tag{2.12}$$

where $A_{\hat{\alpha}}(x, \theta)$ is the $\mathcal{N} = 1$, $\mathrm{d} = 10$ SYM fermionic connection, whose properties are reviewed in Appendix C. To show that Q annihilates this vertex operator, note that the OPE's (2.2) imply

$$d_{\hat{\alpha}}(\zeta)\, \mathcal{F}(x(\xi), \theta(\xi)) \sim \frac{1}{\zeta - \xi} D_{\hat{\alpha}} \mathcal{F}(x(\xi), \theta(\xi))\,, \tag{2.13}$$

*Here $[A, B\}$ denotes an anticommutator if both A and B are fermionic, or a commutator otherwise.

where $D_{\hat\alpha} = \frac{\partial}{\partial\theta^{\hat\alpha}} + (\theta\gamma^\mu)_{\hat\alpha}\partial_\mu$ and $\mathcal{F}(x,\theta)$ is any superfield. Hence, we write $[Q,\mathcal{F}\} = \lambda^{\hat\alpha} D_{\hat\alpha}\mathcal{F}$. Therefore,

$$\{Q, V\} = \lambda^{\hat\alpha}\lambda^{\hat\beta} D_{\hat\alpha} A_{\hat\beta} = (\lambda\gamma^\mu\lambda)A_\mu = 0\,, \tag{2.14}$$

where we used (C.3) and (2.7). Moreover, adding a BRST-trivial term to V is equivalent to performing the usual gauge transformation on $A_{\hat\alpha}$, since

$$\delta V = [Q,\omega] = \lambda^{\hat\alpha} D_{\hat\alpha}\omega \iff \delta A_{\hat\alpha} = D_{\hat\alpha}\omega\,. \tag{2.15}$$

When computing scattering amplitudes, the integrated form of the vertex operator is also needed. It can be found by looking for an object U satisfying the same relation as in the NSR formalism, namely

$$[Q, U] = \partial V\,. \tag{2.16}$$

This implies that $\int \mathrm{d}\zeta\, U(\zeta)$ is BRST-invariant. Using the OPE's (2.2) and

$$N^{\mu\nu}(\zeta)\lambda^{\hat\alpha}(\xi) \sim \frac{1}{2}\frac{(\gamma^{\mu\nu}\lambda)^{\hat\alpha}}{\zeta-\xi}\,, \tag{2.17}$$

which can be obtained by imposing that $\lambda^{\hat\alpha}$ transform as a spinor under the action of the SO(9, 1) Lorentz currents, as well as the SYM equations of motion (C.3) and (C.14), one can show that

$$U = \partial\theta^{\hat\alpha} A_{\hat\alpha} + \Pi^\mu A_\mu - \frac{1}{2} d_{\hat\alpha} W^{\hat\alpha} - \frac{1}{2} N^{\mu\nu} F_{\mu\nu} \tag{2.18}$$

satisfies (2.16), and thus is the integrated vertex operator corresponding to massless open superstring states.

Given vertex operators, one can compute scattering amplitudes. The pure-spinor prescription for the computation of tree-level amplitudes states that the n-point function is given by

$$\mathcal{A}_n = \left\langle V^{[1]}(\zeta_1)\, V^{[2]}(\zeta_2)\, V^{[3]}(\zeta_3) \int \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-3}\, U^{[1]}(\xi_1)\cdots U^{[n-3]}(\xi_{n-3}) \right\rangle, \tag{2.19}$$

where we have made use of the disk SL(2, $\mathbb{R}$) symmetry to fix the positions of three vertex operators. The angle brackets in the above equation contain integrations over the x, λ and θ zero modes, and are defined such that the only non-vanishing contributions are those coming from three λ's and five θ's in the combination

$$(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta)\,. \tag{2.20}$$

This measure factor is special since it is the unique (up to an overall factor) SO(9, 1) scalar which can be built out of three λ's and five θ's, and it is also in the cohomology of Q. Scattering amplitudes defined in this way are thus BRST-invariant, and can also be shown to be supersymmetric — see [28], for example.

2.2 Non-minimal pure-spinor formalism

The formalism briefly reviewed in the previous section is presently known as the minimal pure-spinor formalism. This is because in 2005 an extension of the pure-spinor formalism was proposed [29]. New variables, called non-minimal pure-spinor variables, were introduced. They consist of a bosonic spinor $\widetilde{\lambda}_{\hat{\alpha}}$ and a fermionic spinor $r_{\hat{\alpha}}$, as well as their conjugates $\widetilde{w}^{\hat{\alpha}}$ and $s^{\hat{\alpha}}$. These variables satisfy the constraints

$$\widetilde{\lambda}\gamma^{\mu}\widetilde{\lambda} = 0\,, \qquad \widetilde{\lambda}\gamma^{\mu}r = 0\,. \tag{2.21}$$

The first of these equations implies $\widetilde{\lambda}_{\hat{\alpha}}$ is a pure spinor.

The action is modified to

$$S_{\mathrm{NM}} = S_{\mathrm{PS}} + \frac{1}{2\pi}\int \mathrm{d}^2\zeta \left(\widetilde{w}^{\hat{\alpha}}\bar{\partial}\widetilde{\lambda}_{\hat{\alpha}} - s^{\hat{\alpha}}\bar{\partial}r_{\hat{\alpha}}\right). \tag{2.22}$$

In addition to $\delta w_{\hat{\alpha}} = \Lambda^{\mu}(\gamma_{\mu}\lambda)_{\hat{\alpha}}$, the action (2.22) is also invariant under the gauge transformations

$$\delta\widetilde{w}^{\hat{\alpha}} = \widetilde{\Lambda}^{\mu}(\gamma_{\mu}\widetilde{\lambda})^{\hat{\alpha}} - \phi^{\mu}(\gamma_{\mu}r)^{\hat{\alpha}}\,, \qquad \delta s^{\hat{\alpha}} = \phi^{\mu}(\gamma_{\mu}\widetilde{\lambda})^{\hat{\alpha}}\,, \tag{2.23}$$

for arbitrary $\widetilde{\Lambda}^{\mu}$ and ϕ^{μ}. One can construct several independent gauge-invariant quantities out of $\widetilde{w}^{\hat{\alpha}}$ and $s^{\hat{\alpha}}$:

$$\begin{aligned}
\widetilde{N}_{\mu\nu} &= \frac{1}{2}(\widetilde{w}\gamma_{\mu\nu}\widetilde{\lambda} - s\gamma_{\mu\nu}r)\,, \\
J_{\widetilde{\lambda}} &= \widetilde{w}\widetilde{\lambda}\,, \\
T_{\widetilde{\lambda}} &= \widetilde{w}\partial\widetilde{\lambda} - s\partial r\,, \\
S_{\mu\nu} &= \frac{1}{2}(s\gamma_{\mu\nu}\widetilde{\lambda})\,, \\
\Phi &= \widetilde{w}r\,,
\end{aligned} \tag{2.24}$$

among others.

It is possible to show that the introduction of the non-minimal variables in the action does not affect the good features of the pure-spinor formalism as compared to Siegel's approach, namely the Lorentz currents' OPE's are still the same as in the NSR formalism and the new variables do not contribute to the central charge of the theory.

The action (2.22) is invariant under the transformations generated by the modified BRST charge

$$Q_{\mathrm{NM}} = Q + \frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}\zeta\, \widetilde{w}^{\hat{\alpha}}r_{\hat{\alpha}}\,. \tag{2.25}$$

The addition of this term implies, by the quartet argment, that the BRST cohomology is independent of $\widetilde{\lambda}_{\hat{\alpha}}$, $r_{\hat{\alpha}}$ and their conjugate momenta. Thus, one can always choose a gauge in which the physical vertex operators of the theory do not depend on the non-minimal variables. However, it is sometimes convenient to allow dependence on these variables, as will be seen for example in the next chapter, when we present the supergravity vertex operators in $\mathrm{AdS}_5 \times \mathrm{S}^5$ that were found in [18].

The introduction of non-minimal pure-spinor variables also allows one to write a relatively simple expression for the b ghost, which is not a fundamental field in this formalism. This object is defined such that $\{Q_{\mathrm{NM}}, b\} = T_{\mathrm{NM}}$, where T_{NM} is the energy-momentum tensor obtained from (2.22). The b ghost is important for multiloop amplitude computations, which are out of the scope of the present thesis. For more information on the b ghost and loop amplitudes, see [30, 3].

2.3 The pure-spinor formalism in curved backgrounds

We conclude this brief review of the pure-spinor formalism with some words on the description of superstrings in an arbitrary curved background. The first work on the pure-spinor superstring in a general type II background was [31]. There the authors have studied the most general action constructed from the worldsheet variables which is classically invariant under worldsheet conformal transformations and has vanishing ghost number.

In order to write such an action, we first need to introduce the right-moving variables that so far we have omitted for simplicity. We have $\widehat{\theta}^{\hat{\alpha}}$ and its conjugate momentum $\widehat{p}_{\hat{\alpha}}$, as well as the pure-spinor ghost variables $\widehat{\lambda}^{\hat{\alpha}}$ and $\widehat{w}_{\hat{\alpha}}$. Note that we have written these variables as $\mathrm{SO}(9,1)$ Majorana-Weyl spinors of the same chiralities as those of their left-moving counterparts. This means that we will be describing the particular case of type IIB backgrounds, of which $\mathrm{AdS}_5 \times \mathrm{S}^5$ is an example.

The available building blocks one can use to construct a classically conformally invariant action and their conformal weights are listed in the following table:

weight	fields
$(0,0)$	$Z^{\boldsymbol{\mathcal{M}}}$
$(1,0)$	$\partial Z^{\boldsymbol{\mathcal{M}}}$, $d_{\hat{\alpha}}$, $\lambda^{\hat{\alpha}} w_{\hat{\beta}}$
$(0,1)$	$\bar{\partial} Z^{\boldsymbol{\mathcal{M}}}$, $\widehat{d}_{\hat{\alpha}}$, $\widehat{\lambda}^{\hat{\alpha}} \widehat{w}_{\hat{\beta}}$
$(1,1)$	$w_{\hat{\beta}} \bar{\partial} \lambda^{\hat{\alpha}}$, $\widehat{w}_{\hat{\beta}} \partial \widehat{\lambda}^{\hat{\alpha}}$, $\bar{\partial} d_{\hat{\alpha}}$, $\partial \widehat{d}_{\hat{\alpha}}$

Here $Z^{\boldsymbol{\mathcal{M}}} := (x^{\boldsymbol{\mu}}, \theta^{\boldsymbol{\hat{\alpha}}}, \widehat{\theta}^{\boldsymbol{\hat{\beta}}})$ are $\mathcal{N} = 2$, d=10 superspace variables and $\widehat{d}_{\hat{\alpha}}$ is defined in complete analogy with (2.4). Note the notation in which indices highlighted in bold correspond to curved superspace indices. Since the worldsheet measure $\mathrm{d}^2\zeta$

has conformal weight $(-1,-1)$, these building blocks have to be combined to form objects with conformal weight $(1,1)$. One can either use a $(1,1)$-block or combine a $(1,0)$-block with a $(0,1)$ one. In each case, the resulting object can also be multiplied by an arbitrary function of $Z^{\mathcal{M}}$, which is called a background superfield.

The action is thus given by

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\pi}\int \mathrm{d}^2\zeta \Big[&\frac{1}{2}\Big(G_{\mathcal{MN}}(Z) + B_{\mathcal{MN}}(Z)\Big)\partial Z^{\mathcal{M}}\bar{\partial}Z^{\mathcal{N}} \\ &+ E_{\mathcal{M}}{}^{\hat{\alpha}}(Z) d_{\hat{\alpha}}\bar{\partial}Z^{\mathcal{M}} + \Omega_{\mathcal{M}\hat{\alpha}}{}^{\hat{\beta}}(Z)\lambda^{\hat{\alpha}}w_{\hat{\beta}}\bar{\partial}Z^{\mathcal{M}} + C_{\hat{\alpha}}{}^{\hat{\beta}\hat{\gamma}}(Z)\lambda^{\hat{\alpha}}w_{\hat{\beta}}\widehat{d}_{\hat{\gamma}} \\ &+ E_{\mathcal{M}}{}^{\hat{\alpha}}(Z)\widehat{d}_{\hat{\alpha}}\partial Z^{\mathcal{M}} + \widehat{\Omega}_{\mathcal{M}\hat{\alpha}}{}^{\hat{\beta}}(Z)\widehat{\lambda}^{\hat{\alpha}}\widehat{w}_{\hat{\beta}}\partial Z^{\mathcal{M}} + \widehat{C}_{\hat{\alpha}}{}^{\hat{\beta}\hat{\gamma}}(Z)\widehat{\lambda}^{\hat{\alpha}}\widehat{w}_{\hat{\beta}}d_{\hat{\gamma}} \\ &+ \mathcal{P}^{\hat{\alpha}\hat{\beta}}(Z)d_{\hat{\alpha}}\widehat{d}_{\hat{\beta}} + \mathcal{S}_{\hat{\alpha}\hat{\gamma}}{}^{\hat{\beta}\hat{\delta}}(Z)\lambda^{\hat{\alpha}}w_{\hat{\beta}}\widehat{\lambda}^{\hat{\gamma}}\widehat{w}_{\hat{\delta}} \\ &+ w_{\hat{\alpha}}\bar{\partial}\lambda^{\hat{\alpha}} + \widehat{w}_{\hat{\alpha}}\partial\widehat{\lambda}^{\hat{\alpha}}\Big]\,. \end{aligned} \tag{2.26}$$

All the background superfields appearing in this action have physical interpretations. The superfield $E_{\mathcal{M}}{}^{\mathcal{M}}$ is the supervielbein ($G_{\mathcal{MN}} = \eta_{\mu\nu}E_{\mathcal{M}}{}^{\mu}E_{\mathcal{N}}{}^{\nu}$), $B_{\mathcal{MN}}$ is the 2-form potential, $\mathcal{P}^{\hat{\alpha}\hat{\beta}}$ is the superfield whose lowest components are the Type IIB Ramond-Ramond field strengths, $C_{\hat{\alpha}}{}^{\hat{\beta}\hat{\gamma}}$ and $\widehat{C}_{\hat{\alpha}}{}^{\hat{\beta}\hat{\gamma}}$ are related to the two gravitini and dilatini field strengths, $\Omega_{\mathcal{M}\hat{\alpha}}{}^{\hat{\beta}}$ and $\widehat{\Omega}_{\mathcal{M}\hat{\alpha}}{}^{\hat{\beta}}$ are the spin connections and finally $\mathcal{S}_{\hat{\alpha}\hat{\gamma}}{}^{\hat{\beta}\hat{\delta}}$ is related to the curvature.

The first line of (2.26) is the standard type IIB GS action, but the other lines are needed for BRST invariance. The BRST charge of the theory is given by

$$Q = Q_{\mathrm{L}} + Q_{\mathrm{R}}\,, \tag{2.27}$$

where

$$Q_{\mathrm{L}} = \frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}\zeta\, \lambda^{\hat{\alpha}}d_{\hat{\alpha}}\,, \qquad Q_{\mathrm{R}} = \frac{1}{2\pi \mathrm{i}}\oint \mathrm{d}\bar{\zeta}\, \widehat{\lambda}^{\hat{\alpha}}\widehat{d}_{\hat{\alpha}}\,. \tag{2.28}$$

Since the BRST charge must be nilpotent, we must have

$$\{Q_{\mathrm{L}}, Q_{\mathrm{L}}\} = \{Q_{\mathrm{L}}, Q_{\mathrm{R}}\} = \{Q_{\mathrm{R}}, Q_{\mathrm{R}}\} = 0\,, \tag{2.29}$$

which can be easily obtained from $Q^2 = \frac{1}{2}\{Q,Q\} = 0$. Moreover, in order for these charges to be well defined, the BRST current $\lambda^{\hat{\alpha}}d_{\hat{\alpha}}$ (resp. $\widehat{\lambda}^{\hat{\alpha}}\widehat{d}_{\hat{\alpha}}$) has to be holomorphic (resp. antiholomorphic), i.e.

$$\bar{\partial}\Big(\lambda^{\hat{\alpha}}d_{\hat{\alpha}}\Big) = \partial\Big(\widehat{\lambda}^{\hat{\alpha}}\widehat{d}_{\hat{\alpha}}\Big) = 0\,. \tag{2.30}$$

In [31], it was shown that (2.29) and (2.30) imply the supergravity equations of motion for the background superfields in (2.26).

In the next chapter, we will present the $\mathrm{AdS}_5 \times \mathrm{S}^5$ action, which is a particular case of the general action introduced here. The background superfields will be described by means of coset space techniques, making the symmetries of the background manifest.

Chapter 3

Zero-mode cohomology in $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this chapter, we briefly review the computation which led to the leading-order expressions for the behavior of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ supergravity vertex operators near the AdS_5 boundary. The details and proofs omitted here can be found in the thesis by Fleury [32].

As seen in the previous chapter, in the pure-spinor formalism, physical vertex operators must be in the cohomology of the BRST operator Q. In [18], the authors computed Q for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background using the coset

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(6)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)}\,. \tag{3.1}$$

The AdS_5 superspace is parameterized by five bosonic variables denoted z and x^a for $a = 0$ to 3 and thirty-two fermionic variables denoted $\theta^{\alpha i}, \bar\theta_i^{\dot\alpha}, \psi_j^\beta, \bar\psi^{\dot\beta j}$ for $\alpha, \dot\alpha = 1$ to 2 and $i, j = 1$ to 4. We use the standard d=4 two-component spinor notation as described in Appendix A.1. These variables appear in the $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(6)}$ coset representative as

$$g = \exp\left(x^a P_a + \mathrm{i}\,\theta^{\alpha i} q_{\alpha i} + \mathrm{i}\,\bar\theta_{\dot\alpha i}\bar q^{\dot\alpha i}\right)\,\exp\left(\mathrm{i}\,\psi_j^\beta s_\beta^j + \mathrm{i}\,\bar\psi_{\dot\beta}^j \bar s_j^{\dot\beta}\right)\,z^\Delta\,, \tag{3.2}$$

where $P_a, q_{\alpha i}, \bar q^{\dot\alpha i}$ are generators of the $\mathcal{N}=4$, $d=4$ supersymmetry algebra, $s_\beta^j, \bar s_j^{\dot\beta}$ are the $\mathcal{N}=4$, $d=4$ superconformal generators and Δ is the generator of dilatations. With this choice of coset representative, it is easy to show that the AdS_5 metric has the form

$$\mathrm{d}s^2 = \frac{\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \mathrm{d}z^2}{z^2} \tag{3.3}$$

and thus the boundary of AdS_5 is located at $z=0$. At the boundary, the variables $x^a, \theta^{\alpha i}, \bar\theta_i^{\dot\alpha}$ transform in the usual $\mathcal{N}=4$, $d=4$ superconformal manner under the action of global $\mathrm{PSU}(2,2|4)$ transformations.

The S^5 is parameterized by an SO(6) vector y^I for $I = 1$ to 6 satisfying $y_I y^I = 1$. The SO(6) Pauli matrices $(\rho_I)_{ij}$ described in Appendix A.3 can be used to define $y_{ij} := (\rho_I)_{ij}\, y^I$, which satisfies $y_{ij} = \frac{1}{2}\varepsilon_{ijk\ell}\, y^{k\ell}$ and $y_{ij}y^{jk} = \delta_i^k$.

The final ingredients needed to study the pure-spinor string in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are the left- and right-moving ghost variables $\lambda, \widehat{\lambda}$ and their conjugate momenta $w, \widehat{w}$. The λ's are pure spinors, i.e. they satisfy $\lambda\gamma^\mu\lambda = 0$ and $\widehat{\lambda}\gamma^\mu\widehat{\lambda} = 0$ for $\mu = 0$ to 9. Note these expressions have been written in ten-dimensional notation where $\lambda^{\hat{\alpha}}$ and $\widehat{\lambda}^{\hat{\alpha}}$ are chiral spinors ($\hat{\alpha} = 1$ to 16) which can be decomposed into $\mathrm{SO}(3,1) \times \mathrm{SO}(6)$ spinors $(\lambda^{\alpha i}, \bar{\lambda}^{\dot{\alpha}}_j)$ and $(\widehat{\lambda}^{\alpha i}, \bar{\widehat{\lambda}}^{\dot{\alpha}}_j)$ in the usual manner. As explained in the previous chapter, the gauge invariance under $\delta w = (\gamma^\mu\lambda)\,\Lambda_\mu$ for any Λ_μ implies that w can only appear in combinations of the gauge-invariant quantities

$$N^{\mu\nu} = \frac{1}{2} w\gamma^{\mu\nu}\lambda\,, \qquad J_{\mathrm{g}} = w\lambda\,, \tag{3.4}$$

which are respectively the $\mathrm{SO}(9,1)$ Lorentz currents in the ghost sector and the ghost-number current. Of course, similar expressions hold for the hatted quantities.

3.1 Worldsheet action

To construct the BRST-invariant worldsheet action using the coset (3.1), one needs to define the left-invariant current $J = g^{-1}\mathrm{d}g$, taking values in the $\mathrm{PSU}(2,2|4)$ Lie algebra. Here $\mathrm{d} = \mathrm{d}\zeta\frac{\partial}{\partial\zeta} + \mathrm{d}\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}$ and the variables $\zeta, \bar{\zeta}$ parameterize the string worldsheet. The components of this current are defined via

$$g^{-1}\frac{\partial}{\partial\zeta}g = J^a\frac{1}{2}(P_a + K_a) + J^\Delta\Delta + J^k_j U^j_k + J^{\alpha i}q_{\alpha i} + J_{\dot{\alpha}i}\bar{q}^{\dot{\alpha}i} + J^\beta_j s^j_\beta + J^j_{\dot{\beta}}\bar{s}^{\dot{\beta}}_j + \cdots\,, \tag{3.5}$$

where K_a are the generators of special conformal transformations in four dimensions, U^j_k are the $\mathrm{SU}(4)$ R-symmetry generators and the dots stand for terms proportional to generators in the isotropy group of AdS_5, i.e. in $\mathrm{SO}(4,1)$. Analogously, one can define $\bar{J}^a, \bar{J}^\Delta, \ldots$ through the calculation of $g^{-1}\frac{\partial}{\partial\bar{\zeta}}g$. Note that when computing the left-hand side of (3.5) one should really look for the terms proportional to the combinations $\frac{1}{2}(P_a + K_a)$ and $\frac{1}{2}(P_a - K_a)$, as opposed to P_a and K_a alone, since the former are generators of the AdS_5 isometry group $\mathrm{SO}(4,2)$.

The components of the left-invariant current are obtained by means of the following formula for the derivation of an exponential

$$\mathrm{d}e^A = \left(\mathrm{d}A + \frac{1}{2!}[A, \mathrm{d}A] + \frac{1}{3!}[A,[A,\mathrm{d}A]] + \cdots\right)e^A\,, \tag{3.6}$$

as well as the Hadamard lemma

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots\,. \tag{3.7}$$

The commutators appearing in the above equations can be calculated using the $\mathrm{psu}(2,2|4)$ algebra shown in Appendix E.

In terms of these currents, the matter part of the worldsheet action is given by

$$\begin{aligned} S_{\rm matter} &= \frac{1}{2\pi}\int \mathrm{d}^2\zeta \left[\frac{1}{2}\eta_{ab}J^a\bar{J}^b + \frac{1}{2}J^{\Delta}\bar{J}^{\Delta} - \frac{1}{8}(\nabla y)_{jk}(\bar{\nabla}y)^{jk}\right. \\ &\qquad - 2J^{\alpha i}\bar{J}_{\alpha i} - 2J_{\dot\alpha i}\bar{J}^{\dot\alpha i} - 2J^{\beta}_{j}\bar{J}^{j}_{\beta} - 2J^{j}_{\dot\beta}\bar{J}^{\dot\beta}_{j} \\ &\qquad \left. - y_{jk}J^{\alpha j}\bar{J}^{k}_{\alpha} - y^{jk}J^{\alpha}_{j}\bar{J}_{\alpha k} + y^{jk}J_{\dot\alpha j}\bar{J}^{\dot\alpha}_{k} + y_{jk}J^{j}_{\dot\alpha}\bar{J}^{\dot\alpha k}\right], \end{aligned} \tag{3.8}$$

where $(\nabla y)_{jk} = \partial y_{jk} - J^{\ell}_{j}y_{\ell k} - J^{\ell}_{k}y_{j\ell}$. This action was found by looking for an expression which reduces to the one written in terms of the more usual coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ when one uses the SO(6) gauge invariance to gauge-fix $y^I = \delta^I_6$, for example. The latter action in turn can be found by demanding that the Lagrangian be invariant under global $\mathrm{PSU}(2,2|4)$ and local $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ transformations.

The complete action has also a contribution coming from the ghosts:

$$\begin{aligned} S_{\rm ghost} &= \frac{1}{2\pi}\int \mathrm{d}^2\zeta \left[w\cdot\bar{\nabla}\lambda - \widehat{w}\cdot\nabla\widehat{\lambda} + \frac{1}{2}y^{j\ell}(\bar{\nabla}y)_{\ell k}w_{\mathcal{A}j}\lambda^{\mathcal{A}k} - \frac{1}{2}y^{j\ell}(\bar{\nabla}y)_{\ell k}\widehat{w}_{\mathcal{A}j}\widehat{\lambda}^{\mathcal{A}k}\right. \\ &\qquad \left. - 2N_{ab}\widehat{N}^{ab} - 4(y^J N_{Ja})(y_K\widehat{N}^{Ka}) + 2N_{IJ}\widehat{N}^{IJ} - 4(y^L N_{LJ})(y_M\widehat{N}^{MJ})\right], \end{aligned} \tag{3.9}$$

where $\lambda^{\mathcal{A}k}, w_{\mathcal{A}j}$ and their hatted counterparts are $\mathrm{SO}(4,1)\times\mathrm{SO}(6)$ spinors given by

$$\lambda^{\mathcal{A}k} = \begin{pmatrix} \lambda^k_\alpha \\ y^{k\ell}\bar{\lambda}^{\dot\alpha}_{\ell} \end{pmatrix}, \qquad w_{\mathcal{A}j} = \begin{pmatrix} w^\alpha_j \\ -y_{j\ell}\bar{w}^{\ell}_{\dot\alpha} \end{pmatrix}, \tag{3.10}$$

and

$$\begin{aligned} w\cdot\bar{\nabla}\lambda &:= w^\alpha_j\bar{\partial}\lambda^j_\alpha + \bar{w}^k_{\dot\alpha}y_{ki}\bar{\partial}(y^{ij}\bar{\lambda}^{\dot\alpha}_j) - w^\alpha_j\bar{J}^\beta_\alpha\lambda^j_\beta - \bar{w}^k_{\dot\alpha}\bar{J}^{\dot\alpha}_{\dot\beta}\bar{\lambda}^{\dot\beta}_k \\ &\quad + 2w^\alpha_j\bar{J}_{\alpha\dot\alpha}y^{jk}\bar{\lambda}^{\dot\alpha}_k - 2\bar{w}^k_{\dot\alpha}y_{k\ell}\bar{J}^{\dot\alpha\alpha}\lambda^\ell_\alpha + w^\alpha_j\bar{J}^j_k\lambda^k_\alpha + \bar{w}^k_{\dot\alpha}y_{ki}\bar{J}^i_m y^{mn}\bar{\lambda}^{\dot\alpha}_n. \end{aligned} \tag{3.11}$$

Note that the covariant derivative above has contributions coming from all the $\mathrm{SO}(4,1)\times\mathrm{SO}(6)$ connections, corresponding to U^j_k, $\frac{1}{2}(P_a - K_a)$ and the four-dimensional Lorentz generators M_{ab}. An analogous definiton holds for $\widehat{w}\cdot\nabla\widehat{\lambda}$.

3.2 The BRST operator and its cohomology

In the last section we introduced the worldsheet action for the pure-spinor superstring in an $\mathrm{AdS}_5\times\mathrm{S}^5$ background, which is given by $S = S_{\rm matter} + S_{\rm ghost}$. This action is invariant under the BRST transformations generated by

$$\begin{aligned} Q &= \int \mathrm{d}\zeta \left[\lambda^{\alpha i}(J_{\alpha i} - y_{ij}J^j_\alpha) - \bar{\lambda}_{\dot\alpha i}(J^{\dot\alpha i} + y^{ij}J^{\dot\alpha}_j)\right] \\ &\quad - \int \mathrm{d}\bar{\zeta}\left[\widehat{\lambda}^{\alpha i}(\bar{J}_{\alpha i} + y_{ij}\bar{J}^j_\alpha) + \widehat{\bar{\lambda}}_{\dot\alpha i}(\bar{J}^{\dot\alpha i} - y^{ij}\bar{J}^{\dot\alpha}_j)\right]. \end{aligned} \tag{3.12}$$

This BRST operator was also found by comparing with the one written in terms of the more usual coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ in [13].

To simplify the analysis of the cohomology, it is convenient to express the BRST charge in terms of the worldsheet variables and their canonical momenta, defined as $P_x = \frac{\delta S}{\delta(\partial_\tau x)}$, $P_z = \frac{\delta S}{\delta(\partial_\tau z)}$ and so on. Here τ is the variable associated with the time direction of the worldsheet, whereas σ is associated with the space direction. In our conventions,

$$\partial = \frac{1}{2}(\partial_\sigma - \partial_\tau)\,, \qquad \bar{\partial} = \frac{1}{2}(\partial_\sigma + \partial_\tau)\,. \tag{3.13}$$

After substituting the currents, the operator Q can be organized in the form

$$Q = Q_{-\frac{1}{2}} + Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \cdots\,, \tag{3.14}$$

where $Q_n \propto z^n$. Near the boundary of AdS_5, it is also possible to expand a physical vertex operator V as

$$V = V_{d_0} + V_{d_0+1} + \cdots\,, \tag{3.15}$$

where $V_n \propto z^n$ and d_0 is the minimum degree of V, i.e. V_{d_0} is the leading-order term in the expansion of V near the boundary.

Equations (3.14) and (3.15) imply that, if one is only interested in the leading-order contribution to the vertex operator, then one can first compute the cohomology of $Q_{-\frac{1}{2}}$, then compute the cohomology of $Q_{\frac{1}{2}}$ restricted to that of $Q_{-\frac{1}{2}}$, then compute the cohomology of $Q_{\frac{3}{2}}$ restricted to those of $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$, and so on. The reason is that, collecting the terms with the same power of z, one has*

$$QV = 0 \iff \begin{cases} Q_{-\frac{1}{2}}V_{d_0} = 0\,, \\ Q_{\frac{1}{2}}V_{d_0} + Q_{-\frac{1}{2}}V_{d_0+1} = 0\,, \\ Q_{\frac{3}{2}}V_{d_0} + Q_{\frac{1}{2}}V_{d_0+1} + Q_{-\frac{1}{2}}V_{d_0+2} = 0\,, \\ \vdots \end{cases} \tag{3.16}$$

This procedure is well defined since the complete BRST operator Q is nilpotent, which implies, in particular,

$$\begin{aligned} \{Q_{-\frac{1}{2}}, Q_{-\frac{1}{2}}\} &= 0\,, \\ \{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\} + 2\,\{Q_{-\frac{1}{2}}, Q_{\frac{3}{2}}\} &= 0\,, \end{aligned} \tag{3.17}$$

that is, $Q_{-\frac{1}{2}}$ is nilpotent and $Q_{\frac{1}{2}}$ is nilpotent when acting on operators in the cohomology of $Q_{-\frac{1}{2}}$. Indeed, acting with (3.17) on a $Q_{-\frac{1}{2}}$-closed operator $\mathcal{O}$, we get

$$\{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\}\mathcal{O} = -2\,\{Q_{-\frac{1}{2}}, Q_{\frac{3}{2}}\}\mathcal{O} = -2\,Q_{-\frac{1}{2}}Q_{\frac{3}{2}}\mathcal{O} = 0\,, \tag{3.18}$$

*Here and in the following the juxtaposition of the BRST operator and the operator it acts on should be understood as a shorthand notation for the generalized commutator, i.e. $QV \equiv [Q, V\}$.

up to a $Q_{-\frac{1}{2}}$-trivial term. A similar nilpotency argument applies to $Q_{\frac{3}{2}}, Q_{\frac{5}{2}}, \ldots$.

The computation of $Q_{-\frac{1}{2}}$ gives

$$Q_{-\frac{1}{2}} = \frac{1}{\sqrt{z}} \left(\lambda^{+\gamma m} y_{mi} P_{\psi_i^\gamma} + \bar{\lambda}_j^{+\dot{\alpha}} y^{ji} P_{\bar{\psi}^{i\dot{\alpha}}} \right) + \partial_\sigma\text{--terms}\,, \tag{3.19}$$

where

$$\lambda^{+\alpha i} := -\mathrm{i}\,(\lambda^{\alpha i} - \widehat{\lambda}^{\alpha i})\,, \qquad \bar{\lambda}^+_{\dot{\alpha}j} := \mathrm{i}\,(\bar{\lambda}_{\dot{\alpha}j} + \bar{\widehat{\lambda}}_{\dot{\alpha}j})\,. \tag{3.20}$$

We do not consider terms in Q containing σ-derivatives, since we are only interested in its zero-mode cohomology. In other words, we take the limit in which the string length goes to zero (or the tension goes to infinity), which corresponds to supergravity (massless states).

Because of the usual quartet argument, we assume the zero-mode cohomology of $Q_{-\frac{1}{2}}$ is independent of λ^+.† Moreover, the states in the cohomology depend on ψ only through $\lambda^- \gamma_\mu \hat{\psi}$, where $\hat{\psi} := y_{ij}\,(\gamma^{[ij]}\psi)$ and

$$\lambda^{-\alpha i} := -\mathrm{i}\,(\lambda^{\alpha i} + \widehat{\lambda}^{\alpha i})\,, \qquad \bar{\lambda}^-_{\dot{\alpha}j} := \mathrm{i}\,(\bar{\lambda}_{\dot{\alpha}j} - \bar{\widehat{\lambda}}_{\dot{\alpha}j})\,. \tag{3.21}$$

The reason is that $\lambda^- \gamma_\mu \hat{\psi}$ is annihilated by $Q_{-\frac{1}{2}}$, since $Q_{-\frac{1}{2}} \left(\lambda^- \gamma_\mu \hat{\psi} \right) \propto \lambda^- \gamma_\mu \lambda^+$, which vanishes because of the pure-spinor conditions for λ and $\widehat{\lambda}$. One can also show that

$$\lambda^+ \gamma_\mu \lambda^+ + \lambda^- \gamma_\mu \lambda^- = 0\,. \tag{3.22}$$

This identity implies that, when considering states in the cohomology of $Q_{-\frac{1}{2}}$, we have $\lambda^- \gamma_\mu \lambda^- = 0$, i.e. λ^- is a pure spinor.

The next step is to compute the zero-mode cohomology of $Q_{\frac{1}{2}} + Q_{\frac{3}{2}} + \cdots$ restricted to states in the cohomology of $Q_{-\frac{1}{2}}$. This means we can neglect terms containing λ^+ and we can consider λ^- a pure spinor. It turns out that $Q_{\frac{3}{2}}, Q_{\frac{5}{2}}, \ldots$ act as zero on states in the cohomology of $Q_{-\frac{1}{2}}$. This is because the terms depending on ψ in their expansions cannot be expressed in terms of the $\lambda^- \gamma_\mu \hat{\psi}$. Thus, the zero-mode cohomology of Q near the boundary of AdS_5 is determined by $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$ only.

Writing the canonical momenta as derivatives, the part of $Q_{\frac{1}{2}}$ that is relevant for us is, then,

$$Q_{\frac{1}{2}} = \sqrt{z} \left[\lambda^{\hat{\alpha}} D_{\hat{\alpha}} + 4\,(\lambda \gamma^{[ij]} \hat{\psi}) \frac{\partial}{\partial y^{ij}} + y_{ij} (\lambda \gamma^{[ij]} \hat{\psi}) \left(2z \frac{\partial}{\partial z} + y^{k\ell} \frac{\partial}{\partial y^{k\ell}} - \lambda^{\hat{\alpha}} \frac{\partial}{\partial \lambda^{\hat{\alpha}}} \right) \right], \tag{3.23}$$

†Actually, this argument is too naive, and the cohomology is only independent of λ^+ after allowing dependence on non-mininal pure-spinor variables. See section 3.3 of [32] for a more detailed discussion.

where $D_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{\hat{\alpha}}} + (\theta\gamma^a)_{\hat{\alpha}}\partial_a$ is the dimensional reduction of the d=10 supersymmetric derivative and we have dropped the minus superscript from λ^-. The second term in (3.23) is understood not to act on $\lambda\gamma_\mu\hat{\psi}$, even though $\hat{\psi}$ depends on y. This is because there are other terms in $Q_{\frac{1}{2}}$ containing ψ-derivatives (conjugate momenta) which we have omitted. It turns out the action of these extra terms on $\lambda\gamma_\mu\hat{\psi}$ cancels out the action of the second term in (3.23). Since $Q_{\frac{1}{2}}$ should only act on objects in the cohomology of $Q_{-\frac{1}{2}}$, it would be pointless to write those extra terms.

In order to express the vertex operators in a convenient way, we need to allow dependence on the non-minimal pure-spinor variables introduced in section 2.2. After introducing the non-minimal variables, we need to modify the BRST operator $Q_{\frac{1}{2}}$ as:

$$Q_{\frac{1}{2}} \longmapsto Q_{\frac{1}{2}} + \widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}} \, . \tag{3.24}$$

Then, for arbitrary N, it was shown in [18] that $Q_{\frac{1}{2}} + \widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}$ annihilates the following ghost-number +2 (closed superstring) vertex operator:

$$V_{(N)} = z^{2-N} \int \mathrm{d}u \sum_{n=0}^{4} 8^n P_n(N)\, (yuu)^{N-n-1}\, \Omega_{(n)} T^{(4-N)}(x, \theta, u, \overline{u}) \, , \tag{3.25}$$

where $P_n(N) = \frac{1}{N}\prod_{m=0}^{n}(N-m)$ is a polynomial of degree n in N, $T^{(4-N)}$ is a G-analytic superfield of harmonic U(1) charge $4-N$, and $\int \mathrm{d}u$ denotes an integral over the compact space SU(4)/S(U(2)×U(2)) parameterized by the harmonic variables $u_{\mathcal{I}}{}^i$. See appendix B for more information on harmonic superspace. The operators $\Omega_{(n)}$ in (3.25) are defined by

$$\begin{aligned}
\Omega_{(0)} &= \frac{1}{16}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D}) \\
&\quad + \frac{1}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_{\mu\nu\rho}\widetilde{\lambda})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D}) \, , &(3.26\text{a}) \\
\Omega_{(1)} &= -\frac{1}{2}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D}) \\
&\quad - \frac{6}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_{\mu\nu\rho}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D}) \, , &(3.26\text{b}) \\
\Omega_{(2)} &= \frac{3}{2}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D}) \\
&\quad + \frac{12}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_{\mu\nu\rho}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\rho\widetilde{D}) \, , &(3.26\text{c}) \\
\Omega_{(3)} &= -2\,(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\rho\hat{\psi})(\lambda\gamma^\sigma\widetilde{D}) \\
&\quad - \frac{8}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_{\mu\nu\rho}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\rho\hat{\psi}) \, , &(3.26\text{d}) \\
\Omega_{(4)} &= (uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\rho\hat{\psi})(\lambda\gamma^\sigma\hat{\psi}) \, , &(3.26\text{e})
\end{aligned}$$

with $\widetilde{D}^{\hat\alpha} := -\frac{1}{2}(\overline{uu})_{ij}(\gamma^{[ij]}D)^{\hat\alpha}$. Note this definition has an extra factor of $-\frac{1}{2}$ as compared to Fleury's. This leads to the different numerical factors in (3.26).

It is easy to check that the $\left(2z\frac{\partial}{\partial z} + y^{k\ell}\frac{\partial}{\partial y^{k\ell}} - \lambda^{\hat\alpha}\frac{\partial}{\partial\lambda^{\hat\alpha}}\right)$-part of $Q_{\frac{1}{2}}$ annihilates $V_{(N)}$. Moreover, the $\Omega_{(n)}$'s have been designed so that the following equations are satisfied:

$$\begin{aligned}
\left(\sqrt{z}\lambda^{\hat\alpha}D_{\hat\alpha} + \tilde{w}^{\hat\alpha}r_{\hat\alpha}\right)\Omega_{(0)}T^{(4-N)} &= 0\,, && \text{(3.27a)}\\
\left(\sqrt{z}\lambda^{\hat\alpha}D_{\hat\alpha} + \tilde{w}^{\hat\alpha}r_{\hat\alpha}\right)\Omega_{(1)}T^{(4-N)} &= -\sqrt{z}\,(\lambda\gamma^{[ij]}\hat\psi)(uu)_{ij}\Omega_{(0)}T^{(4-N)}\,, && \text{(3.27b)}\\
\left(\sqrt{z}\lambda^{\hat\alpha}D_{\hat\alpha} + \tilde{w}^{\hat\alpha}r_{\hat\alpha}\right)\Omega_{(2)}T^{(4-N)} &= -\sqrt{z}\,(\lambda\gamma^{[ij]}\hat\psi)(uu)_{ij}\Omega_{(1)}T^{(4-N)}\,, && \text{(3.27c)}\\
\left(\sqrt{z}\lambda^{\hat\alpha}D_{\hat\alpha} + \tilde{w}^{\hat\alpha}r_{\hat\alpha}\right)\Omega_{(3)}T^{(4-N)} &= -\sqrt{z}\,(\lambda\gamma^{[ij]}\hat\psi)(uu)_{ij}\Omega_{(2)}T^{(4-N)}\,, && \text{(3.27d)}\\
\left(\sqrt{z}\lambda^{\hat\alpha}D_{\hat\alpha} + \tilde{w}^{\hat\alpha}r_{\hat\alpha}\right)\Omega_{(4)}T^{(4-N)} &= -\sqrt{z}\,(\lambda\gamma^{[ij]}\hat\psi)(uu)_{ij}\Omega_{(3)}T^{(4-N)}\,, && \text{(3.27e)}\\
\sqrt{z}\,(\lambda\gamma^{[ij]}\hat\psi)(uu)_{ij}\Omega_{(4)}T^{(4-N)} &= 0\,. && \text{(3.27f)}
\end{aligned}$$

3.3 Connection to $\mathcal{N} = 4$ SYM

If the operators (3.25) correspond to supergravity states in $\mathrm{AdS}_5\times\mathrm{S}^5$, then AdS/CFT predicts they should be dual to half-BPS states in $\mathcal{N} = 4$ SYM. Indeed, we can make this relation explicit by making use of the harmonic superspace. Consider the following family of gauge-invariant operators introduced in [19]:

$$W^{(N)} := \mathrm{tr}\, W^N\,, \qquad N = 1, 2, \ldots\,, \tag{3.28}$$

where $W := (uu)^{ij}W_{ij}$, W_{ij} is the Sohnius field strength of $\mathcal{N} = 4$ SYM [22] and the trace is taken over the gauge group. In components, we have

$$W_{ij}(x,\theta,\bar\theta) = \phi_{ij}(x) - \varepsilon_{ijk\ell}\theta^k\xi^\ell(x) + 2\bar\theta_{[i}\bar\xi_{j]}(x) + \frac{1}{4}\varepsilon_{ijk\ell}\theta^k\sigma^{ab}\theta^\ell f_{ab}(x) - \frac{1}{2}\bar\theta_i\tilde\sigma^{ab}\bar\theta_j f_{ab}(x) + \cdots\,, \tag{3.29}$$

where ϕ_{ij}, ξ, $\bar\xi$ and f_{ab} are, respectively, the $\mathcal{N} = 4$ SYM scalars, chiral and anti-chiral gluinos and gluon field-strength.

It is easy to see $W^{(N)}$ describes a gauge-invariant half-BPS operator constructed from N SYM fields. Taking the $\theta = 0$ component of W_{ij}, we get

$$\phi^{(N)} := (uu)^{i_1j_1}\ldots(uu)^{i_Nj_N}\,\mathrm{tr}\,[\phi_{i_1j_1}(x)\ldots\phi_{i_Nj_N}(x)]\,. \tag{3.30}$$

Since the u's are bosonic (commuting) and satisfy $(uu)_{ij}(uu)^{ij} = 0$, this is a totally symmetric and traceless product of ϕ's, i.e. it is a chiral primary operator.

For each value of N, one can show that $W^{(N)}$ is an analytic superfield. In particular, this implies that it depends only on half the sixteen θ's. Thus, it is possible to define a superfield dual to $W^{(N)}$ through the coupling

$$\int \mathrm{d}^4x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(N)}(x,\theta,u)\, U^{(4-N)}(x,\theta,u,\overline{u})\right]\,, \tag{3.31}$$

where $D'^4, \bar{D}^4$, defined in Appendix B, can be regarded as Berezin integrals over the eight fermionic variables $(u\theta)^{\alpha A'}, (u\bar{\theta})^{\dot{\alpha}}_{A}$. Here $U^{(4-N)}$ is a (otherwise unconstrained) G-analytic superfield of harmonic U(1) charge $4-N$. In [20], these superfields $U^{(4-N)}$ were shown to be in one-to-one correspondence with the chiral superfields describing type IIB supergravity states in $\mathrm{AdS}_5 \times \mathrm{S}^5$ — see [33] for an introduction to this work.

It is natural to identify $U^{(4-N)}$ with the $T^{(4-N)}$ of (3.25), since they have exactly the same properties. One way to see it is consistent is by first noting that the coupling (3.31) is invariant under the transformation

$$U^{(N)} \longmapsto U^{(N)} + u_A{}^i \frac{\partial}{\partial \bar{u}_{A'}{}^i} \Xi^{(N-1)\,A}_{A'} , \tag{3.32}$$

with $\Xi^{(N-1)\,A}_{A'}$ some G-analytic superfield of harmonic U(1) charge $N-1$. Then one can show that, when $T^{(4-N)}$ changes according to (3.32), $V_{(N)}$ changes by a BRST-trivial amount.

Thus, the superfield $T^{(4-N)}$ appearing in the expression for the vertex operator $V_{(N)}$ is dual to the half-BPS operator $W^{(N)}$ in the sense of (3.31). This in turn implies a correspondence between the supergravity state itself and $W^{(N)}$. For example, when $T^{(2)} = \theta^{\alpha i}(uu)_{ij}\theta^{\beta j}\theta^k_\alpha(uu)_{k\ell}\theta^\ell_\beta$, one can show $V_{(2)} \propto y_{ij}(\lambda^i\lambda^j)$ (up to BRST-trivial terms). This $\mathrm{PSU}(2,2|4)$ scalar is the zero-momentum dilaton vertex operator, which is dual to the linearized SYM action $\int \mathrm{d}^4x \int \mathrm{d}u\, \bar{D}^4 W^{(2)}$, as can be seen from (3.31).

Chapter 4

Open-closed amplitudes

Because of the duality presented at the end of the last chapter, and also because of symmetry arguments, the disk scattering amplitude with the supergravity vertex operator $V_{(N)}$ and N massless open superstring (SYM) vertex operators was conjectured in [18] to be proportional to the coupling (3.31). The SYM vertex operators would be located on a D3-brane parallel and close to the AdS_5 boundary, at some fixed value of y^{ij} and z near 0. Since the disk has an $\mathrm{SL}(2,\mathbb{R})$ symmetry which allows us to fix the positions of one open and one closed superstring vertex operator, the disk amplitude has the form

$$\mathcal{M}_N := \left\langle V_{(N)}\, V_{\mathrm{SYM}} \int \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1}\, U_{\mathrm{SYM}}(\xi_1) \cdots U_{\mathrm{SYM}}(\xi_{N-1}) \right\rangle_{\mathrm{D3-brane}}, \tag{4.1}$$

where V_{SYM} is the unintegrated vertex operator of SYM and U_{SYM} is the integrated one.

The angle brackets in the above equation contain integrations over the x, λ and θ zero modes, but they do not contain integrations over the z and y^{ij} zero modes, since the position of the D3-brane is fixed. Schematically, one has

$$\left\langle \lambda^{\hat\alpha}\lambda^{\hat\beta}\lambda^{\hat\gamma}\, f(x,z,y,\theta) \right\rangle_{\mathrm{D3-brane}} = \int \frac{\mathrm{d}^4 x}{z^4} \int \frac{(\mathrm{d}^5\theta)^{\hat\alpha\hat\beta\hat\gamma}}{z^{-5/2}} f(x,z,y,\theta)\,, \tag{4.2}$$

where the powers of z ensure the measure is $\mathrm{PSU}(2,2|4)$-invariant, since d^4x and $\mathrm{d}^5\theta$ have dimension -4 and $\frac{5}{2}$, respectively. More details on the integration of the λ and θ zero modes will be given later in this chapter.

Proving that (4.1) is indeed proportional to (3.31) is the main purpose of this thesis, and what will be done in this chapter.

4.1 Preliminary calculations

In this section we perform some preliminary calculations whose results will be useful for the amplitude computations we will do later in this chapter. We begin by showing how the operators $\Omega_{(0)}$ and $\Omega_{(1)}$ that appear in the expression for the supergravity vertex operator can be simplified through the addition of BRST-trivial terms to the vertex operator $V_{(2)}$.

Next, we investigate the pure spinor measure factor for the λ and θ zero-mode integrations taking into account the $\mathrm{SO}(3,1) \times \mathrm{SU}(4)$ symmetry of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ boundary. Following [34], we show the measure factor to be unique up to BRST transformations and an overall factor.

4.1.1 Simplification of $\Omega_{(0)}$ and $\Omega_{(1)}$

In this subsection, we will show that, by adding a BRST-trivial quantity to the vertex operator $V_{(2)}$, one can effectively simplify the form of the operators $\Omega_{(0)}$ and $\Omega_{(1)}$. More precisely, we will find an object $\chi_{(2)}$ such that we can write a BRST-equivalent $N=2$ vertex operator as

$$V'_{(2)} = V_{(2)} + \left(Q_{\frac{1}{2}} + \widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) \chi_{(2)} = \int \mathrm{d}u \left[(yuu)\widetilde{\Omega}_{(0)} + 8\,\widetilde{\Omega}_{(1)}\right] T^{(2)}\,, \tag{4.3}$$

with simpler, non-minimal-variable-free operators $\widetilde{\Omega}_{(0)}$ and $\widetilde{\Omega}_{(1)}$.

First ingredients

The first step is to decompose $\lambda^{\hat{\alpha}} D_{\hat{\alpha}}$ in the following manner:

$$\begin{aligned}
\lambda^{\hat{\alpha}} D_{\hat{\alpha}} &= -\frac{1}{2}(-2)\lambda^{\hat{\alpha}}\delta_{\hat{\alpha}}^{\hat{\beta}} D_{\hat{\beta}} \\
&= -\frac{1}{2}(uu)_{ij}(\overline{uu})^{ij}\lambda^{\hat{\alpha}}\delta_{\hat{\alpha}}^{\hat{\beta}} D_{\hat{\beta}} \\
&= -\frac{1}{4}(uu)_{ij}(\overline{uu})_{k\ell}\lambda^{\hat{\alpha}}\varepsilon^{ijk\ell}\delta_{\hat{\alpha}}^{\hat{\beta}} D_{\hat{\beta}} \\
&= \frac{1}{4}(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\{\gamma^{[ij]},\gamma^{[k\ell]}\}D) \\
&= -\frac{1}{2}(uu)_{ij}(\lambda\gamma^{[ij]}\widetilde{D}) + \frac{1}{4}(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D)\,,
\end{aligned} \tag{4.4}$$

where we used the definition $\widetilde{D}^{\hat{\alpha}} := -\frac{1}{2}(\overline{uu})_{ij}(\gamma^{[ij]}D)^{\hat{\alpha}}$. Note the second term in (4.4) annihilates G-analytic superfields.

We will also need the following results:

$$\{(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D), (\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})\} = 0\,. \tag{4.5}$$

and

$$\begin{aligned}
&\{(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D), (\lambda\gamma^{\mu}\hat{\psi})(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})\} = \\
&\qquad = 16\,(\overline{uu})_{mn}(\lambda\widetilde{\lambda})(\lambda\gamma^{[mn]}\hat{\psi})(\lambda\gamma^{a}\widetilde{D})\partial_a\,.
\end{aligned} \tag{4.6}$$

Let us prove the above anticommutation relations. First, note that $\{D_{\hat\alpha}, \widetilde{D}^{\hat\beta}\} = -\frac{1}{2}(\overline{uu})_{ij}(\gamma^{[ij]})^{\hat\beta\hat\gamma}\{D_{\hat\alpha}, D_{\hat\gamma}\} = (\overline{uu})_{ij}(\gamma^{[ij]a})_{\hat\alpha}{}^{\hat\beta}\partial_a$ implies

$$\begin{aligned}\{(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D), (\lambda\gamma^{\mu}\widetilde{D})\} &= (uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]})^{\hat\alpha}(\lambda\gamma^{\mu})_{\hat\beta}\{D_{\hat\alpha}, \widetilde{D}^{\hat\beta}\} \\ &= (uu)_{ij}(\overline{uu})_{k\ell}(\overline{uu})_{mn}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}\gamma^{[mn]a}\gamma^{\mu}\lambda)\partial_a \\ &= (uu)_{ij}(\overline{uu})_{k\ell}(\overline{uu})_{mn}(\lambda\gamma^{[k\ell][ij][mn]a\mu}\lambda)\partial_a \\ &= 0\,, \end{aligned} \tag{4.7}$$

since $(\overline{uu})_{k\ell}(\overline{uu})_{mn}$ is symmetric under $[k\ell] \leftrightarrow [mn]$. Therefore,

$$\{(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D), (\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})\} =$$

$$\begin{aligned} &= (uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[k\ell]}\gamma^{[ij]})^{\hat\alpha}(\widetilde{\lambda}\gamma_{\mu\nu})_{\hat\beta}\{D_{\hat\alpha}, \widetilde{D}^{\hat\beta}\} \\ &= -(uu)_{ij}(\overline{uu})_{k\ell}(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[k\ell]}\gamma^{[ij]}\gamma^{[mn]a}\gamma_{\mu}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &= -(uu)_{ij}(\overline{uu})_{k\ell}(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\{\gamma^{[k\ell]}, \gamma^{[ij]}\}\gamma^{[mn]}\gamma^{a}\gamma_{\mu}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &= \varepsilon^{k\ell ij}(uu)_{ij}(\overline{uu})_{k\ell}(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[mn]}\gamma^{a}\gamma_{\mu}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &= -4\,(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[mn]}\gamma^{a}\gamma_{\mu}\gamma_{\nu}\widetilde{\lambda})\partial_a \end{aligned} \tag{4.8}$$

where we used $(\overline{uu})_{k\ell}(\overline{uu})_{mn}\gamma^{[k\ell]}\gamma^{[mn]} = \frac{1}{2}(\overline{uu})_{k\ell}(\overline{uu})_{mn}\{\gamma^{[k\ell]}, \gamma^{[mn]}\} = -(\overline{uu})_{k\ell}(\overline{uu})^{k\ell} = 0$ and $\varepsilon^{k\ell ij}(uu)_{ij}(\overline{uu})_{k\ell} = 2\,(uu)^{k\ell}(\overline{uu})_{k\ell} = -4$. Then, using $\gamma^{\nu}\gamma_{\rho} = -2\delta^{\nu}_{\rho} - \gamma_{\rho}\gamma^{\nu}$ a few times and recalling $(\lambda\gamma_{\nu})_{\hat\alpha}(\lambda\gamma^{\nu})_{\hat\beta} = 0$ for a pure spinor, we get

$$-4\,(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[mn]}\gamma^{a}\gamma_{\mu}\gamma_{\nu}\widetilde{\lambda})\partial_a =$$

$$\begin{aligned} &= 8\,(\overline{uu})_{mn}(\lambda\gamma^{a}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[mn]}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &\quad + 4\,(\overline{uu})_{mn}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{[mn]}\gamma_{\mu}\gamma^{a}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &= -16\,(\overline{uu})_{mn}(\lambda\gamma^{a}\widetilde{D})(\lambda\gamma^{[mn]}\widetilde{D})(\lambda\widetilde{\lambda})\partial_a \\ &\quad - 8\,(\overline{uu})_{mn}(\lambda\gamma^{[mn]}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\lambda\gamma^{a}\gamma_{\nu}\widetilde{\lambda})\partial_a \\ &= 0\,, \end{aligned} \tag{4.9}$$

since $(\overline{uu})_{mn}(\gamma^{[mn]}\widetilde{D}) \propto (\overline{uu})_{mn}(\overline{uu})^{mn}D = 0$. Hence, we have proven equation (4.5).

Now note that $(\lambda\gamma^{\mu}\hat\psi)(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D}) = (\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\hat\psi)(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})$. Then, carrying out the same steps as in the previous case, one obtains the proof of (4.6).

Alternative form of $\Omega_{(0)}$

The next step is to write $\Omega_{(0)}$ and $\Omega_{(1)}$ in alternative forms which will be useful when comparing with BRST-trivial terms. Let us begin with $\Omega_{(0)}$. For the r-dependent part, we have (cf. (3.26))

$$\Omega_{(0)}\big|_r := \frac{1}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_{\mu\nu\rho}\widetilde{\lambda})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})$$
$$= -\frac{1}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_\mu\gamma_\nu\lambda)(\lambda\gamma^\mu\widetilde{D})(\widetilde{\lambda}\gamma_\rho\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})$$
$$+ \frac{1}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(r\gamma_\mu\gamma_\nu\widetilde{D})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\gamma_\rho\widetilde{\lambda})(\lambda\gamma^\rho\widetilde{D}), \quad (4.10)$$

where we used $(\gamma^\nu)_{\hat{\alpha}(\hat{\beta}|}(\gamma_\nu)_{|\hat{\gamma}\hat{\delta})} = 0$. Now one can substitute $\gamma_\mu\gamma_\nu$ in the first term and $\gamma^\nu\gamma_\rho$ in the second term with the anticommutators $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$ and $\{\gamma^\nu, \gamma_\rho\} = -2\delta^\nu_\rho$ to obtain

$$\Omega_{(0)}\big|_r = \frac{2}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-2}(\lambda r)(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\rho}\widetilde{D})$$
$$- \frac{2}{\sqrt{z}}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(r\gamma_{\mu\rho}\widetilde{D}). \quad (4.11)$$

The r-independent part is more involved. Using $(\gamma^\nu)_{\hat{\alpha}(\hat{\beta}|}(\gamma_\nu)_{|\hat{\gamma}\hat{\delta})} = 0$ once again, we get

$$\Omega_{(0)}\big|_{r=0} := \frac{1}{16}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_{\mu\nu\rho\sigma[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D})$$
$$= -\frac{1}{16}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\widetilde{\lambda}\gamma_\mu\gamma_\nu\lambda)(\lambda\gamma^\mu\widetilde{D})(\widetilde{D}\gamma^\nu\gamma_\rho\gamma_\sigma\gamma_{[ij]}\widetilde{\lambda})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D})$$
$$- \frac{1}{16}(uu)^{ij}(\lambda\widetilde{\lambda})^{-2}(\lambda\gamma^\nu\gamma_\rho\gamma_\sigma\gamma_{[ij]}\widetilde{\lambda})(\lambda\gamma^\mu\widetilde{D})(\widetilde{\lambda}\gamma_\mu\gamma_\nu\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D}).$$

The second term in the above expression vanishes, as can be easily seen by writing $\gamma^\nu\gamma_\rho = -2\delta^\nu_\rho - \gamma_\rho\gamma^\nu$ and using $(\lambda\gamma_\sigma)_{\hat{\alpha}}(\lambda\gamma^\sigma)_{\hat{\beta}} = 0$. In the first term, one can substitute $\gamma_\mu\gamma_\nu$ with $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$ to obtain

$$\Omega_{(0)}\big|_{r=0} = \frac{1}{8}(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_\nu\widetilde{D})(\widetilde{D}\gamma^\nu\gamma_\rho\gamma_\sigma\gamma_{[ij]}\widetilde{\lambda})(\lambda\gamma^\rho\widetilde{D})(\lambda\gamma^\sigma\widetilde{D}).$$

Then, after one more round of $(\gamma^\sigma)_{\hat{\alpha}(\hat{\beta}|}(\gamma_\sigma)_{|\hat{\gamma}\hat{\delta})} = 0$ and $\gamma^\nu\gamma_\rho = -2\delta^\nu_\rho - \gamma_\rho\gamma^\nu$, we get

$$\Omega_{(0)}\big|_{r=0} = -\frac{1}{4}(uu)^{ij}(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D})$$
$$+ \frac{1}{8}(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma_{[ij]}\gamma^\sigma\widetilde{\lambda})(\lambda\gamma^\rho\widetilde{D})(\widetilde{D}\gamma_{\nu\sigma\rho}\widetilde{D})$$
$$=: \widetilde{\Omega}_{(0)} + \frac{1}{8}(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\nu\widetilde{D})(\lambda\gamma_{[ij]}\gamma^\sigma\widetilde{\lambda})(\lambda\gamma^\rho\widetilde{D})(\widetilde{D}\gamma_{\nu\sigma\rho}\widetilde{D}). \quad (4.12)$$

Finally, using $(\gamma^\sigma)^{\hat\alpha(\hat\beta}(\gamma_\sigma)^{\hat\gamma\hat\delta)}=0$ in the last term of the above equation, we arrive at

$$\begin{aligned}
\Omega_{(0)}\big|_{r=0}-\widetilde\Omega_{(0)} &= \frac{1}{4}(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\widetilde D)(\lambda\gamma_{[ij]}\gamma^\sigma\gamma_\nu\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&= -\frac{1}{4}(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\widetilde D)(\lambda\gamma^\sigma\gamma_{[ij]}\gamma_\nu\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&= \frac{1}{2}(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\widetilde D)(\lambda\gamma^\sigma\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&\quad+\frac{1}{4}(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\widetilde D)(\lambda\gamma^\sigma\gamma_\nu\gamma_{[ij]}\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&= (uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\widetilde D)(\lambda\gamma^\sigma\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\,, \qquad (4.13)
\end{aligned}$$

where again we used $\gamma^\nu\gamma_\rho=-2\delta^\nu_\rho-\gamma_\rho\gamma^\nu$. Therefore,

$$\Omega_{(0)}\big|_{r=0}=\widetilde\Omega_{(0)}+(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\widetilde D)(\lambda\gamma^\sigma\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_{\sigma\rho}\widetilde D)\,. \qquad (4.14)$$

Alternative form of $\Omega_{(1)}$

Let us now derive an alternative form of $\Omega_{(1)}$. For the r-dependent part, the same manipulations which led to (4.11) yield

$$\begin{aligned}
\Omega_{(1)}\big|_r &= -\frac{12}{\sqrt z}(\lambda\widetilde\lambda)^{-2}(\lambda r)(\lambda\gamma^\mu\hat\psi)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_{\mu\rho}\widetilde D)\\
&\quad+\frac{12}{\sqrt z}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\mu\hat\psi)(\lambda\gamma^\rho\widetilde D)(r\gamma_{\mu\rho}\widetilde D)\,. \qquad (4.15)
\end{aligned}$$

For the r-independent part, carrying out the same steps as in the $\Omega_{(0)}\big|_{r=0}$ calculation until (4.12), we obtain

$$\Omega_{(1)}\big|_{r=0}=\widetilde{\widetilde\Omega}_{(1)}-(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma_{[ij]}\gamma^\sigma\widetilde\lambda)(\lambda\gamma^\rho\widetilde D)(\widetilde D\gamma_{\nu\sigma\rho}\widetilde D)\,,$$

with $\widetilde{\widetilde\Omega}_{(1)}:=2\,(uu)^{ij}(\lambda\gamma^\mu\hat\psi)(\lambda\gamma^\nu\widetilde D)(\widetilde D\gamma_{\mu\nu[ij]}\widetilde D)$. Now we use $(\gamma^\sigma)^{\hat\alpha(\hat\beta}(\gamma_\sigma)^{\hat\gamma\hat\delta)}=0$ and get

$$\begin{aligned}
\Omega_{(1)}\big|_{r=0}-\widetilde{\widetilde\Omega}_{(1)} &= -(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma_{[ij]}\gamma^\sigma\gamma_\nu\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&\quad+(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\nu\widetilde D)(\lambda\gamma_{[ij]}\gamma^\sigma\gamma_\rho\widetilde D)\\
&= (uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\sigma\gamma_{[ij]}\gamma_\nu\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&\quad-(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\nu\widetilde D)(\lambda\gamma^\sigma\gamma_{[ij]}\gamma_\rho\widetilde D)\\
&= -2\,(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\hat\psi)(\lambda\gamma^\sigma\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_{\sigma\rho}\widetilde D)\\
&\quad-(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\sigma\gamma_\nu\gamma_{[ij]}\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\rho\widetilde D)\\
&\quad+2\,(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma_{[ij]}\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\nu\widetilde D)(\lambda\gamma^\sigma\widetilde D)\\
&\quad+(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_\sigma\gamma_\nu\widetilde D)(\lambda\gamma^\sigma\gamma_\rho\gamma_{[ij]}\widetilde D)\\
&= -2\,(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\hat\psi)(\lambda\gamma^\sigma\widetilde D)(\lambda\gamma^\rho\widetilde D)(\widetilde\lambda\gamma_{\sigma\rho}\widetilde D)\\
&\quad-6\,(uu)^{ij}(\lambda\widetilde\lambda)^{-1}(\lambda\gamma_{[ij]}\widetilde D)(\lambda\gamma^\nu\hat\psi)(\lambda\gamma^\sigma\widetilde D)(\widetilde\lambda\gamma_{\nu\sigma}\widetilde D)\,, \qquad (4.16)
\end{aligned}$$

after using $\gamma^\nu\gamma_\rho = -2\delta^\nu_\rho - \gamma_\rho\gamma^\nu$ and $(\lambda\gamma_\sigma)_{\hat\alpha}(\lambda\gamma^\sigma)_{\hat\beta} = 0$ several times. Therefore,

$$\begin{aligned}\Omega_{(1)}\big|_{r=0} &= \widetilde{\widetilde{\Omega}}_{(1)} - 2\,(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_{[ij]}\hat{\psi})(\lambda\gamma^\sigma\widetilde{D})(\lambda\gamma^\rho\widetilde{D})(\widetilde{\lambda}\gamma_{\sigma\rho}\widetilde{D}) \\ &\quad - 6\,(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_{[ij]}\widetilde{D})(\lambda\gamma^\nu\hat{\psi})(\lambda\gamma^\sigma\widetilde{D})(\widetilde{\lambda}\gamma_{\nu\sigma}\widetilde{D})\,. \qquad (4.17)\end{aligned}$$

BRST-trivial terms and results

Now we are ready to look for the BRST-trivial terms which will effectively simplify the expressions of $\Omega_{(0)}$ and $\Omega_{(1)}$. We have (cf. (3.25))

$$V_{(2)} = \int \mathrm{d}u\,\left[(yuu)\Omega_{(0)} + 8\,\Omega_{(1)}\right] T^{(2)}$$

and, as stated in the beginning of this subsection, our goal is to find a BRST-equivalent vertex operator

$$V'_{(2)} = V_{(2)} + \left(Q_{\frac{1}{2}} + \widetilde{w}^{\hat\alpha} r_{\hat\alpha}\right)\chi_{(2)} = \int \mathrm{d}u\,\left[(yuu)\widetilde{\Omega}_{(0)} + 8\,\widetilde{\Omega}_{(1)}\right] T^{(2)}\,,$$

with simpler, non-minimal-variable-free operators $\widetilde{\Omega}_{(0)}$ and $\widetilde{\Omega}_{(1)}$.

Recall the expression for the operator $Q_{\frac{1}{2}}$ restricted to act on states in the cohomology of $Q_{-\frac{1}{2}}$:

$$Q_{\frac{1}{2}} = \sqrt{z}\left[\lambda^{\hat\alpha}D_{\hat\alpha} + 4\,(\lambda\gamma^{[ij]}\hat{\psi})\frac{\partial}{\partial y^{ij}} + y_{ij}(\lambda\gamma^{[ij]}\hat{\psi})\left(2z\frac{\partial}{\partial z} + y^{k\ell}\frac{\partial}{\partial y^{k\ell}} - \lambda^{\hat\alpha}\frac{\partial}{\partial\lambda^{\hat\alpha}}\right)\right].$$

Since there is no overall z-dependence in $V_{(2)}$, it is natural to propose an object $\chi_{(2)}$ proportional to $z^{-\frac{1}{2}}$. Furthermore, since $V_{(2)}$ is in the ghost-number +2 cohomology of the BRST operator, $\chi_{(2)}$ should have ghost-number +1. Thus, if we want the last term in $Q_{\frac{1}{2}}$ to annihilate $\chi_{(2)}$, it must be proportional to y^{ij}. Recalling that $\hat{\psi} := y_{ij}(\gamma^{[ij]}\psi)$, it is then not difficult to find a suitable $\chi_{(2)}$:

$$\begin{aligned}\chi_{(2)} &= \frac{1}{\sqrt{z}}\int \mathrm{d}u\,\Big[\kappa_1\,(yuu)(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)} \\ &\qquad + \kappa_2\,(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)}\Big]\,, \qquad (4.18)\end{aligned}$$

for some constants κ_1 and κ_2 to be determined.

The last term in $Q_{\frac{1}{2}}$ annihilates both terms in this expression by construction. Let us see what the action of the other terms yields. When acting with the first term in $Q_{\frac{1}{2}}$ on the first term in $\chi_{(2)}$, we get

$$-\frac{1}{2}\kappa_1\int \mathrm{d}u\,(yuu)(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_{[ij]}\widetilde{D})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)}\,, \qquad (4.19)$$

since, as shown in the beginning of this subsection, $\lambda^{\hat{\alpha}} D_{\hat{\alpha}} = -\frac{1}{2}(uu)_{ij}(\lambda\gamma^{[ij]}\widetilde{D}) + \frac{1}{4}(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D)$ and $(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D)$ anticommutes with $(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})$ and annihilates $T^{(2)}$, since it is a G-analytic superfield.

Next, we act with the first term in $Q_{\frac{1}{2}}$ on the second term in $\chi_{(2)}$ to get

$$-\frac{1}{2}\kappa_2\int \mathrm{d}u\,(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_{[ij]}\widetilde{D})(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)} + 4\kappa_2\int \mathrm{d}u\,(\overline{uu})_{mn}(\lambda\gamma^{[mn]}\hat{\psi})(\lambda\gamma^a\widetilde{D})\partial_a T^{(2)}\,, \tag{4.20}$$

since the anticommutator of $(uu)_{ij}(\overline{uu})_{k\ell}(\lambda\gamma^{[k\ell]}\gamma^{[ij]}D)$ and $(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})$ is $16\,(\overline{uu})_{mn}(\lambda\widetilde{\lambda})(\lambda\gamma^{[mn]}\hat{\psi})(\lambda\gamma^a\widetilde{D})\partial_a$ (cf. (4.5)).

The second term in $Q_{\frac{1}{2}}$ annihilates the second term in $\chi_{(2)}$, since the latter has no explicit y-dependence.* When acting on the first term in $\chi_{(2)}$, it gives

$$8\kappa_1\int \mathrm{d}u\,(uu)^{ij}(\lambda\widetilde{\lambda})^{-1}(\lambda\gamma_{[ij]}\hat{\psi})(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)}\,. \tag{4.21}$$

Finally, it is left to compute the action of $\widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}$ on $\chi_{(2)}$. This is easily done by thinking of $\widetilde{w}$ as a $\widetilde{\lambda}$-derivative, and recalling $r_{\hat{\alpha}}$ is a fermionic spinor. We get

$$\begin{aligned}\widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\chi_{(2)} \;=\; & -\frac{1}{\sqrt{z}}\kappa_1\int \mathrm{d}u\,(yuu)\Big[(\lambda\widetilde{\lambda})^{-2}(\lambda r)(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)} \\ & \qquad - (\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(r\gamma_{\mu\nu}\widetilde{D})T^{(2)}\Big] \\ & -\frac{1}{\sqrt{z}}\kappa_2\int \mathrm{d}u\,\Big[(\lambda\widetilde{\lambda})^{-2}(\lambda r)(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D})T^{(2)} \\ & \qquad - (\lambda\widetilde{\lambda})^{-1}(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(r\gamma_{\mu\nu}\widetilde{D})T^{(2)}\Big]\,.\end{aligned} \tag{4.22}$$

Hence, comparing the above equations with the alternative forms of $\Omega_{(0)}$ and $\Omega_{(1)}$ obtained in equations (4.11), (4.14), (4.15) and (4.17), it is easy to see that, if we choose $\kappa_1 = 2$ and $\kappa_2 = -96$, we have

$$V'_{(2)} = V_{(2)} + \left(Q_{\frac{1}{2}} + \widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right)\chi_{(2)} = \int \mathrm{d}u\,\Big[(yuu)\widetilde{\Omega}_{(0)} + 8\,\widetilde{\Omega}_{(1)}\Big]\,T^{(2)}\,,$$

with

$$\widetilde{\Omega}_{(0)} = -\frac{1}{4}(uu)^{ij}(\lambda\gamma^\mu\widetilde{D})(\lambda\gamma^\nu\widetilde{D})(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D})\,, \tag{4.23a}$$
$$\widetilde{\Omega}_{(1)} = 2\,(uu)^{ij}(\lambda\gamma^\mu\hat{\psi})(\lambda\gamma^\nu\widetilde{D})(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D}) - 48\,(\overline{uu})_{ij}(\lambda\gamma^{[ij]}\hat{\psi})(\lambda\gamma^a\widetilde{D})\partial_a\,, \tag{4.23b}$$

which are relatively simple expressions that do not depend on the non-minimal pure-spinor variables, as promised.

*Recall $4\,(\lambda\gamma^{[ij]}\hat{\psi})\frac{\partial}{\partial y^{ij}}$ is understood not to act on $\lambda\gamma_\mu\hat{\psi}$, even though $\hat{\psi}$ depends on y.

Note that we could have obtained the same expression for $\widetilde{\Omega}_{(0)}$ by adding the BRST-trivial term

$$\left(Q_{\frac{1}{2}} + \widetilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) \left[2\sqrt{z} \int \mathrm{d}u \, \frac{1}{(\lambda\widetilde{\lambda})} (\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{\lambda}\gamma_{\mu\nu}\widetilde{D}) T^{(3)}(x,\theta,u,\overline{u})\right] \tag{4.24}$$

to $V_{(1)}$. This method could also be used to find an equivalent expression for $\Omega_{(2)}$ without the non-minimal variables. However, for $N > 3$, it seems that $V_{(N)}$ can only be written in a λ^{+}-independent manner if one introduces non-minimal variables.

4.1.2 The $\mathcal{N} = 4$, $d = 4$ pure spinor measure factor

The other preliminary calculation we will perform in this section concerns the pure spinor measure factor. In order to compute superstring scattering amplitudes in an $\mathcal{N} = 4$, d $= 4$ theory — such as the gauge theory describing the effective world-volume degrees of freedom of a D3-brane, or the boundary of the AdS_5 superspace — using the pure-spinor formalism, one needs to know how to perform the integrations of the λ and θ zero modes in that case. In other words, one needs to find a BRST-invariant measure factor analogous to the one introduced in section 2.1:

$$(\lambda\gamma^{\mu}\theta)(\lambda\gamma^{\nu}\theta)(\lambda\gamma^{\rho}\theta)(\theta\gamma_{\mu\nu\rho}\theta) \, . \tag{4.25}$$

At first, it might seem to be just a matter of dimensional reduction. However, although the particular combination of λ's and θ's of (4.25) is special in ten flat dimensions, there is no reason why its dimensional reduction should be preferred over any other BRST-invariant, $\mathrm{SO}(3,1)\times\mathrm{SU}(4)$ scalar in four dimensions. Therefore, it is important to investigate whether there is any ambiguity in the definition of the $\mathcal{N} = 4$, d $= 4$ measure factor.

In this subsection, this issue is studied in detail, following [34]. First, we will write the most general $\mathrm{SO}(3,1)\times\mathrm{SU}(4)$ scalar expression with three λ's and five θ's and derive the conditions for it to be BRST invariant. Then, we will find the independent BRST-trivial combinations of the terms previously introduced. Finally, we will present the main result of this subsection, namely we will show that the $\mathcal{N} = 4$, d $= 4$ measure factor is unique up to BRST-trivial terms and an overall factor.

BRST equations

In four-dimensional notation, the most general SO(3,1)×SU(4)-invariant, real expression one can write with three λ's and five θ's is

$$\begin{aligned}(\lambda^3\theta^5) := {} & c_1\,\varepsilon_{mnj\ell}(\bar\lambda_i\bar\lambda_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell)\\ & + c_2\,(\bar\lambda_j\bar\lambda_k)(\bar\lambda_\ell\bar\theta_i)(\theta^i\theta^j)(\theta^k\theta^\ell) + c_3\,\varepsilon_{mnj\ell}(\bar\lambda_i\bar\theta_k)(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\theta^j)\\ & + c_4\,(\bar\lambda_\ell\bar\lambda_k)(\lambda^j\theta^\ell)(\theta^i\theta^k)(\bar\theta_i\bar\theta_j) + c_5\,(\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\theta_\ell)(\lambda^k\theta^\ell)(\theta^i\theta^j)\\ & + c_6\,\varepsilon_{mn\ell k}(\lambda^i\theta^k)(\lambda^j\theta^\ell)(\lambda^m\theta^n)(\bar\theta_i\bar\theta_j)\\ & + \text{H.c.}\,,\end{aligned} \tag{4.26}$$

where $c_1,\ldots,c_6$ are arbitrary constants and "H.c." means "Hermitian conjugate". One can convince oneself these are the only non-zero independent terms which can be constructed, keeping in mind that

$$\lambda^{\alpha i}\bar\lambda_i^{\dot\alpha} = 0 \qquad \text{and} \qquad (\lambda^i\lambda^j) = \frac{1}{2}\varepsilon^{ijk\ell}(\bar\lambda_k\bar\lambda_\ell)\,, \tag{4.27}$$

which are the dimensional reduction of $\lambda\gamma^\mu\lambda = 0$. More details on dimensional reduction can be found in Appendix A.3.

The notation we are going to use throughout this subsection is such that

$$(\lambda^3\theta^5) =: \sum_{n=1}^{6} c_n\mathbf{T}_n + \text{H.c.}\,, \tag{4.28}$$

i.e. we define $\mathbf{T}_1,\ldots,\mathbf{T}_6$ to be the independent possible terms as appearing in (4.26). For example, $\mathbf{T}_3 \equiv \varepsilon_{mnj\ell}(\bar\lambda_i\bar\theta_k)(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\theta^j)$. The $\mathbf{T}_n$ and their Hermitian conjugates $\mathbf{T}_n^\dagger$ form a basis for four-dimensional expressions made of three λ's and five θ's. For example, it is not difficult to show the dimensional reduction of (4.25) gives (4.26) with $c_1 = 1$, $c_2 = c_4 = 4$, $c_3 = 3$, $c_5 = 12$ and $c_6 = 2$, up to an overall factor.

Since we are looking for the pure spinor measure, we are interested in expressions which are annihilated by $\lambda^{\hat\alpha}D_{\hat\alpha} = \lambda^{\alpha p}D_{\alpha p} + \bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}$. This requirement yields equations for the constants in (4.26). We begin with

$$\left[\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5)\right]_{\theta^4\bar\theta^0} = \lambda^{\alpha p}D_{\alpha p}[c_1\mathbf{T}_1] + \bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}[c_2\mathbf{T}_2 + c_3\mathbf{T}_3]\,, \tag{4.29}$$

where the subscript $\theta^4\bar\theta^0$ means "contributions with four θ's and no $\bar\theta$." The explicit calculation gives:

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_1 &= \varepsilon_{mnj\ell}(\bar\lambda_i\bar\lambda_k)\Big[(\lambda^m\lambda^n)(\theta^i\theta^j)(\theta^k\theta^\ell) + (\lambda^m\theta^n)(\theta^i\lambda^j)(\theta^k\theta^\ell)\\ &\qquad + (\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\lambda^\ell)\Big]\\ &= 4\,(\bar\lambda_i\bar\lambda_k)(\bar\lambda_j\bar\lambda_\ell)(\theta^i\theta^j)(\theta^k\theta^\ell)\,,\end{aligned} \tag{4.30}$$

where we used (4.27) and $\lambda^{\alpha[i|}\lambda^{\beta|j]} = -\frac{1}{2}\varepsilon^{\alpha\beta}(\lambda^i\lambda^j)$. Moreover,

$$\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_2 = (\bar{\lambda}_j\bar{\lambda}_k)(\bar{\lambda}_\ell\bar{\lambda}_i)(\theta^i\theta^j)(\theta^k\theta^\ell) \tag{4.31}$$

and $\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_3 = 0$. Therefore

$$\boxed{\big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\big]_{\theta^4\bar{\theta}^0} = 0 \iff c_2 = 4c_1\,.} \tag{4.32}$$

Proceeding to the next order, we have

$$\big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\big]_{\theta^3\bar{\theta}^1} = \lambda^{\alpha p}D_{\alpha p}[c_2\mathbf{T}_2 + c_3\mathbf{T}_3] + \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}[c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6]\,. \tag{4.33}$$

The $\mathbf{T}_2$-contribution is easy to compute. We get

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_2 = -(\bar{\lambda}_j\bar{\lambda}_k)(\bar{\lambda}_\ell\bar{\theta}_i)(\lambda^i\theta^j)(\theta^k\theta^\ell)\,. \tag{4.34}$$

For $\mathbf{T}_3$, we obtain

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_3 &= -\varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\theta}_k)\Big[(\lambda^k\lambda^\ell)(\lambda^m\theta^n)(\theta^i\theta^j) - (\lambda^k\theta^\ell)(\lambda^m\lambda^n)(\theta^i\theta^j)\\ &\qquad\qquad - (\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\lambda^j)\Big]\\ &= 4\,(\bar{\lambda}_j\bar{\lambda}_\ell)(\bar{\lambda}_i\bar{\theta}_k)(\lambda^k\theta^\ell)(\theta^i\theta^j)\,.\end{aligned} \tag{4.35}$$

The $\mathbf{T}_4$- and $\mathbf{T}_5$-contributions are also simple to calculate:

$$\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_4 = -(\bar{\lambda}_\ell\bar{\lambda}_k)(\lambda^j\theta^\ell)(\theta^i\theta^k)(\bar{\lambda}_i\bar{\theta}_j)\,, \tag{4.36}$$

$$\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_5 = -(\bar{\lambda}_i\bar{\theta}_k)(\bar{\lambda}_j\bar{\lambda}_\ell)(\lambda^k\theta^\ell)(\theta^i\theta^j)\,. \tag{4.37}$$

Finally, $\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_6 = 0$. In all, we get our second equation for the coefficients:

$$\boxed{\big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\big]_{\theta^3\bar{\theta}^1} = 0 \iff c_2 + 4c_3 = c_4 + c_5\,.} \tag{4.38}$$

We now analyze the contributions with equal number of θ's and $\bar{\theta}$'s:

$$\big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\big]_{\theta^2\bar{\theta}^2} = \lambda^{\alpha p}D_{\alpha p}[c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6] + \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}[\bar{c}_4\mathbf{T}_4^\dagger + \bar{c}_5\mathbf{T}_5^\dagger + \bar{c}_6\mathbf{T}_6^\dagger]\,. \tag{4.39}$$

Again, it is straightforward to compute the contributions from $\mathbf{T}_4$ and $\mathbf{T}_5$:

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_4 = -(\bar{\lambda}_\ell\bar{\lambda}_k)(\lambda^j\theta^\ell)(\lambda^i\theta^k)(\bar{\theta}_i\bar{\theta}_j)\,, \tag{4.40}$$

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_5 = (\bar{\lambda}_i\bar{\theta}_k)(\bar{\lambda}_j\bar{\theta}_\ell)(\lambda^k\lambda^\ell)(\theta^i\theta^j)\,. \tag{4.41}$$

The $\mathbf{T}_6$-contribution yields

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_6 &= \varepsilon_{mn\ell k}(\bar{\theta}_i\bar{\theta}_j)\Big[(\lambda^i\lambda^k)(\lambda^j\theta^\ell)(\lambda^m\theta^n) - (\lambda^i\theta^k)(\lambda^j\lambda^\ell)(\lambda^m\theta^n)\\ &\qquad\qquad + (\lambda^i\theta^k)(\lambda^j\theta^\ell)(\lambda^m\lambda^n)\Big]\\ &= 4\,(\bar{\lambda}_\ell\bar{\lambda}_k)(\lambda^i\theta^k)(\lambda^j\theta^\ell)(\bar{\theta}_i\bar{\theta}_j)\,.\end{aligned} \tag{4.42}$$

These in turn imply, by Hermitian conjugation,

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_4^\dagger = -(\lambda^\ell\lambda^k)(\bar\lambda_j\bar\theta_\ell)(\bar\lambda_i\bar\theta_k)(\theta^i\theta^j)\,, \tag{4.43}$$
$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_5^\dagger = (\lambda^i\theta^k)(\lambda^j\theta^\ell)(\bar\lambda_k\bar\lambda_\ell)(\bar\theta_i\bar\theta_j)\,, \tag{4.44}$$
$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_6^\dagger = 4\,(\lambda^\ell\lambda^k)(\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\theta_\ell)(\theta^i\theta^j)\,. \tag{4.45}$$

Thus we obtain our last equation:

$$\boxed{\left[\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5)\right]_{\theta^2\bar\theta^2} = 0 \iff \bar c_5 = c_4 + 4c_6\,,} \tag{4.46}$$

as well as its complex conjugate.

Note that the vanishing of the orders $\theta^1\bar\theta^3$ and $\theta^0\bar\theta^4$ implies the complex conjugates of (4.32) and (4.38), since they are just the Hermitian conjugates of the orders $\theta^3\bar\theta^1$ and $\theta^4\bar\theta^0$, respectively.

In summary, we have the following system of equations:

$$\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5) = 0 \iff \begin{cases} c_2 = 4c_1 \\ c_2 + 4c_3 = c_4 + c_5 \\ \bar c_5 = c_4 + 4c_6 \end{cases}, \tag{4.47}$$

as well as their complex conjugates.

BRST-trivial combinations

In the last subsubsection, we found the equations which the constants in (4.26) have to satisfy for the expression to be BRST-invariant. Because there are less equations than constants, one might think the $\mathcal{N} = 4$, d $= 4$ pure spinor measure factor is then not unambiguously defined. Fortunately, that is not the case, and the seemingly independent expressions are actually related by BRST-trivial terms, as we show in the following.

In order to find the independent BRST-trivial combinations of the $\mathbf{T}_n$, i.e. the combinations which equal $\lambda^{\hat\alpha}D_{\hat\alpha}$ of something, we start by looking for all independent possible terms with two λ's and six θ's. Keeping (4.27) in mind, we find that there are five:

$$\chi_1 := (\lambda^i\theta^j)(\theta^k\theta^\ell)(\bar\lambda_k\bar\theta_\ell)(\bar\theta_i\bar\theta_j)\,, \tag{4.48a}$$
$$\chi_2 := \varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^m\theta^k)(\theta^\ell\theta^n)(\bar\theta_m\bar\theta_n)\,, \tag{4.48b}$$
$$\chi_3 := \varepsilon^{ijk\ell}(\bar\lambda_i\bar\theta_j)(\bar\lambda_m\bar\theta_k)(\bar\theta_\ell\bar\theta_n)(\theta^m\theta^n)\,, \tag{4.48c}$$
$$\chi_4 := \varepsilon_{mnj\ell}(\bar\lambda_i\bar\theta_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell)\,, \tag{4.48d}$$
$$\chi_5 := \varepsilon^{mnj\ell}(\lambda^i\theta^k)(\bar\lambda_m\bar\theta_n)(\bar\theta_i\bar\theta_j)(\bar\theta_k\bar\theta_\ell)\,. \tag{4.48e}$$

Acting with $\lambda^{\hat{\alpha}} D_{\hat{\alpha}} = \lambda^{\alpha p} D_{\alpha p} + \bar{\lambda}_{\dot{\alpha}p} \bar{D}^{\dot{\alpha}p}$ on these terms, we obtain BRST-trivial expressions made of three λ's and five θ's. We begin with the first one:

$$\begin{aligned} \lambda^{\alpha p} D_{\alpha p} \chi_1 &= (\lambda^i \theta^j)(\theta^k \lambda^\ell)(\bar{\lambda}_k \bar{\theta}_\ell)(\bar{\theta}_i \bar{\theta}_j) \\ &= \frac{1}{2}\left[\mathbf{T}_4^\dagger - \mathbf{T}_5^\dagger\right], \end{aligned} \tag{4.49}$$

$$\begin{aligned} \bar{\lambda}_{\dot{\alpha}p} \bar{D}^{\dot{\alpha}p} \chi_1 &= -(\lambda^i \theta^j)(\theta^k \theta^\ell)(\bar{\lambda}_k \bar{\theta}_\ell)(\bar{\theta}_i \bar{\lambda}_j) \\ &= \frac{1}{2}\left[\mathbf{T}_5 - \mathbf{T}_4\right]. \end{aligned} \tag{4.50}$$

Thus we find the first BRST-trivial expression:

$$\boxed{\mathbf{T}_4^\dagger - \mathbf{T}_5^\dagger + \mathbf{T}_5 - \mathbf{T}_4 = \lambda^{\hat{\alpha}} D_{\hat{\alpha}}\left[2\chi_1\right].} \tag{4.51}$$

Of course, we could multiply the expression on the left-hand side of this equation by any constant and it would remain BRST-trivial. The same applies to the other boxed expressions we find in the following.

For the second term in (4.48), we have

$$\begin{aligned} \lambda^{\alpha p} D_{\alpha p} \chi_2 &= \varepsilon_{ijk\ell}(\bar{\theta}_m \bar{\theta}_n)\Big[(\lambda^i \lambda^j)(\lambda^m \theta^k)(\theta^\ell \theta^n) - (\lambda^i \theta^j)(\lambda^m \lambda^k)(\theta^\ell \theta^n) \\ &\qquad + (\lambda^i \theta^j)(\lambda^m \theta^k)(\lambda^\ell \theta^n) - (\lambda^i \theta^j)(\lambda^m \theta^k)(\theta^\ell \lambda^n)\Big] \\ &= 4\mathbf{T}_4 - \mathbf{T}_6, \end{aligned} \tag{4.52}$$

$$\begin{aligned} \bar{\lambda}_{\dot{\alpha}p} \bar{D}^{\dot{\alpha}p} \chi_2 &= -\varepsilon_{ijk\ell}(\lambda^i \theta^j)(\lambda^m \theta^k)(\theta^\ell \theta^n)(\bar{\theta}_m \bar{\lambda}_n) \\ &= \mathbf{T}_3. \end{aligned} \tag{4.53}$$

Therefore,

$$\boxed{\mathbf{T}_3 + 4\mathbf{T}_4 - \mathbf{T}_6 = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_2.} \tag{4.54}$$

Since the third term in (4.48) is equal to $\chi_2^\dagger$,

$$\boxed{\mathbf{T}_3^\dagger + 4\mathbf{T}_4^\dagger - \mathbf{T}_6^\dagger = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_3.} \tag{4.55}$$

For the fourth term,

$$\begin{aligned} \lambda^{\alpha p} D_{\alpha p} \chi_4 &= -\varepsilon_{mnj\ell}(\bar{\lambda}_i \bar{\theta}_k)\Big[(\lambda^m \lambda^n)(\theta^i \theta^j)(\theta^k \theta^\ell) + (\lambda^m \theta^n)(\theta^i \lambda^j)(\theta^k \theta^\ell) \\ &\qquad - (\lambda^m \theta^n)(\theta^i \theta^j)(\lambda^k \theta^\ell) + (\lambda^m \theta^n)(\theta^i \theta^j)(\theta^k \lambda^\ell)\Big] \\ &= 4\mathbf{T}_2 - \mathbf{T}_3, \end{aligned} \tag{4.56}$$

$$\begin{aligned}\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\chi_4 &= \varepsilon_{mnj\ell}(\bar\lambda_i\bar\lambda_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell)\\ &= \mathbf{T}_1\,. \end{aligned} \tag{4.57}$$

Therefore,

$$\boxed{\mathbf{T}_1 + 4\mathbf{T}_2 - \mathbf{T}_3 = \lambda^{\hat\alpha}D_{\hat\alpha}\chi_4\,.} \tag{4.58}$$

Finally, the last term in (4.48) is equal to $\chi_4^\dagger$, so

$$\boxed{\mathbf{T}_1^\dagger + 4\mathbf{T}_2^\dagger - \mathbf{T}_3^\dagger = \lambda^{\hat\alpha}D_{\hat\alpha}\chi_5\,.} \tag{4.59}$$

The $\mathcal{N}$ = 4, d = 4 measure factor in a simple form

We are now in position to show the $\mathcal{N}$ = 4, d = 4 measure factor is unique up to BRST-trivial terms and an overall factor. Consider once again the most general real expression with three λ's and five θ's of (4.26). One has

$$(\lambda^3\theta^5) = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6 + \text{H.c.}\,. \tag{4.60}$$

If this is BRST-invariant, then the constants satisfy the equations (4.47) and their complex conjugates. We are free to add BRST-trivial terms to the above expression. If we add

$$-c_1\left[\mathbf{T}_1 + 4\mathbf{T}_2 - \mathbf{T}_3\right] + \text{H.c.}$$

to $(\lambda^3\theta^5)$, we get[†]

$$\begin{aligned}(\lambda^3\theta^5) &= (c_2 - 4c_1)\mathbf{T}_2 + (c_3 + c_1)\mathbf{T}_3 + c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6 + \text{H.c.}\\ &= \frac{1}{4}(c_4 + c_5)\mathbf{T}_3 + c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + \frac{1}{4}(\bar c_5 - c_4)\mathbf{T}_6 + \text{H.c.}\,,\end{aligned} \tag{4.61}$$

where we used (4.47). Furthermore, we can add the BRST-trivial term

$$-\frac{1}{4}(c_4 + c_5)\left[\mathbf{T}_3 + 4\mathbf{T}_4 - \mathbf{T}_6\right] + \text{H.c.}$$

to get

$$(\lambda^3\theta^5) = -c_5\mathbf{T}_4 + c_5\mathbf{T}_5 + \frac{1}{4}(c_5 + \bar c_5)\mathbf{T}_6 + \text{H.c.}\,. \tag{4.62}$$

Finally, if $c_5 = \alpha + \mathrm{i}\beta$, with $\alpha, \beta \in \mathbb{R}$, then we can add the BRST-trivial term

$$-\mathrm{i}\beta\left[\mathbf{T}_5 - \mathbf{T}_5^\dagger - \mathbf{T}_4 + \mathbf{T}_4^\dagger\right]$$

to $(\lambda^3\theta^5)$, thus obtaining

$$\boxed{(\lambda^3\theta^5) = -\alpha\left[\mathbf{T}_4 - \mathbf{T}_5 - \frac{1}{2}\mathbf{T}_6 + \text{H.c.}\right].} \tag{4.63}$$

This shows that the measure is unique up to BRST-trivial terms and an overall factor.

[†]Note the equal sign here means "equal up to BRST-trivial terms."

4.2 The case $N = 1$

We now proceed to the computation of the amplitude (4.1) for the case $N = 1$. This is the simplest case, not only because the amplitude does not involve integrated vertex operators, but also because $V_{(1)}$ is simpler than any other supergravity vertex operator in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Indeed, from (3.25) we have

$$V_{(1)} = z \int \mathrm{d}u \, \Omega_{(0)} T^{(3)}(x, \theta, u, \overline{u}) \,, \tag{4.64}$$

which has no y- or ψ-dependence. These operators are the duals to SYM "singleton" operators, i.e. the duals to abelian SYM fields.

In fact, as shown in the previous section, the expression for $V_{(1)}$ can be further simplified. By adding a BRST-trivial quantity to $V_{(1)}$, we get an equivalent expression which does not depend on the non-minimal pure spinor variables. It looks the same as (4.64), but with $\Omega_{(0)}$ replaced by

$$\widetilde{\Omega}_{(0)} = -\frac{1}{4}(uu)^{ij}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D}) \,. \tag{4.65}$$

Based on the conjecture referred to at the beginning of this chapter, we expect the following relation to hold:

$$\mathcal{M}_1 = \left\langle V_{(1)} \, V_{\mathrm{SYM}} \right\rangle_{\mathrm{D3-brane}} \propto \int \mathrm{d}^4 x \int \mathrm{d}u \, \bar{D}^4 D'^4 \left[W^{(1)}(x, \theta, u) \, T^{(3)}(x, \theta, u, \overline{u}) \right] , \tag{4.66}$$

where $V_{\mathrm{SYM}} = \sqrt{z} \, \lambda^{\hat{\alpha}} A_{\hat{\alpha}}(x, \theta)$ and $A_{\hat{\alpha}}$ is the dimensional reduction of the $\mathcal{N} = 1$, $\mathrm{d} = 10$ SYM superfield, whose properties are reviewed in Appendix C. This is the same vertex operator as the one introduced in section 2.1, but with an extra factor of $\sqrt{z}$ which is needed for BRST-invariance. Substituting the expressions for the vertex operators, one can write the amplitude as

$$\mathcal{M}_1 = \int \mathrm{d}^4 x \int \mathrm{d}u \, \left\langle \lambda^{\hat{\alpha}} A_{\hat{\alpha}} \, \widetilde{\Omega}_{(0)} T^{(3)} \right\rangle , \tag{4.67}$$

where we used (4.64) and replaced $\Omega_{(0)}$ by the $\widetilde{\Omega}_{(0)}$ of (4.23a). Note that the factors of z coming from the vertex operators cancel the ones in (4.2).

Now the angle brackets denote the integrations of the λ and θ zero modes only. In order to perform these integrations, we need a $\lambda^{\hat{\alpha}} D_{\hat{\alpha}}$-invariant, $\mathrm{SO}(3,1) \times \mathrm{SU}(4)$ scalar measure factor, since this is the symmetry of the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$. Since, as shown in the previous section, all such measure factors are equivalent up to BRST-trivial terms and an overall factor, we can use the usual

$$(\lambda\gamma^{\mu}\theta)(\lambda\gamma^{\nu}\theta)(\lambda\gamma^{\rho}\theta)(\theta\gamma_{\mu\nu\rho}\theta) \,,$$

or, more precisely, its dimensional reduction.

Using this measure factor, the pure-spinor prescription for the computation of tree-level scattering amplitudes states that [2]

$$\left\langle \lambda^{\hat\alpha_1}\lambda^{\hat\alpha_2}\lambda^{\hat\alpha_3} f_{\hat\alpha_1\hat\alpha_2\hat\alpha_3}(x,\theta,u,\overline{u})\right\rangle \propto (\mathcal{T}D^5)^{\hat\alpha_1\hat\alpha_2\hat\alpha_3} f_{\hat\alpha_1\hat\alpha_2\hat\alpha_3}(x,\theta,u,\overline{u})\,, \tag{4.68}$$

where

$$(\mathcal{T}D^5)^{\hat\alpha_1\hat\alpha_2\hat\alpha_3} := \mathcal{T}^{\hat\alpha_1\hat\alpha_2\hat\alpha_3}_{\hat\beta_1\hat\beta_2\hat\beta_3}(\gamma^\mu D)^{\hat\beta_1}(\gamma^\nu D)^{\hat\beta_2}(\gamma^\rho D)^{\hat\beta_3}(D\gamma_{\mu\nu\rho}D)\,, \tag{4.69}$$

$$\mathcal{T}^{\hat\alpha_1\hat\alpha_2\hat\alpha_3}_{\hat\beta_1\hat\beta_2\hat\beta_3} := \delta^{(\hat\alpha_1}_{\hat\beta_1}\delta^{\hat\alpha_2}_{\hat\beta_2}\delta^{\hat\alpha_3)}_{\hat\beta_3} + \frac{3}{20}(\gamma_\mu)^{(\hat\alpha_1\hat\alpha_2}\delta^{\hat\alpha_3)}_{(\hat\beta_1}(\gamma^\mu)_{\hat\beta_2\hat\beta_3)}\,. \tag{4.70}$$

This definition ensures that $(\mathcal{T}D^5)^{\hat\alpha_1\hat\alpha_2\hat\alpha_3}$ is totally symmetric and γ-traceless, i.e.

$$(\gamma^\mu)_{\hat\alpha_1\hat\alpha_2}(\mathcal{T}D^5)^{\hat\alpha_1\hat\alpha_2\hat\alpha_3} = 0\,, \tag{4.71}$$

as is the product of three pure spinors. The proportionality constant in (4.68) depends on the choice of normalization, and will not be important for us.

In the case at hand,

$$f_{\hat\alpha\hat\beta\hat\gamma}(x,\theta,u,\overline{u}) \propto A_{\hat\gamma}(x,\theta)\,\Omega_{(0)\hat\alpha\hat\beta}T^{(3)}(x,\theta,u,\overline{u})\,, \tag{4.72}$$

with

$$\Omega_{(0)\hat\alpha\hat\beta} := (uu)^{ij}(\gamma^\mu\widetilde{D})_{\hat\alpha}(\gamma^\nu\widetilde{D})_{\hat\beta}(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D})\,. \tag{4.73}$$

Thus, we arrive at the final form of the amplitude:

$$\mathcal{M}_1 \propto \int \mathrm{d}^4x \int \mathrm{d}u\,(\mathcal{T}D^5)^{\hat\alpha\hat\beta\hat\gamma}\left[A_{\hat\gamma}(x,\theta)\,\Omega_{(0)\hat\alpha\hat\beta}T^{(3)}(x,\theta,u,\overline{u})\right]. \tag{4.74}$$

4.2.1 Possible contributions

Computing (4.74) explicitly would be very complicated, mostly because of the dimensional reduction of $(\mathcal{T}D^5)$ — see Appendix D.2 —, but fortunately we need not do that. Instead, we can use symmetry arguments and equations of motion to determine the form of the terms that might appear in the computation, and then use supersymmetry to obtain the relative coefficients.

Let us see what kinds of terms one can expect to find when computing $\mathcal{M}_1$. Recalling that

$$\widetilde{D}^{\hat\alpha} := -\frac{1}{2}(\overline{uu})_{k\ell}(\gamma^{[k\ell]}D)^{\hat\alpha} = \begin{pmatrix} \widetilde{D}^{\alpha i} \\ \bar{\widetilde{D}}{}^{\dot\alpha}_j \end{pmatrix} = \begin{pmatrix} (\overline{uu})^{ik}D^\alpha_k \\ (\overline{uu})_{j\ell}\bar{D}^{\dot\alpha\ell} \end{pmatrix}, \tag{4.75}$$

we see that the amplitude has the schematic form

$$\mathcal{M}_1 \sim \int \mathrm{d}^4x \int \mathrm{d}u\,\overline{u}^6(D_{\hat\alpha})^5\left[A_{\hat\beta}\,(D_{\hat\gamma})^4T^{(3)}\right], \tag{4.76}$$

where the index contractions need to be worked out. Depending on the number of $D_{\hat{\alpha}}$'s that act on $A_{\hat{\beta}}$, there can be several possible contributions, as we show in the following.

In our search for the possible contributions to (4.74), we are guided by dimensional analysis, SO(3, 1)×SU(4)-invariance, the SYM equations of motion and gauge invariance. One can see $\mathcal{M}_1$ is gauge-invariant because, under a gauge transformation $\delta A_{\hat{\alpha}} = D_{\hat{\alpha}}\Lambda$, one has

$$\delta\mathcal{M}_1 \propto \Big\langle V_{(1)}\,\sqrt{z}\lambda^{\hat{\alpha}}D_{\hat{\alpha}}\Lambda\Big\rangle = -\Big\langle\Big(\sqrt{z}\lambda^{\hat{\alpha}}D_{\hat{\alpha}}V_{(1)}\Big)\,\Lambda\Big\rangle = 0\,, \tag{4.77}$$

since BRST-exact terms decouple and since $Q_{\frac{1}{2}}V_{(1)} = 0$ implies $\sqrt{z}\lambda^{\hat{\alpha}}D_{\hat{\alpha}}V_{(1)} = 0$. We also take into account that $T^{(3)}$ is a G-analytic superfield, such that independent contributions only contain $(\overline{u}D_{\hat{\alpha}})$ derivatives acting on it.

When the five $D_{\hat{\alpha}}$'s in (4.76) act on $A_{\hat{\beta}}$, we get a dimension-3 superfield. So this term could in principle give a contribution proportional to $\partial_a\partial_b W_{ij}$ or $\partial_a F_{bc}$. We do not consider $\partial_a\partial_b A_c$ because this term is not gauge-invariant. It is easy to convince oneself that one cannot construct a non-zero term out of $\partial_a F_{bc}$, and hence the only possible contribution is

$$\overline{u}^6\left[(D_{\hat{\alpha}})^5A_{\hat{\beta}}\right](D_{\hat{\gamma}})^4T^{(3)} \sim (\overline{uu})^{\ell m}\partial_{\beta\dot{\beta}}\partial_{\alpha\dot{\alpha}}W_{\ell m}\,\bar{D}^{\dot{\beta}n}\bar{\widetilde{D}}{}^{\dot{\alpha}}_{n}\widetilde{D}^{\alpha k}D^{\beta}_{k}T^{(3)}\,, \tag{4.78}$$

where we used the definition $\partial_{\alpha\dot{\alpha}} := (\sigma^a)_{\alpha\dot{\alpha}}\partial_a$.

When four $D_{\hat{\alpha}}$'s hit $A_{\hat{\beta}}$, we get a dimension-$\frac{5}{2}$ superfield. This could be either $\partial_a W^{\beta j}$ or its conjugate, $\partial_a\bar{W}^{\dot{\beta}}_{j}$. Keeping in mind the SYM equation of motion $\partial_{\alpha\dot{\alpha}}W^{\alpha i} = 0$, we are left with two possibilities:

$$\begin{aligned}\overline{u}^6\left[(D_{\hat{\alpha}})^4A_{\hat{\beta}}\right](D_{\hat{\gamma}})^5T^{(3)} \;\sim\;& \partial_{\alpha\dot{\alpha}}W^{\beta j}\,\bar{\widetilde{D}}{}^{\dot{\alpha}}_{k}(\bar{D}^{k}\bar{\widetilde{D}}_{j})\widetilde{D}^{\alpha r}D_{\beta r}T^{(3)}\\ &+\partial_{\alpha\dot{\alpha}}\bar{W}^{\dot{\beta}}_{j}\,\bar{D}^{r}_{\dot{\beta}}\bar{\widetilde{D}}{}^{\dot{\alpha}}_{r}(D_k\widetilde{D}^{j})\widetilde{D}^{\alpha k}T^{(3)}\,.\end{aligned} \tag{4.79}$$

Going on with this analysis, we find all the possible gauge-invariant contributions coming from different numbers of derivatives acting on $A_{\hat{\beta}}$, until the last case:

$$\overline{u}^6\left[D_{\hat{\alpha}}A_{\hat{\beta}}\right](D_{\hat{\gamma}})^8T^{(3)} \sim W^{(1)}\,\bar{D}^4D'^4T^{(3)}\,. \tag{4.80}$$

Note that, since $\mathcal{M}_1$ is gauge-invariant, any contributions coming partly from a term in which no $D_{\hat{\alpha}}$ acts on $A_{\hat{\beta}}$ can be expressed as a linear combination of gauge-invariant terms in which at least one $D_{\hat{\alpha}}$ acts on $A_{\hat{\beta}}$.

In summary, we get the following list of possible terms:

1. $W^{(1)}\,\bar{D}^4D'^4T^{(3)}$,

2. $(uu)_{\ell m} W^{\ell}_{\beta}\, \bar{D}^4 D^{\beta}_{k} (\widetilde{D}^k \widetilde{D}^m) T^{(3)}$,

3. $(uu)^{\ell m} \bar{W}^{\dot\beta}_{\ell}\, (\bar{\widetilde{D}}_k \bar{\widetilde{D}}_m) \bar{D}^{k}_{\dot\beta} D'^4 T^{(3)}$,

4. $(\sigma^{ab})_{\gamma}{}^{\beta} F_{ab}\, \bar{D}^4 D^{\gamma}_{j} \widetilde{D}^{j}_{\beta} T^{(3)} =: F_{\gamma}{}^{\beta}\, \bar{D}^4 D^{\gamma}_{j} \widetilde{D}^{j}_{\beta} T^{(3)}$,

5. $(\tilde\sigma^{ab})^{\dot\beta}{}_{\dot\alpha} F_{ab}\, \bar{\widetilde{D}}_{\dot\beta j} \bar{D}^{\dot\alpha j} D'^4 T^{(3)} =: F^{\dot\beta}{}_{\dot\alpha}\, \bar{\widetilde{D}}_{\dot\beta j} \bar{D}^{\dot\alpha j} D'^4 T^{(3)}$,

6. $(uu)^{mj} \partial_{\alpha\dot\alpha} W_{\ell m}\, (\bar{D}^n \bar{\widetilde{D}}_j) \bar{\widetilde{D}}^{\dot\alpha}_{n} \widetilde{D}^{\alpha k} (D_k \widetilde{D}^{\ell}) T^{(3)}$,

7. $\partial_{\alpha\dot\alpha} W^{\beta j}\, \bar{\widetilde{D}}^{\dot\alpha}_{k} (\bar{D}^k \bar{\widetilde{D}}_j) \widetilde{D}^{\alpha r} D_{\beta r} T^{(3)}$,

8. $\partial_{\alpha\dot\alpha} \bar{W}^{\dot\beta}_{j}\, \bar{D}^{r}_{\dot\beta} \bar{\widetilde{D}}^{\dot\alpha}_{r} (D_k \widetilde{D}^j) \widetilde{D}^{\alpha k} T^{(3)}$,

9. $(\overline{uu})^{\ell m} \partial_{\beta\dot\beta} \partial_{\alpha\dot\alpha} W_{\ell m}\, \bar{D}^{\dot\beta n} \bar{\widetilde{D}}^{\dot\alpha}_{n} \widetilde{D}^{\alpha k} D^{\beta}_{k} T^{(3)}$.

To conclude this subsection, let us argue that

$$\bar{D}^4 D'^4 \left[W^{(1)}(x,\theta,u)\, T^{(3)}(x,\theta,u,\overline{u}) \right]$$

is also given by a linear combination of the terms listed above. This expression contains eight derivatives of the type $(\overline{u} D_{\hat\alpha})$. When all these derivatives hit $T^{(3)}$, one obviously gets the first term in the list. If only one derivative acts on $W^{(1)}$, the resulting expression is proportional to either the second or the third term in the list, depending on the chirality of the derivative ($D_{\alpha i}$ or $\bar{D}^{i}_{\dot\alpha}$). And so on until the case in which four $(\overline{u} D_{\hat\alpha})$'s hit $W^{(1)}$ to give either zero or something proportional to the last term. Acting with five or more derivatives on $W^{(1)}$ gives zero, as can be seen from the fact that $(\overline{u} D_{\hat\alpha})^5 W^{(1)}$ has U(1) charge $-\frac{3}{2}$, while the SYM fields $\{\phi_{ij}, \xi^{\alpha i}, \bar\xi^{\dot\alpha}_{i}, f_{ab}\}$ have U(1) charges ranging from -1 to $+1$.

4.2.2 Proof using supersymmetry

Defining $\mathbf{t}_n$ ($n = 1, \ldots, 9$) to be the n-th possible term as listed at the end of the previous subsection, the amplitude we are computing must have the form

$$\mathcal{M}_1 = \int \mathrm{d}^4 x \int \mathrm{d}u \sum_{n=1}^{9} c_n\, \mathbf{t}_n\,, \qquad (4.81)$$

for some constants $c_1, \ldots, c_9$. In this subsection, we will prove that these constants are uniquely determined (up to an overall factor) by supersymmetry. Because both sides of (4.66) are supersymmetric, this proves our conjecture, since both the left- and right-hand sides of (4.66) must be proportional to (4.81). Note that the left-hand side of (4.66) is supersymmetric (up to total derivatives), because $\lambda^{\hat\alpha} A_{\hat\alpha}\, \widetilde{\Omega}_{(0)} T^{(3)}$ is annihilated by $\lambda^{\hat{\alpha}} D_{\hat{\alpha}}$ so BRST invariance of the pure-spinor measure factor implies supersymmetry as usual [2]. To see that the right-hand side of (4.66) is also supersymmetric (up to total derivatives), it suffices to write the supersymmetry generators as $q_{\hat{\alpha}} = D_{\hat{\alpha}} - (\theta\gamma^a)_{\hat{\alpha}}\partial_a$ (cf. (2.3)) and note that $D_{\alpha i}\bar{D}^4 D'^4 \mathcal{F}$ and $\bar{D}^j_{\dot{\alpha}}\bar{D}^4 D'^4\mathcal{F}$ vanish for any G-analytic superfield $\mathcal{F}$ (since there are only four independent $(\overline{u}D)_{\alpha A'}$ and they are fermionic).

In order to uniquely determine the constants $c_1, \ldots, c_9$, we have to impose supersymmetry which implies that $D_{\hat{\alpha}}\left(\sum_{n=1}^{9} c_n\, \mathbf{t}_n\right) = 0$. We begin by acting on the possible terms with $D_{\alpha i}$. We have:

1.

$$\begin{aligned} D_{\alpha i}\left[W^{(1)}\,\bar{D}^4 D'^4 T^{(3)}\right] &= \left[D_{\alpha i}W^{(1)}\right]\bar{D}^4 D'^4 T^{(3)} + W^{(1)}\, D_{\alpha i}\bar{D}^4 D'^4 T^{(3)} \\ &= -2\,(uu)_{ij} W^j_\alpha \bar{D}^4 D'^4 T^{(3)} + 8\,\partial_{\alpha\dot{\alpha}} W^{(1)}\,(\bar{\tilde{D}}_i \bar{\tilde{D}}_j)\bar{D}^{\dot{\alpha}j} D'^4 T^{(3)}\,, \end{aligned}$$

where we used $D_{\alpha i}W_{jk} = -\varepsilon_{ijk\ell}W^\ell_\alpha$ and $[D_{\alpha i}, \bar{D}^4] = -8\,(\bar{\tilde{D}}_i\bar{\tilde{D}}_j)\bar{D}^{\dot{\alpha}j}\partial_{\alpha\dot{\alpha}}$ and integrated by parts.

2.

$$\begin{aligned} D_{\alpha i}\left[(uu)_{\ell m} W^\ell_\beta\, \bar{D}^4 D^\beta_k(\widetilde{D}^k \widetilde{D}^m) T^{(3)}\right] &= \left[D_{\alpha i}W^\ell_\beta\right]\bar{D}^4 D^\beta_k(\widetilde{D}^k D_\ell)T^{(3)} - W^\ell_\beta\, D_{\alpha i}\bar{D}^4 D^\beta_k(\widetilde{D}^k D_\ell)T^{(3)} \\ &= \frac{1}{2}F_\alpha{}^\beta\,\bar{D}^4 D_{\beta k}(\widetilde{D}^k D_i)T^{(3)} + \frac{1}{4}(uu)_{ij}W^j_\alpha \bar{D}^4 D'^4 T^{(3)} \\ &\quad - 8\,\partial_{\alpha\dot{\alpha}}W^\ell_\beta\,(\bar{\tilde{D}}_i\bar{\tilde{D}}_j)\bar{D}^{\dot{\alpha}j}D^\beta_k(\widetilde{D}^k D_\ell)T^{(3)}\,, \end{aligned}$$

where we used $D_{\alpha i}W^{\beta\ell} = \frac{1}{2}\delta^\ell_i F_\alpha{}^\beta$ and

$$D_{\alpha i}D^\beta_k(\widetilde{D}^k D_\ell)T^{(3)} = -\frac{1}{4}\delta^\beta_\alpha (uu)_{i\ell}D'^4 T^{(3)}\,.$$

3.

$$\begin{aligned} D_{\alpha i}\left[(uu)^{\ell m}\bar{W}^{\dot{\beta}}_\ell\,(\bar{\tilde{D}}_k\bar{\tilde{D}}_m)\bar{D}^k_{\dot{\beta}}D'^4 T^{(3)}\right] &= (uu)^{\ell m}\left[D_{\alpha i}\bar{W}^{\dot{\beta}}_\ell\right](\bar{\tilde{D}}_k\bar{\tilde{D}}_m)\bar{D}^k_{\dot{\beta}}D'^4 T^{(3)} \\ &\quad - (uu)^{\ell m}\bar{W}^{\dot{\beta}}_\ell\, D_{\alpha i}(\bar{\tilde{D}}_k\bar{\tilde{D}}_m)\bar{D}^k_{\dot{\beta}}D'^4 T^{(3)} \\ &= -2\,(uu)^{\ell m}\partial_{\alpha\dot{\alpha}}W_{i\ell}\,(\bar{\tilde{D}}_k\bar{\tilde{D}}_m)\bar{D}^{\dot{\alpha}k}D'^4 T^{(3)} \\ &\quad - 2\,(uu)^{\ell m}\partial_{\alpha\dot{\alpha}}\bar{W}^{\dot{\beta}}_\ell\,\bar{\tilde{D}}^{\dot{\alpha}}_m\bar{\tilde{D}}_{\dot{\beta}i}D'^4 T^{(3)} \\ &\quad + 2\,(uu)^{\ell m}(\overline{uu})_{mi}\,\partial_{\alpha\dot{\alpha}}\bar{W}^{\dot{\beta}}_\ell\,\bar{\tilde{D}}^{\dot{\alpha}}_k\bar{D}^k_{\dot{\beta}}D'^4 T^{(3)} \\ &= -2\,(uu)^{\ell m}\partial_{\alpha\dot{\alpha}}W_{i\ell}\,(\bar{\tilde{D}}_k\bar{\tilde{D}}_m)\bar{D}^{\dot{\alpha}k}D'^4 T^{(3)} \\ &\quad + 3\,(uu)^{\ell m}(\overline{uu})_{mi}\,\partial_{\alpha\dot{\alpha}}\bar{W}^{\dot{\beta}}_\ell\,\bar{\tilde{D}}^{\dot{\alpha}}_k\bar{D}^k_{\dot{\beta}}D'^4 T^{(3)}\,, \end{aligned}$$

where we used $D_{\alpha i}\bar{W}_{\dot{\alpha}j} = -2\,\partial_{\alpha\dot{\alpha}}W_{ij}$ and the identities $(\overline{uu})^{i[j}(\overline{uu})^{k\ell]} = 0$ and $\varepsilon^{\dot{\alpha}[\dot{\beta}}\varepsilon^{\dot{\gamma}\dot{\delta}]} = 0$.

4.

$$D_{\alpha i}\Big[F_\gamma{}^\beta\,\bar D^4 D^\gamma_j \widetilde D^j_\beta T^{(3)}\Big] \;=\; F_\gamma{}^\beta\,\bar D^4 D_{\alpha i} D^\gamma_j \widetilde D^j_\beta T^{(3)} + 8\,\partial_{\alpha\dot\alpha}F_\gamma{}^\beta\,(\bar{\widetilde D}_i \bar{\widetilde D}_j)\bar D^{\dot\alpha j} D^\gamma_k \widetilde D^k_\beta T^{(3)}\,,$$

where we used $D_{\alpha i}F_\gamma{}^\beta = 0$.

5.

$$D_{\alpha i}\Big[F^{\dot\beta}{}_{\dot\alpha}\,\bar{\widetilde D}_{\dot\beta j}\bar D^{\dot\alpha j} D'^4 T^{(3)}\Big] \;=\; -4\,\partial_{\alpha\dot\alpha}\bar W^{\dot\beta}_i\,\bar{\widetilde D}_{\dot\beta j}\bar D^{\dot\alpha j} D'^4 T^{(3)}\,,$$

where we used $D_{\alpha i}F^{\dot\beta}{}_{\dot\alpha} = -4\,\partial_{\alpha\dot\alpha}\bar W^{\dot\beta}_i$ and $\partial_{\alpha\dot\beta}F^{\dot\beta}{}_{\dot\alpha} = 0$.

6.

$$\begin{aligned}
&D_{\alpha i}\Big[(uu)^{mj}\partial_{\beta\dot\beta}W_{\ell m}\,(\bar D^n \bar{\widetilde D}_j)\bar{\widetilde D}^{\dot\beta}_n \widetilde D^{\beta k}(D_k \widetilde D^\ell)T^{(3)}\Big] = \\
&= -\varepsilon_{i\ell mp}(uu)^{mj}\partial_{\beta\dot\beta}W^p_\alpha\,(\bar D^n \bar{\widetilde D}_j)\bar{\widetilde D}^{\dot\beta}_n \widetilde D^{\beta k}(D_k \widetilde D^\ell)T^{(3)} \\
&\quad + (uu)^{mj}\partial_{\beta\dot\beta}W_{\ell m}\,D_{\alpha i}(\bar D^n \bar{\widetilde D}_j)\bar{\widetilde D}^{\dot\beta}_n \widetilde D^{\beta k}(D_k \widetilde D^\ell)T^{(3)} \\
&= -\varepsilon_{i\ell mp}(uu)^{mj}\partial_{\beta\dot\beta}W^p_\alpha\,(\bar D^n \bar{\widetilde D}_j)\bar{\widetilde D}^{\dot\beta}_n \widetilde D^{\beta k}(D_k \widetilde D^\ell)T^{(3)} \\
&\quad - 3\,(uu)^{mj}(\overline{uu})_{ji}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{\ell m}\,\bar D^{\dot\alpha n}\bar{\widetilde D}^{\dot\beta}_n \widetilde D^{\beta k}(D_k \widetilde D^\ell)T^{(3)} \\
&\quad + \frac{1}{4}(uu)^{mj}(\overline{uu})^{\ell p}(uu)_{pi}\partial_{\alpha\dot\alpha}W_{\ell m}\,(\bar D^n \bar{\widetilde D}_j)\bar{\widetilde D}^{\dot\alpha}_n D'^4 T^{(3)}\,.
\end{aligned}$$

7.

$$\begin{aligned}
D_{\alpha i}\Big[\partial_{\beta\dot\beta}W^{\gamma j}\,\bar{\widetilde D}^{\dot\beta}_k(\bar D^k \bar{\widetilde D}_j)\widetilde D^{\beta r}D_{\gamma r}T^{(3)}\Big] &= \big[D_{\alpha i}\partial_{\beta\dot\beta}W^{\gamma j}\big]\,\bar{\widetilde D}^{\dot\beta}_k(\bar D^k \bar{\widetilde D}_j)\widetilde D^{\beta r}D_{\gamma r}T^{(3)} \\
&\quad - \partial_{\beta\dot\beta}W^{\gamma j}\,D_{\alpha i}\bar{\widetilde D}^{\dot\beta}_k(\bar D^k \bar{\widetilde D}_j)\widetilde D^{\beta r}D_{\gamma r}T^{(3)} \\
&= \frac{1}{2}\partial_{\beta\dot\beta}F_\alpha{}^\gamma\,\bar{\widetilde D}^{\dot\beta}_k(\bar D^k \bar{\widetilde D}_i)\widetilde D^{\beta r}D_{\gamma r}T^{(3)} \\
&\quad + 3\,(\overline{uu})_{ij}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W^{\gamma j}\,\bar{\widetilde D}^{\dot\beta}_k \bar D^{\dot\alpha k}\widetilde D^{\beta r}D_{\gamma r}T^{(3)} \\
&\quad + \partial_{\beta\dot\beta}W^{\gamma j}\,\bar{\widetilde D}^{\dot\beta}_k(\bar D^k \bar{\widetilde D}_j)D_{\alpha i}\widetilde D^{\beta r}D_{\gamma r}T^{(3)}\,.
\end{aligned}$$

8.

$$\begin{aligned}
D_{\alpha i}\Big[\partial_{\beta\dot\beta}\bar W^{\dot\gamma}_j\,\bar D^r_{\dot\gamma}\bar{\widetilde D}^{\dot\beta}_r(D_k \widetilde D^j)\widetilde D^{\beta k}T^{(3)}\Big] &= -2\,\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{ij}\,\bar D^{\dot\alpha r}\bar{\widetilde D}^{\dot\beta}_r(D_k \widetilde D^j)\widetilde D^{\beta k}T^{(3)} \\
&\quad + \frac{1}{4}(\overline{uu})^{jm}(uu)_{mi}\partial_{\alpha\dot\alpha}\bar W^{\dot\gamma}_j\,\bar D^r_{\dot\gamma}\bar{\widetilde D}^{\dot\alpha}_r D'^4 T^{(3)}\,.
\end{aligned}$$

9.

$$\begin{aligned}
D_{\alpha i}\Big[(\overline{uu})^{\ell m}\partial_{\beta\dot\beta}\partial_{\gamma\dot\gamma}W_{\ell m}\,\bar D^{\dot\beta n}\bar{\widetilde D}^{\dot\gamma}_n \widetilde D^{\gamma k}D^\beta_k T^{(3)}\Big] &= -2\,(\overline{uu})_{ij}\partial_{\beta\dot\beta}\partial_{\gamma\dot\gamma}W^j_\alpha\,\bar D^{\dot\beta n}\bar{\widetilde D}^{\dot\gamma}_n \widetilde D^{\gamma k}D^\beta_k T^{(3)} \\
&\quad + (\overline{uu})^{\ell m}\partial_{\beta\dot\beta}\partial_{\gamma\dot\gamma}W_{\ell m}\,\bar D^{\dot\beta n}\bar{\widetilde D}^{\dot\gamma}_n \widetilde D^{\gamma k}D^\beta_k D_{\alpha i}T^{(3)}\,.
\end{aligned}$$

We see that acting with $D_{\alpha i}$ on $\mathbf{t}_n$ produces various terms. In order to impose that the amplitude is supersymmetric, we need to collect the terms which should cancel independently. In the following, we organize the results according to the superfields appearing in each term. The imposition that they cancel gives rise to a system of many equations for the constants c_n, which have to be solved at the same time.

- Terms proportional to $W^{\beta j}$ (without x-derivatives):

$$(D_{\alpha i}\mathcal{M}_1)|_{W^{\beta j}} = \left(\frac{1}{4}c_2 - 2c_1\right)(uu)_{ij} W^j_\alpha \bar{D}^4 D'^4 T^{(3)} .$$

  Hence we get our first equation:

$$\frac{1}{4}c_2 - 2c_1 = 0 . \tag{4.82}$$

- Terms proportional to $F_\gamma{}^\beta$:

$$\begin{aligned}(D_{\alpha i}\mathcal{M}_1)|_{F_\gamma{}^\beta} &= \frac{1}{2}c_2\, F_\alpha{}^\beta\, \bar{D}^4 D_{\beta k}(\widetilde{D}^k D_i)T^{(3)} + c_4\, F_\gamma{}^\beta\, \bar{D}^4 D_{\alpha i} D^\gamma_j \widetilde{D}^j_\beta T^{(3)} \\ &= \left(\frac{1}{2}c_2 + \frac{2}{3}c_4\right) F_\alpha{}^\beta\, \bar{D}^4 D_{\beta k}(\widetilde{D}^k D_i)T^{(3)} ,\end{aligned}$$

  where we used $F_\gamma{}^\beta \bar{D}^4 D_{\alpha i} D^\gamma_j \widetilde{D}^j_\beta T^{(3)} = \frac{2}{3} F_\alpha{}^\beta \bar{D}^4 D_{\beta k}(\widetilde{D}^k D_i)T^{(3)}$. Therefore,

$$\frac{1}{2}c_2 + \frac{2}{3}c_4 = 0 . \tag{4.83}$$

- Terms proportional to $\partial F_\gamma{}^\beta$:

$$\begin{aligned}(D_{\alpha i}\mathcal{M}_1)|_{\partial F_\gamma{}^\beta} &= 8c_4\, \partial_{\alpha\dot\alpha} F_\gamma{}^\beta\, (\bar{\widetilde{\bar{D}}}_i \bar{\widetilde{\bar{D}}}_j)\bar{D}^{\dot\alpha j} D^\gamma_k \widetilde{D}^k_\beta T^{(3)} \\ &\quad + \frac{1}{2}c_7\, \partial_{\beta\dot\beta} F_\alpha{}^\gamma\, \bar{\widetilde{\bar{D}}}{}^{\dot\beta}_k (\bar{D}^k \bar{\widetilde{\bar{D}}}_i)\widetilde{D}^{\beta r} D_{\gamma r} T^{(3)} \\ &= \left(8c_4 + \frac{1}{2}c_7\right) \partial_{\alpha\dot\alpha} F_\gamma{}^\beta\, (\bar{\widetilde{\bar{D}}}_i \bar{\widetilde{\bar{D}}}_j)\bar{D}^{\dot\alpha j} D^\gamma_k \widetilde{D}^k_\beta T^{(3)} ,\end{aligned}$$

  where we used $\partial_{\alpha\dot\alpha} F_\beta{}^\gamma = \partial_{\beta\dot\alpha} F_\alpha{}^\gamma$. So we get

$$8c_4 + \frac{1}{2}c_7 = 0 . \tag{4.84}$$

- Terms proportional to $\partial \bar{W}^{\dot\alpha}_j$:

$$\begin{aligned}(D_{\alpha i}\mathcal{M}_1)|_{\partial \bar{W}^{\dot\alpha}_j} &= 3c_3\, (uu)^{\ell m} (\overline{uu})_{mi}\, \partial_{\alpha\dot\alpha} \bar{W}^{\dot\beta}_\ell\, \bar{\widetilde{\bar{D}}}{}^{\dot\alpha}_k \bar{D}^k_{\dot\beta} D'^4 T^{(3)} \\ &\quad - 4c_5\, \partial_{\alpha\dot\alpha} \bar{W}^{\dot\beta}_i\, \bar{\widetilde{\bar{D}}}_{\dot\beta j} \bar{D}^{\dot\alpha j} D'^4 T^{(3)} \\ &\quad + \frac{1}{4}c_8\, (\overline{uu})^{jm} (uu)_{mi} \partial_{\alpha\dot\alpha} \bar{W}^{\dot\gamma}_j\, \bar{D}^r_{\dot\gamma} \bar{\widetilde{\bar{D}}}{}^{\dot\alpha}_r D'^4 T^{(3)} \\ &= (3c_3 - 4c_5)\, (uu)^{\ell m} (\overline{uu})_{mi}\, \partial_{\alpha\dot\alpha} \bar{W}^{\dot\beta}_\ell\, \bar{\widetilde{\bar{D}}}{}^{\dot\alpha}_k \bar{D}^k_{\dot\beta} D'^4 T^{(3)} \\ &\quad - \left(\frac{1}{4}c_8 + 4c_5\right) (\overline{uu})^{\ell m} (uu)_{mi}\, \partial_{\alpha\dot\alpha} \bar{W}^{\dot\beta}_\ell\, \bar{\widetilde{\bar{D}}}{}^{\dot\alpha}_k \bar{D}^k_{\dot\beta} D'^4 T^{(3)} ,\end{aligned}$$

where we used $(uu)^{\ell m}(\overline{uu})_{mi} + (\overline{uu})^{\ell m}(uu)_{mi} = \delta^\ell_i$. Thus we obtain two equations:

$$\left\{ \begin{array}{rcl} 3c_3 - 4c_5 &=& 0\,, \\ \frac{1}{4}c_8 + 4c_5 &=& 0\,. \end{array} \right. \tag{4.85}$$

- Terms proportional to $\partial W^{\beta j}$:

$$\begin{aligned} (D_{\alpha i}\mathcal{M}_1)|_{\partial W^{\beta j}} &= -8c_2\, \partial_{\alpha\dot\alpha} W^\ell_\beta\, (\bar{\tilde{D}}_i \bar{\tilde{D}}_j)\bar{D}^{\dot\alpha j} D^\beta_k (\tilde{D}^k D_\ell) T^{(3)} \\ &\quad - c_6\, \varepsilon_{i\ell mp}(uu)^{mj} \partial_{\beta\dot\beta} W^p_\alpha\, (\bar{D}^n \bar{\tilde{D}}_j) \bar{\tilde{D}}^{\dot\beta}_n \tilde{D}^{\beta k} (D_k \tilde{D}^\ell) T^{(3)} \\ &\quad + c_7\, \partial_{\beta\dot\beta} W^{\gamma j}\, \bar{\tilde{D}}^{\dot\beta}_k (\bar{D}^k \bar{\tilde{D}}_j) D_{\alpha i} \tilde{D}^{\beta r} D_{\gamma r} T^{(3)} \\ &= (c_6 - 8c_2)\, \partial_{\alpha\dot\alpha} W^\ell_\beta\, (\bar{\tilde{D}}_i \bar{\tilde{D}}_j)\bar{D}^{\dot\alpha j} D^\beta_k (\tilde{D}^k D_\ell) T^{(3)} \\ &\quad + \left(\frac{3}{2}c_6 - c_7\right) \partial_{\gamma\dot\alpha} W^\ell_\beta\, (\bar{\tilde{D}}_\ell \bar{\tilde{D}}_j) \bar{D}^{\dot\alpha j} \tilde{D}^{\gamma k} D^\beta_k D_{\alpha i} T^{(3)}\,, \end{aligned}$$

where we used the identities $\varepsilon_{i\ell mp} = -3\,(uu)_{[i\ell}(\overline{uu})_{m]p} - 3\,(\overline{uu})_{[i\ell}(uu)_{m]p}$, $\varepsilon_{\alpha[\beta}\varepsilon_{\gamma\delta]} = 0$ and $(uu)_{[i\ell}(uu)_{m]p} = 0$. Hence we get two more equations:

$$\left\{ \begin{array}{rcl} c_6 - 8c_2 &=& 0\,, \\ \frac{3}{2}c_6 - c_7 &=& 0\,. \end{array} \right. \tag{4.86}$$

- Terms proportional to $\partial^2 W^{\beta j}$:

$$\begin{aligned} (D_{\alpha i}\mathcal{M}_1)|_{\partial^2 W^{\beta j}} &= 3c_7\, (\overline{uu})_{ij} \partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta} W^{\gamma j}\, \bar{\tilde{D}}^{\dot\beta}_k \bar{D}^{\dot\alpha k} \tilde{D}^{\beta r} D_{\gamma r} T^{(3)} \\ &\quad - 2c_9\, (\overline{uu})_{ij} \partial_{\beta\dot\beta}\partial_{\gamma\dot\gamma} W^j_\alpha\, \bar{D}^{\dot\beta n} \bar{\tilde{D}}^{\dot\gamma}_n \tilde{D}^{\gamma k} D^\beta_k T^{(3)} \\ &= (3c_7 + 2c_9)\, (\overline{uu})_{ij} \partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta} W^{\gamma j}\, \bar{\tilde{D}}^{\dot\beta}_k \bar{D}^{\dot\alpha k} \tilde{D}^{\beta r} D_{\gamma r} T^{(3)}\,, \end{aligned}$$

whence

$$3c_7 + 2c_9 = 0\,. \tag{4.87}$$

- Terms proportional to ∂W_{ij}:

$$\begin{aligned} (D_{\alpha i}\mathcal{M}_1)|_{\partial W ij} &= 8c_1\, \partial_{\alpha\dot\alpha} W^{(1)}\, (\bar{\tilde{D}}_i \bar{\tilde{D}}_j) \bar{D}^{\dot\alpha j} D'^4 T^{(3)} \\ &\quad - 2c_3\, (uu)^{\ell m} \partial_{\alpha\dot\alpha} W_{i\ell}\, (\bar{\tilde{D}}_k \bar{\tilde{D}}_m) \bar{D}^{\dot\alpha k} D'^4 T^{(3)} \\ &\quad + \frac{1}{4}c_6\, (uu)^{mj} (\overline{uu})^{\ell p} (uu)_{pi} \partial_{\alpha\dot\alpha} W_{\ell m}\, (\bar{D}^n \bar{\tilde{D}}_j) \bar{\tilde{D}}^{\dot\alpha}_n D'^4 T^{(3)} \\ &= (8c_1 + c_3)\, \partial_{\alpha\dot\alpha} W^{(1)}\, (\bar{\tilde{D}}_i \bar{\tilde{D}}_j) \bar{D}^{\dot\alpha j} D'^4 T^{(3)} \\ &\quad + \left(2c_3 + \frac{1}{4}c_6\right) (uu)^{mj} (\overline{uu})^{\ell p} (uu)_{pi} \partial_{\alpha\dot\alpha} W_{\ell m}\, (\bar{D}^n \bar{\tilde{D}}_j) \bar{\tilde{D}}^{\dot\alpha}_n D'^4 T^{(3)}\,, \end{aligned}$$

so that we must have

$$\left\{ \begin{array}{rcl} 8c_1 + c_3 &=& 0\,, \\ 2c_3 + \frac{1}{4}c_6 &=& 0\,. \end{array} \right. \tag{4.88}$$

- Terms proportional to $\partial^2 W_{ij}$:

$$\begin{aligned}(D_{\alpha i}\mathcal{M}_1)|_{\partial^2 W ij} &= -3c_6\,(uu)^{mj}(\overline{uu})_{ji}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{\ell m}\,\bar D^{\dot\alpha n}\bar{\widetilde D}{}^{\dot\beta}_{n}\widetilde D^{\beta k}(D_k\widetilde D^{\ell})T^{(3)}\\ &\quad - 2c_8\,\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{ij}\,\bar D^{\dot\alpha r}\bar{\widetilde D}{}^{\dot\beta}_{r}(D_k\widetilde D^{j})\widetilde D^{\beta k}T^{(3)}\\ &\quad + c_9\,(\overline{uu})^{\ell m}\partial_{\beta\dot\beta}\partial_{\gamma\dot\gamma}W_{\ell m}\,\bar D^{\dot\beta n}\bar{\widetilde D}{}^{\dot\gamma}_{n}\widetilde D^{\gamma k}D^{\beta}_{k}D_{\alpha i}T^{(3)}\\ &= (3c_6-2c_8)\,(uu)^{mj}(\overline{uu})_{ji}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{m\ell}\,\bar D^{\dot\alpha n}\bar{\widetilde D}{}^{\dot\beta}_{n}\widetilde D^{\beta k}(D_k\widetilde D^{\ell})T^{(3)}\\ &\quad - \left(\frac{4}{3}c_9+2c_8\right)(\overline{uu})^{mj}(uu)_{ji}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}W_{m\ell}\,\bar D^{\dot\alpha n}\bar{\widetilde D}{}^{\dot\beta}_{n}\widetilde D^{\beta k}(D_k\widetilde D^{\ell})T^{(3)}\,,\end{aligned}$$

then we find the last pair of equations:

$$\begin{cases} 3c_6-2c_8 = 0\,,\\ \frac{4}{3}c_9+2c_8 = 0\,.\end{cases} \tag{4.89}$$

Putting together (4.82), ... , (4.89), we get the following system of equations:

$$\begin{cases} \frac{1}{4}c_2-2c_1 = 0\\ \frac{1}{2}c_2+\frac{2}{3}c_4 = 0\\ 8c_4+\frac{1}{2}c_7 = 0\\ 3c_3-4c_5 = 0\\ \frac{1}{4}c_8+4c_5 = 0\\ c_6-8c_2 = 0\\ \frac{3}{2}c_6-c_7 = 0\\ 3c_7+2c_9 = 0\\ 8c_1+c_3 = 0\\ 2c_3+\frac{1}{4}c_6 = 0\\ 3c_6-2c_8 = 0\\ \frac{4}{3}c_9+2c_8 = 0\,.\end{cases} \tag{4.90}$$

We see there are a few more equations than unknowns, but they turn out to be not all independent. Setting $c_1 = 1$ as our normalization,[‡] this system can be solved to

[‡]Here we are implicitly neglecting the case in which all the constants c_n vanish, which of course is also a solution to the system of equations. It is straightforward to show the right-hand side of (4.66) is non-vanishing, and the left-hand side can also be shown to be non-zero by direct computation of one of the possible terms, for example by choosing the gauge $A_{\hat\alpha} = (\theta\gamma^{[ij]})_{\hat\alpha}W_{ij}$ with W_{ij} constant.

give

$$
\begin{aligned}
c_2 = -c_3 &= 8\,, \\
c_4 = c_5 &= -6\,, \\
c_6 &= 64\,, \\
c_7 = c_8 &= 96\,, \\
c_9 &= -144\,.
\end{aligned}
\tag{4.91}
$$

Note that, assuming $T^{(3)}$ is harmonic real (as it should be in order for the vertex operator to be real), this solution implies $\sum_{n=1}^{9} c_n\,\mathbf{t}_n$ is harmonic real, since $\widetilde{\mathbf{t}_1} = \mathbf{t}_1$, $\widetilde{\mathbf{t}_2} = -\mathbf{t}_3$, $\widetilde{\mathbf{t}_4} = \mathbf{t}_5$, $\widetilde{\mathbf{t}_6} = \mathbf{t}_6$, $\widetilde{\mathbf{t}_7} = \mathbf{t}_8$ and $\widetilde{\mathbf{t}_9} = \mathbf{t}_9$. Let us show $\mathbf{t}_6$ is harmonic real, since this is not as easy to see as the other relations. We have

$$
\begin{aligned}
\widetilde{\mathbf{t}_6} &= \frac{1}{2}\varepsilon^{\ell mpq}(uu)_{mj}\partial_{\alpha\dot\alpha}W_{pq}\,(D_n\widetilde{\bar D}^j)\widetilde{D}^{\alpha n}\widetilde{\bar{\bar D}}{}^{\dot\alpha}_{k}(\bar D^k\widetilde{\bar{\bar D}}_\ell)T^{(3)} \\
&= -\frac{3}{2}(uu)^{[pq}(\overline{uu})^{\ell]m}(uu)_{mj}\partial_{\alpha\dot\alpha}W_{pq}\,(D_n\widetilde{\bar D}^j)\widetilde{D}^{\alpha n}\widetilde{\bar{\bar D}}{}^{\dot\alpha}_{k}(\bar D^k\widetilde{\bar{\bar D}}_\ell)T^{(3)} \\
&= -(uu)^{\ell p}(\overline{uu})^{qm}(uu)_{mj}\partial_{\alpha\dot\alpha}W_{pq}\,(D_n\widetilde{\bar D}^j)\widetilde{D}^{\alpha n}\widetilde{\bar{\bar D}}{}^{\dot\alpha}_{k}(\bar D^k\widetilde{\bar{\bar D}}_\ell)T^{(3)} \\
&= (uu)^{\ell p}\partial_{\alpha\dot\alpha}W_{pj}\,(\bar D^k\widetilde{\bar{\bar D}}_\ell)\widetilde{\bar{\bar D}}{}^{\dot\alpha}_{k}\widetilde{D}^{\alpha n}(D_n\widetilde{\bar D}^j)T^{(3)} \\
&= \mathbf{t}_6\,,
\end{aligned}
\tag{4.92}
$$

where we have used $(W_{\ell m})^\dagger = \frac{1}{2}\varepsilon^{\ell mpq}W_{pq}$, $\varepsilon^{\ell mpq} = -3\,(uu)^{[pq}(\overline{uu})^{\ell]m} - 3\,(\overline{uu})^{[pq}(uu)^{\ell]m}$, $(uu)^{\ell m}(uu)_{mj} = 0$ and $(\overline{uu})^{qm}(uu)_{mj} = \delta^q_j - (uu)^{qm}(\overline{uu})_{mj}$. Therefore,

$$
D_{\alpha i}\left(\sum_{n=1}^{9} c_n\,\mathbf{t}_n\right) = 0 \iff \bar D^i_{\dot\alpha}\left(\sum_{n=1}^{9} c_n\,\mathbf{t}_n\right) = 0\,. \tag{4.93}
$$

Hence we have proved that, up to an overall factor, there is only one combination of the possible terms which is supersymmetric. This in turn shows that equation (4.66) is indeed true.

4.3 The cases $N > 1$

In the last section, we have shown that

$$
V_{(1)} = z\int \mathrm{d}u\,\Omega_{(0)}T^{(3)}(x,\theta,u,\overline{u})
$$

implies

$$
\mathcal{M}_1 = \Big\langle V_{\mathrm{SYM}}\,V_{(1)}\Big\rangle_{\mathrm{D3-brane}} \propto \int \mathrm{d}^4x \int \mathrm{d}u\,\bar D^4 D'^4\left[W^{(1)}\,T^{(3)}\right]. \tag{4.94}
$$

Now we want to generalize this result to any N, i.e. we want to show

$$\begin{aligned}\mathcal{M}_N &:= \int \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{N-1} \left\langle V_{\mathrm{SYM}}\, V_{(N)}\, U_{\mathrm{SYM}}(\xi_1) \ldots U_{\mathrm{SYM}}(\xi_{N-1}) \right\rangle_{\mathrm{D3-brane}} \\ &\propto \int \mathrm{d}^4 x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(N)}\, T^{(4-N)} \right] , \end{aligned} \tag{4.95}$$

where we recall U_{SYM} is the integrated version of $V_{\mathrm{SYM}} = \sqrt{z}\, \lambda^{\hat{\alpha}} A_{\hat{\alpha}}$ and $V_{(N)}$ was defined in (3.25).

4.3.1 The $N = 2$ case

As a natural first step, let us analyze the case $N = 2$. In this case, we want to compute the following scattering amplitude:

$$\mathcal{M}_2 = \int \mathrm{d}\xi \left\langle V_{\mathrm{SYM}}(\infty)\, V_{(2)}(\mathrm{i}\epsilon, -\mathrm{i}\epsilon)\, U_{\mathrm{SYM}}(\xi) \right\rangle_{\mathrm{D3-brane}} , \tag{4.96}$$

where we have fixed the worldsheet positions of the unintegrated vertex operators (here ϵ is a positive infinitesimal).

The easiest way to do this computation is by taking the OPE of $V_{(2)}(\mathrm{i}\epsilon, -\mathrm{i}\epsilon)$ with $U_{\mathrm{SYM}}(\xi)$ and looking for terms which can contribute to the dual to V_{SYM}. Here we use the word "dual" meaning an object O such that $\langle V_{\mathrm{SYM}}\, O \rangle$ is nonzero. This object must be in the ghost-number +2 cohomology of the BRST operator Q and its product with V_{SYM} should include terms proportional to the measure factor

$$(\lambda \gamma^{\mu} \theta)(\lambda \gamma^{\nu} \theta)(\lambda \gamma^{\rho} \theta)(\theta \gamma_{\mu\nu\rho} \theta) .$$

Moreover, the amplitude must of course be PSU$(2, 2|4)$-invariant, and in particular SU(4)-invariant. A superfield in the $[0, p, 0]$ representation of SU(4) should couple to another superfield in the same representation, so that a scalar ($[0, 0, 0]$) is present in their tensor product decomposition. The Dynkin label p is related to the number of y's (n_y) in a vertex operator by $p = n_y + 1$, as can be seen from the coupling (3.31). The argument goes as follows: $T^{(4-N)}$ couples to $(W_{ij})^N$ and hence $V_{(N)}$ corresponds to $(W_{ij})^N$. Now, $(W_{ij})^N$ transforms in the $[0, N, 0]$ of SU(4), while $V_{(N)} \sim y^{N-1}$. Therefore, since V_{SYM} is independent of y, so must O be.

As seen in subsection 3.2, any object in the ghost-number +2 cohomology of Q has the form

$$O_{(N)} = z^{2-N} \int \mathrm{d}u \sum_{n=0}^{4} 8^n P_n(N)\, (yuu)^{N-n-1}\, \Omega_{(n)} G^{(4-N)}(x, \theta, u, \overline{u}) ,$$

where $G^{(4-N)}(x, \theta, u, \overline{u})$ is some G-analytic superfield of harmonic U(1) charge $4-N$ and the operators $\Omega_{(n)}$ were defined in (3.26). Since, by the argument given in the previous paragraph, the dual to V_{SYM} must be independent of y, it follows that it can be written as

$$O \equiv O_{(1)} = z \int \mathrm{d}u \, \Omega_{(0)} G^{(3)} + Q\chi \,, \tag{4.97}$$

where we have explicitly included a possible BRST-trivial term. Thus our problem is equivalent to finding the superfield $G^{(3)}$.

To find terms in the OPE of $V_{(2)}(\mathrm{i}\epsilon, -\mathrm{i}\epsilon)$ with $U_{\text{SYM}}(\xi)$ which can contribute to O, one first considers the OPE's coming from the conformal-weight +1 operators of the integrated vertex operator. Since we are only interested in the $z \to 0$ limit, we can consider the flat-space expression for the integrated vertex operator corresponding to states propagating in a D3-brane world-volume, i.e. (cf. (2.18))

$$U_{\text{SYM}} = \partial\theta^{\hat{\alpha}} A_{\hat{\alpha}} + (\partial x^a + \theta\gamma^a\partial\theta) A_a + z\partial y^{ij} W_{ij} + \frac{1}{2} z P_{\hat{\psi}^{\hat{\alpha}}} W^{\hat{\alpha}} + \mathcal{O}(z^2) \,, \tag{4.98}$$

where $P_{\hat{\psi}^{\hat{\alpha}}}$ is the momentum conjugate to $\hat{\psi}^{\hat{\alpha}}$. Note the decomposition of $\partial x^\mu A_\mu$ into $\partial x^a A_a + z\partial y^{ij} W_{ij}$.

The term $\frac{1}{2} z P_{\hat{\psi}^{\hat{\alpha}}} W^{\hat{\alpha}}$ is just the $d_{\hat{\alpha}} W^{\hat{\alpha}}$ term written in $\mathrm{AdS}_5 \times \mathrm{S}^5$ notation. The easiest way to see $P_{\hat{\psi}^{\hat{\alpha}}}$ corresponds to $d_{\hat{\alpha}}$ in the $z \to 0$ limit is by recalling the expression for $Q_{-\frac{1}{2}}$, the lowest term in the Q expansion in powers of z, which is reproduced below:

$$Q_{-\frac{1}{2}} \propto \frac{1}{\sqrt{z}} \left(\lambda^{+\gamma m} y_{mi} P_{\psi_i^\gamma} + \bar{\lambda}_j^{+\dot{\alpha}} y^{ji} P_{\bar{\psi}^{i\dot{\alpha}}} \right) .$$

Since this expression can be written in ten-dimensional notation as $z^{-\frac{1}{2}} \lambda^{+\hat{\alpha}} P_{\hat{\psi}^{\hat{\alpha}}}$ and the expression in flat space would be $\lambda^{\hat{\alpha}} d_{\hat{\alpha}}$, it follows that $P_{\hat{\psi}^{\hat{\alpha}}}$ corresponds to $d_{\hat{\alpha}}$. The factor of z in $\frac{1}{2} z P_{\hat{\psi}^{\hat{\alpha}}} W^{\hat{\alpha}}$ enters by dimensional analysis, and the numerical factor is needed for BRST-invariance.

The terms of order z^2 in (4.98) will not contribute. For example, one of these terms is $z^2 N^{ab} F_{ab}$. Because the OPE of N^{ab} and λ is independent of z, this term cannot give a contribution of order z, and therefore cannot contribute to the dual of V_{SYM} (cf. (4.97)) in the $z \to 0$ limit.

Another term in U_{SYM} which will not contribute is $\partial x^a A_a$. Since the kinetic term for x^a in the Lagrangian is

$$\frac{1}{4\pi} \frac{\partial x^a \, \bar{\partial} x_a}{z^2} \,,$$

it turns out the OPE of ∂x^a and a superfield depending on x is also of order z^2.

In fact, since the dual to V_{SYM} does not depend on y or $\hat{\psi}$, only the terms $z \, \partial y^{ij} W_{ij}$ and $\frac{1}{2} z P_{\hat{\psi}^{\hat{\alpha}}} W^{\hat{\alpha}}$ in U_{SYM} may contribute to $\mathcal{M}_2$. The reason is that these are the only terms which can remove the y- and $\hat{\psi}$-dependence of $V_{(2)}$ via the OPE's

$$P_{\hat{\psi}\hat{\alpha}}(\xi)\,\hat{\psi}^{\hat{\beta}}(\zeta,\bar{\zeta}) \sim \frac{\delta_{\hat{\alpha}}^{\hat{\beta}}}{\xi-\zeta} + \frac{\delta_{\hat{\alpha}}^{\hat{\beta}}}{\xi-\bar{\zeta}} \tag{4.99}$$

and

$$\partial y^{ij}(\xi)\,y^{k\ell}(\zeta,\bar{\zeta}) \sim \frac{2\varepsilon^{ijk\ell}}{\xi-\zeta} + \frac{2\varepsilon^{ijk\ell}}{\xi-\bar{\zeta}} + \cdots\,, \tag{4.100}$$

where the dots include terms depending on y.

The closed superstring vertex operator for $N = 2$ is given by (cf. (3.25))

$$V_{(2)} = \int \mathrm{d}u\,\left[(yuu)\,\Omega_{(0)} + 8\,\Omega_{(1)}\right] T^{(2)}\,. \tag{4.101}$$

Hence, there are two contributions to the amplitude. The $\Omega_{(0)}$-term in $V_{(2)}$ is contracted with the $z\,\partial y^{ij} W_{ij}$ in U_{SYM} to give

$$4\left[\frac{1}{\xi-\mathrm{i}\epsilon} + \frac{1}{\xi+\mathrm{i}\epsilon}\right] z \int \mathrm{d}u\, W^{(1)}\,\Omega_{(0)} T^{(2)}\,,$$

while the $\Omega_{(1)}$-term in $V_{(2)}$ is contracted with the $\frac{1}{2} z P_{\hat{\psi}\hat{\alpha}} W^{\hat{\alpha}}$ in U_{SYM} to give

$$-4\left[\frac{1}{\xi-\mathrm{i}\epsilon} + \frac{1}{\xi+\mathrm{i}\epsilon}\right] z \int \mathrm{d}u\, \widehat{\Omega}_{(1)} T^{(2)}\,,$$

where $\widehat{\Omega}_{(n)}$ is equal to $\Omega_{(n)}$ with the substitution $\hat{\psi}^{\hat{\alpha}} \longmapsto W^{\hat{\alpha}}$.

So, performing the integral over $\mathrm{d}\xi$ in the complex plane, choosing a contour that encloses the pole $\xi = \mathrm{i}\epsilon$, we obtain

$$\mathcal{M}_2 \propto \left\langle V_{\mathrm{SYM}}\, z \int \mathrm{d}u \left[W^{(1)}\,\Omega_{(0)} - \widehat{\Omega}_{(1)}\right] T^{(2)}\right\rangle. \tag{4.102}$$

Since $\sqrt{z}\lambda^{\hat{\alpha}} D_{\hat{\alpha}}(\lambda\gamma^{\mu} W) = 0$, the $\widehat{\Omega}_{(n)}$'s also satisfy the equations (3.27), with the substitution $\hat{\psi}^{\hat{\alpha}} \longmapsto W^{\hat{\alpha}}$. Hence, it is not difficult to show that

$$z \int \mathrm{d}u \left[W^{(1)}\,\Omega_{(0)} - \widehat{\Omega}_{(1)}\right] T^{(2)}\,, \tag{4.103}$$

and thus the amplitude, is BRST-invariant as it should be. Let us do that. First, note that the $\left(2z\frac{\partial}{\partial z} + y^{k\ell}\frac{\partial}{\partial y^{k\ell}} - \lambda^{\hat{\alpha}}\frac{\partial}{\partial \lambda^{\hat{\alpha}}}\right)$-part of $Q_{\frac{1}{2}}$ annihilates (4.103). Then, since this expression does not depend on y, it is left to show that (4.103) is annihilated by $\sqrt{z}\lambda^{\hat{\alpha}} D_{\hat{\alpha}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}$. Using (3.27a) and the SYM equation of motion $D_{\hat{\alpha}} W_{ij} = -(\gamma_{[ij]} W)_{\hat{\alpha}}$, we get

$$\left(\sqrt{z}\lambda^{\hat{\alpha}} D_{\hat{\alpha}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right)(W^{(1)}\,\Omega_{(0)} T^{(2)}) = -\sqrt{z}\,(\lambda\gamma_{[ij]} W)(uu)^{ij}\Omega_{(0)} T^{(2)}\,, \tag{4.104}$$

while the modified version of (3.27b) gives

$$\left(\sqrt{z}\lambda^{\hat{\alpha}}D_{\hat{\alpha}} + \tilde{w}^{\hat{\alpha}}r_{\hat{\alpha}}\right)\widehat{\Omega}_{(1)}T^{(2)} = -\sqrt{z}\,(\lambda\gamma^{[ij]}W)(uu)_{ij}\Omega_{(0)}T^{(2)}\,. \tag{4.105}$$

Thus the BRST variations of the two terms in (4.103) cancel each other, implying that expression is indeed BRST-invariant.

Previously we argued that any object in the ghost-number +2 cohomology of Q which does not depend on y can be expressed in the form (4.97) for some $G^{(3)}$ which is G-analytic. Therefore, (4.103) can be expressed in the form (4.97). Moreover, when W_{ij} is constant, it is easy to see that (4.103) can be expressed in this form with $G^{(3)} = W^{(1)}\,T^{(2)}$. So $G^{(3)} = W^{(1)}\,T^{(2)} + f$ where f is a G-analytic term involving derivatives of W_{ij}. But $DW^{(1)}$ is not G-analytic and there are no G-analytic terms of the appropriate dimension that can be constructed out of derivatives of W_{ij}. So $G^{(3)}$ must be equal to $W^{(1)}\,T^{(2)}$ even when W_{ij} is not constant, i.e. (4.103) must be equal to

$$z\int \mathrm{d}u\,\Omega_{(0)}\left[W^{(1)}\,T^{(2)}\right] + Q\chi_2\,, \tag{4.106}$$

where the BRST-trivial term $Q\chi_2$ vanishes when W_{ij} is constant.

Indeed, using the SYM equations of motion and performing manipulations similar to those in subsection 4.1.1, one obtains, after a long and tedious computation,

$$\begin{aligned}\widetilde{\Omega}_{(0)}\left[W^{(1)}\,T^{(2)}\right] &= W^{(1)}\,\widetilde{\Omega}_{(0)}T^{(2)} \\ &\quad + 12\,(uu)_{ij}(W\gamma^{\mu}\widetilde{D})(\lambda\gamma^{[ij]}\widetilde{D})(\lambda\gamma_{\mu}\widetilde{D})T^{(2)} \\ &\quad - 2\,(uu)^{ij}(\lambda\gamma^{\mu}W)(\lambda\gamma^{\nu}\widetilde{D})(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D})T^{(2)} \\ &\quad - 48\,(uu)^{ij}(\overline{uu})^{pq}\,(\partial_a W_{pq})\,(\lambda\gamma_{[ij]}\widetilde{D})(\lambda\gamma^{a}\widetilde{D})T^{(2)} \\ &\quad + 48\,F_{\mu\nu}(\lambda\gamma^{\mu}\widetilde{D})(\lambda\gamma^{\nu}\widetilde{D})T^{(2)} \\ &\quad - 96\,(\overline{uu})_{ij}(\lambda\gamma^{[ij]}\partial_a W)(\lambda\gamma^{a}\widetilde{D})T^{(2)}\,. \end{aligned} \tag{4.107}$$

Note that we work in the non-minimal-variable-free gauge for simplicity. In addition, if

$$\begin{aligned}\chi_2 &= \kappa_1\sqrt{z}\int \mathrm{d}u\,(W\gamma^{\mu}\widetilde{D})(\lambda\gamma_{\mu}\widetilde{D})T^{(2)} \\ &\quad + \kappa_2\sqrt{z}\int \mathrm{d}u\,(\overline{uu})^{ij}\,(\partial_a W_{ij})\,(\lambda\gamma^{a}\widetilde{D})T^{(2)}\,, \end{aligned} \tag{4.108}$$

then

$$\begin{aligned}
Q_{\frac{1}{2}}\chi_2 &= -2\,\kappa_1\, z\int \mathrm{d}u\, F_{\mu\nu}(\lambda\gamma^\mu \widetilde{D})(\lambda\gamma^\nu \widetilde{D})T^{(2)} \\
&\quad -\frac{1}{2}\kappa_1\, z\int \mathrm{d}u\,(uu)_{ij}(W\gamma^\mu \widetilde{D})(\lambda\gamma^{[ij]}\widetilde{D})(\lambda\gamma_\mu \widetilde{D})T^{(2)} \\
&\quad +2\,\kappa_1\, z\int \mathrm{d}u\,(\overline{u}\overline{u})_{k\ell}(\lambda\gamma^{[k\ell]}W)(\lambda\gamma^a \widetilde{D})\partial_a T^{(2)} \\
&\quad -\kappa_2\, z\int \mathrm{d}u\,(\overline{u}\overline{u})^{ij}(\lambda\gamma_{[ij]}\partial_a W)(\lambda\gamma^a \widetilde{D})T^{(2)} \\
&\quad -\frac{1}{2}\kappa_2\, z\int \mathrm{d}u\,(\overline{u}\overline{u})^{ij}(uu)_{k\ell}\,(\partial_a W_{ij})\,(\lambda\gamma^{[k\ell]}\widetilde{D})(\lambda\gamma^a \widetilde{D})T^{(2)}\,. \quad (4.109)
\end{aligned}$$

Hence, choosing $\kappa_1 = 24$ and $\kappa_2 = -96$, we have (cf. (4.23b))

$$z\int \mathrm{d}u\,\widetilde{\Omega}_{(0)}\left[W^{(1)}\,T^{(2)}\right] + Q_{\frac{1}{2}}\chi_2 = z\int \mathrm{d}u\left[W^{(1)}\,\widetilde{\Omega}_{(0)} - \widehat{\widetilde{\Omega}}_{(1)}\right]T^{(2)}\,. \qquad (4.110)$$

The conclusion from the above discussion is that

$$\mathcal{M}_2 \propto \left\langle V_{\mathrm{SYM}}\, z\int \mathrm{d}u\,\Omega_{(0)}\left[W^{(1)}\,T^{(2)}\right]\right\rangle. \qquad (4.111)$$

Note this is consistent with the gauge transformation (3.32), since $\delta T^{(2)} = u_A{}^i \frac{\partial}{\partial \overline{u}_{A'}{}^i}\Xi^{(1)\,A}_{A'}$ and the analyticity of $W^{(1)}$ imply

$$\delta\mathcal{M}_2 \propto z\int \mathrm{d}u\,\Omega_{(0)}\left[u_A{}^i\frac{\partial}{\partial \overline{u}_{A'}{}^i}\left(W^{(1)}\,\Xi^{(1)\,A}_{A'}\right)\right],$$

which is BRST-trivial.

Finally, using the $N = 1$ result (cf. (4.94)), we obtain

$$\mathcal{M}_2 \propto \int \mathrm{d}^4x\int \mathrm{d}u\,\bar{D}^4 D'^4\left[W^{(1)}\,W^{(1)}\,T^{(2)}\right] = \int \mathrm{d}^4x\int \mathrm{d}u\,\bar{D}^4 D'^4\left[W^{(2)}\,T^{(2)}\right], \qquad (4.112)$$

thus proving (4.95) in the case $N = 2$. Note that we work in the linearized level, as it is usual when dealing with vertex operators, so that $W^{(1)} = (uu)^{ij}W_{ij}$, without the trace.

4.3.2 Generalization to any $N > 1$

We are now in position to prove (4.95) for any N. Let us copy it here for the sake of readability:

$$\begin{aligned}
\mathcal{M}_N &:= \int \mathrm{d}\xi_1\ldots \mathrm{d}\xi_{N-1}\left\langle V_{\mathrm{SYM}}(\infty)\,V_{(N)}(\mathrm{i}\epsilon,-\mathrm{i}\epsilon)\,U_{\mathrm{SYM}}(\xi_1)\ldots U_{\mathrm{SYM}}(\xi_{N-1})\right\rangle_{\mathrm{D3-brane}} \\
&\propto \int \mathrm{d}^4x\int \mathrm{d}u\,\bar{D}^4 D'^4\left[W^{(N)}\,T^{(4-N)}\right].
\end{aligned}$$

Again, we are looking for the dual to V_{SYM} in the form (4.97), i.e. we are looking for the expression of the G-analytic superfield $G^{(3)}$ in the case of the amplitude $\mathcal{M}_N$.

As argued in the previous subsection, only the terms $z\,\partial y^{ij} W_{ij}$ and $\frac{1}{2} z P_{\hat\psi^{\hat\alpha}} W^{\hat\alpha}$ in the integrated vertex operators can remove the y- and $\hat\psi$-dependence from the supergravity vertex operator through their OPE's and thus contribute to $\mathcal{M}_N$. This also implies that there can be no contribution coming from contractions between two or more integrated vertex operators. Recalling that

$$V_{(N)} = z^{2-N} \int \mathrm{d}u \sum_{n=0}^{4} 8^n P_n(N)\,(yuu)^{N-n-1}\,\Omega_{(n)} T^{(4-N)}(x,\theta,u,\overline{u})\,,$$

the OPE's give, after performing the $(N-1)$ integrations over the $\mathrm{d}\xi$'s,

$$\mathcal{M}_N \propto \Big\langle V_{\text{SYM}}\, z \int \mathrm{d}u \sum_{n=0}^{4} (-1)^n P_n(N)\, W^{(N-n-1)}\, \widehat{\Omega}_{(n)} T^{(4-N)}(x,\theta,u,\overline{u}) \Big\rangle\,. \quad (4.113)$$

Note the factor of z coming from the product of each z in the $N-1$ integrated vertex operators with the z^{2-N} factor of $V_{(N)}$.

Again, one can use equations (3.27) with $\Omega \longmapsto \widehat{\Omega}$ and $\hat\psi^{\hat\alpha} \longmapsto W^{\hat\alpha}$ to show

$$z \int \mathrm{d}u \sum_{n=0}^{4} (-1)^n P_n(N)\, W^{(N-n-1)}\, \widehat{\Omega}_{(n)} T^{(4-N)}(x,\theta,u,\overline{u}) \quad (4.114)$$

is BRST-invariant. It will be useful to write the five terms in the sum of (4.114) explicitly:

$$\begin{aligned}(4.114) = z \int \mathrm{d}u \Big[& W^{(N-1)}\, \widehat{\Omega}_{(0)} - (N-1) W^{(N-2)}\, \widehat{\Omega}_{(1)} \\ & + (N-1)(N-2) W^{(N-3)}\, \widehat{\Omega}_{(2)} \\ & - (N-1)(N-2)(N-3) W^{(N-4)}\, \widehat{\Omega}_{(3)} \\ & + (N-1)(N-2)(N-3)(N-4) W^{(N-5)}\, \widehat{\Omega}_{(4)} \Big] T^{(4-N)}\,. \quad (4.115)\end{aligned}$$

The only part of Q that acts non-trivially in these terms is $\sqrt{z}\lambda^{\hat\alpha} D_{\hat\alpha} + \tilde{w}^{\hat\alpha} r_{\hat\alpha}$. Note that the SYM equation of motion $D_{\hat\alpha} W_{ij} = -(\gamma_{[ij]} W)_{\hat\alpha}$ implies

$$\sqrt{z}\lambda^{\hat\alpha} D_{\hat\alpha} W^{(N)} = -N\sqrt{z}\,(\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-1)}\,.$$

Thus, using the modified version of (3.27), we have

$$\begin{aligned}
\left(Q_{\frac{1}{2}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) (W^{(N-1)}\, \Omega_{(0)} T) &= -(N-1)\sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-2)}\, \Omega_{(0)} T\,, \\
\left(Q_{\frac{1}{2}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) (W^{(N-2)}\, \widehat{\Omega}_{(1)} T) &= -(N-2)\sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-3)}\, \widehat{\Omega}_{(1)} T \\
&\quad - \sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-2)}\, \Omega_{(0)} T\,, \\
\left(Q_{\frac{1}{2}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) (W^{(N-3)}\, \widehat{\Omega}_{(2)} T) &= -(N-3)\sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-4)}\, \widehat{\Omega}_{(2)} T \\
&\quad - \sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-3)}\, \widehat{\Omega}_{(1)} T\,, \\
\left(Q_{\frac{1}{2}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) (W^{(N-4)}\, \widehat{\Omega}_{(3)} T) &= -(N-4)\sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-5)}\, \widehat{\Omega}_{(3)} T \\
&\quad - \sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-4)}\, \widehat{\Omega}_{(2)} T\,, \\
\left(Q_{\frac{1}{2}} + \tilde{w}^{\hat{\alpha}} r_{\hat{\alpha}}\right) (W^{(N-5)}\, \widehat{\Omega}_{(4)} T) &= - \sqrt{z}\, (\lambda\gamma_{[ij]} W)(uu)^{ij} W^{(N-5)}\, \widehat{\Omega}_{(3)} T\,. \qquad (4.116)
\end{aligned}$$

Comparing with (4.115), one can then see that the BRST variation of (4.114) indeed vanishes.

Hence, by arguments completely analogous to the ones given at the end of the previous subsection, we conclude that (4.114) must be equal to

$$z \int \mathrm{d}u\, \Omega_{(0)} \left[W^{(N-1)}\, T^{(4-N)}\right] + Q\chi_N\,, \qquad (4.117)$$

where the BRST-trivial term $Q\chi_N$ vanishes when W_{ij} is constant. This implies $G^{(3)} = W^{(N-1)}\, T^{(4-N)}$ for arbitrary N, and thus

$$\mathcal{M}_N \propto \left\langle V_{\mathrm{SYM}}\, z \int \mathrm{d}u\, \Omega_{(0)} \left[W^{(N-1)}\, T^{(4-N)}\right] \right\rangle. \qquad (4.118)$$

Finally, using (4.94),

$$\mathcal{M}_N \propto \int \mathrm{d}^4x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(1)}\, W^{(N-1)}\, T^{(4-N)}\right] = \int \mathrm{d}^4x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(N)}\, T^{(4-N)}\right], \qquad (4.119)$$

thus proving (4.95) in the general case.

Chapter 5

Conclusion

Despite the remarkable success of the AdS/CFT correspondence, it remains unproved. It is very hard to prove this duality, mainly because, as is often the case, it is of the strong-weak type: the perturbative, weakly-coupled regime on one side corresponds to the non-perturbative, strongly coupled regime on the other. Nevertheless, it is possible to build more and more evidence for the conjecture. One good way to do so is by computing scattering amplitudes in AdS, since they can be compared with correlation functions on the CFT side.

The pure-spinor formalism has proven to be very well suited for amplitude computations in flat space, which makes the computation of amplitudes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ using that formalism seem promising. In this thesis, we have taken a step in that direction. After reviewing some background material, we have explained in detail how the first scattering amplitude involving pure-spinor vertex operators in $\mathrm{AdS}_5 \times \mathrm{S}^5$ was computed. We have verified a conjecture according to which the tree-level scattering amplitude containing a supergravity state and N massless open superstring states close to the boundary of AdS_5 can be written as a harmonic superspace integral involving the supergravity and super-Yang–Mills (SYM) fields. More precisely, we have shown that

$$\begin{aligned}\mathcal{M}_N := &\int \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{N-1} \left\langle V_{\mathrm{SYM}}(\infty)\, V_{(N)}(\mathrm{i}\epsilon, -\mathrm{i}\epsilon)\, U_{\mathrm{SYM}}(\xi_1) \dots U_{\mathrm{SYM}}(\xi_{N-1}) \right\rangle_{\mathrm{D3-brane}} \\ \propto &\int \mathrm{d}^4 x \int \mathrm{d}u\, \bar{D}^4 D'^4 \left[W^{(N)}\, T^{(4-N)} \right],\end{aligned}$$

where $V_{(N)}$ is the supergravity vertex operator defined in (3.25) and the "D3-brane" subscript indicates that the open superstring (SYM) vertex operators are located on a D3-brane parallel and close to the AdS_5 boundary, at some fixed value of y^{ij} and $z \sim 0$.

The harmonic superspace coupling above has been known for some time [20]. Here we have shown that it can be obtained as a superstring scattering amplitude computation involving open and closed superstring vertex operators. This can be seen as a consistency check for the vertex operator found in [18], as well as one more test of the AdS/CFT conjecture, in that the expected relation between supergravity and SYM was found.

In the course of this PhD project, many interesting issues appeared, whose solutions had to be developed. One of these issues gave rise to a "sub-project," namely the investigation of the consistency of the $\mathcal{N} = 4$, $\mathrm{d} = 4$ pure-spinor measure factor [34], which we needed for the computation of the amplitudes involving the effective world-volume degrees of freedom of the D3-brane, and has the potential to be useful in other contexts. Indeed, the main result of that paper is of considerable interest to connect the ten-dimensional formulation of the pure-spinor superstring with four-dimensional superspace approaches to scattering amplitudes and D-brane dynamics. It is very likely to trigger further cross-fertilization between the ten-dimensional methods of e.g. [41, 42, 43] and four-dimensional methods.

Future and perhaps more interesting applications of the vertex operators found in [18] would involve the computation of scattering amplitudes with closed superstring vertex operators only, which could be compared with correlation functions in the SYM side. In principle, the complete expression for the vertex operator would be needed for that computation, but the leading-order boundary behavior might suffice at least in certain limits, such as the one in which the positions of the dual SYM operators are close to each other.

It would also be interesting to compute amplitudes beyond the supergravity limit. To do that, we would need the vertex operators for massive superstring states in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Recently, a supersymmetric DDF-like construction has been achieved within the pure-spinor formalism [44]. Starting with the light-cone massless vertices as input, a systematic derivation of the physical vertex operators at any mass level was given. Since the expressions for the massless vertex operators in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background — their boundary behavior, to be precise — are known, it should be possible to construct a set of DDF-like operators for that background too.

This research can lead to deeper insights into the AdS/CFT duality, as well as the determination of the massive superstring spectrum in $\mathrm{AdS}_5 \times \mathrm{S}^5$ as a by-product.

Appendix A

Notation and conventions

A.1 Two-component spinor notation

The four-dimensional Lorentz group SO(3, 1) is locally isomorphic to SL(2, ℂ), which has two distinct fundamental representations. One of them is described by a pair of complex numbers [35]

$$\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{A.1}$$

with transformation law

$$\psi'_\alpha = \Lambda_\alpha^\beta \psi_\beta \,, \qquad \Lambda \in \mathrm{SL}(2, \mathbb{C}) \,, \tag{A.2}$$

and is called $\left(\frac{1}{2}, 0\right)$ or left-handed chiral representation.

The other fundamental representation, called $\left(0, \frac{1}{2}\right)$ or right-handed chiral, is obtained by complex conjugation:

$$\bar{\psi}'_{\dot\alpha} = \bar{\Lambda}_{\dot\alpha}^{\dot\beta} \bar{\psi}_{\dot\beta} \,, \qquad \bar{\Lambda}_{\dot\alpha}^{\dot\beta} = \overline{(\Lambda_\alpha^\beta)} \,. \tag{A.3}$$

The dot over the indices indicates the representation to which we refer. Note that our convention for complex conjugation is such that the order of the objects remains unchanged.

The indices with and without dot are raised and lowered in the following way:

$$\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta \,, \qquad \bar{\chi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \bar{\chi}_{\dot\beta} \,, \tag{A.4}$$

$$\psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta \,, \qquad \bar{\chi}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \bar{\chi}^{\dot\beta} \,, \tag{A.5}$$

where ε is antisymmetric and such that

$$\varepsilon^{12} = \varepsilon^{\dot1\dot2} = -\varepsilon_{12} = -\varepsilon_{\dot1\dot2} = 1 \quad \Longrightarrow \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \,, \qquad \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma} \,. \tag{A.6}$$

For spinorial derivatives, raising or lowering the indices involve an extra sign. For example, $D_i^\alpha = -\varepsilon^{\alpha\beta} D_{\beta i}$.

The convention for contraction of spinorial indices is

$$\psi^\alpha \lambda_\alpha =: (\psi\lambda)\,, \qquad \bar{\chi}_{\dot\alpha}\bar{\xi}^{\dot\alpha} =: (\bar{\chi}\bar{\xi})\,. \tag{A.7}$$

In SL$(2,\mathbb{C})$ notation, a four-component Dirac spinor is represented by a pair of chiral spinors:

$$\Psi_{\rm D} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}. \tag{A.8}$$

For a Majorana spinor, $\bar{\chi}_{\dot\alpha} = \overline{(\psi_\alpha)}$. The Dirac matrices are

$$\Sigma^a = \begin{pmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{pmatrix}, \tag{A.9}$$

where the matrices σ^a ($a = 0, \ldots, 3$) are defined as

$$(\sigma^a)_{\alpha\dot\alpha} = (-\mathbb{I}_2, \vec\sigma)_{\alpha\dot\alpha}\,, \qquad (\tilde\sigma^a)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma^a)_{\beta\dot\beta} = (-\mathbb{I}_2, -\vec\sigma)^{\dot\alpha\alpha}\,, \tag{A.10}$$

with $\mathbb{I}_2$ the 2×2 identity matrix and $\vec\sigma$ the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.11}$$

and have the following properties:

$$\begin{aligned} &(\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} = -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}\,, \qquad (\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma^b)^{\dot\alpha\alpha} = -2\delta_a^b\,, \\ &\sigma^a\tilde\sigma^b = -\eta^{ab} + \sigma^{ab}\,, \qquad \tilde\sigma^a\sigma^b = -\eta^{ab} + \tilde\sigma^{ab}\,, \\ &\sigma^{ab} = -\sigma^{ba}\,, \qquad \tilde\sigma^{ab} = -\tilde\sigma^{ba}\,, \qquad (\sigma^{ab})_\alpha{}^\alpha = (\tilde\sigma^{ab})^{\dot\alpha}{}_{\dot\alpha} = 0\,, \end{aligned} \tag{A.12}$$

with $\eta^{ab} = {\rm diag}(-1,1,1,1)$. These properties imply $\{\Sigma^a, \Sigma^b\} = -2\eta^{ab}\,\mathbb{I}_4$.

A.2 Ten-dimensional spinors and γ-matrices

In this thesis, the following conventions are used. The 32×32 ten-dimensional Dirac matrices Γ^μ ($\mu = 0$ to 9) are represented in the chiral basis as

$$\Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^{\hat\alpha\hat\beta} \\ (\gamma^\mu)_{\hat\alpha\hat\beta} & 0 \end{pmatrix}, \tag{A.13}$$

where $(\gamma^\mu)_{\hat\alpha\hat\beta} = (\gamma^\mu)_{\hat\beta\hat\alpha}$ and $(\gamma^\mu)^{\hat\alpha\hat\beta} = (\gamma^\mu)^{\hat\beta\hat\alpha}$ ($\hat\alpha, \hat\beta = 1$ to 16) are SO$(9,1)$ Pauli matrices. The latter satisfy

$$(\gamma^\mu)_{\hat\alpha\hat\beta}(\gamma^\nu)^{\hat\beta\hat\gamma} + (\gamma^\nu)_{\hat\alpha\hat\beta}(\gamma^\mu)^{\hat\beta\hat\gamma} = -2\eta^{\mu\nu}\delta_{\hat\alpha}^{\hat\gamma}\,, \tag{A.14}$$

with $\eta^{\mu\nu} = {\rm diag}(-1,1,1,1,1,1,1,1,1,1)$.

It is useful to define antisymmetrized products of γ-matrices as follows:

$$\begin{aligned} (\gamma_{\mu\nu})_{\hat\alpha}{}^{\hat\gamma} &:= (\gamma_{[\mu|})_{\hat\alpha\hat\beta}(\gamma_{|\nu]})^{\hat\beta\hat\gamma}\,, & (\gamma_{\mu\nu})^{\hat\alpha}{}_{\hat\gamma} &:= (\gamma_{[\mu|})^{\hat\alpha\hat\beta}(\gamma_{|\nu]})_{\hat\beta\hat\gamma}\,, \\ (\gamma_{\mu\nu\rho})_{\hat\alpha\hat\beta} &:= (\gamma_{[\mu\nu|})_{\hat\alpha}{}^{\hat\gamma}(\gamma_{|\rho]})_{\hat\gamma\hat\beta}\,, & (\gamma_{\mu\nu\rho})^{\hat\alpha\hat\beta} &:= (\gamma_{[\mu\nu|})^{\hat\alpha}{}_{\hat\gamma}(\gamma_{|\rho]})^{\hat\gamma\hat\beta}\,, \end{aligned} \tag{A.15}$$

and so on, where the antisymmetrization (denoted by the square brackets) is defined such that, if $O_{\mu_1\ldots\mu_n}$ is a totally antisymmetric object, then $O_{\mu_1\ldots\mu_n} = O_{[\mu_1\ldots\mu_n]}$. For example, $(\gamma_{[\mu|})_{\hat\alpha\hat\beta}(\gamma_{|\nu]})^{\hat\beta\hat\gamma} = \frac{1}{2!}\left[(\gamma_\mu)_{\hat\alpha\hat\beta}(\gamma_\nu)^{\hat\beta\hat\gamma} - (\gamma_\nu)_{\hat\alpha\hat\beta}(\gamma_\mu)^{\hat\beta\hat\gamma}\right]$. This antisymmetrization convention is used throughout this thesis, regardless of index kind.

Using these matrices, the product of any two left-handed Majorana-Weyl spinors can be expanded as

$$\chi^{\hat\alpha}\xi^{\hat\beta} = -\frac{1}{16}(\gamma^\mu)^{\hat\alpha\hat\beta}(\chi\gamma_\mu\xi) - \frac{1}{3!16}(\gamma^{\mu\nu\rho})^{\hat\alpha\hat\beta}(\chi\gamma_{\mu\nu\rho}\xi) - \frac{1}{5!32}(\gamma^{\mu\nu\rho\sigma\tau})^{\hat\alpha\hat\beta}(\chi\gamma_{\mu\nu\rho\sigma\tau}\xi)\,, \tag{A.16}$$

where, for example, $(\chi\gamma_\mu\xi) = \chi^{\hat\alpha}(\gamma_\mu)_{\hat\alpha\hat\beta}\xi^{\hat\beta}$. The coefficients in the above equation can be verified by multiplying it by the corresponding matrix and taking the trace with respect to the spinorial indices. The extra factor of $\frac{1}{2}$ in the last term reflects the fact that only half the $\gamma^{\mu\nu\rho\sigma\tau}$ matrices are independent, since $\gamma^{\mu_1\ldots\mu_5} = \frac{1}{5!}\varepsilon^{\mu_1\ldots\mu_{10}}\gamma_{\mu_6\ldots\mu_{10}}$, i.e. they are self-dual. A completely analogous expression holds for the product of two right-handed Majorana-Weyl spinors.

Similarly, for the product of one right- and one left-handed Majorana-Weyl spinor, one has

$$\psi_{\hat\alpha}\xi^{\hat\beta} = \frac{1}{16}\delta_{\hat\alpha}^{\hat\beta}(\psi\xi) + \frac{1}{2!16}(\gamma_{\mu\nu})_{\hat\alpha}{}^{\hat\beta}(\psi\gamma^{\mu\nu}\xi) + \frac{1}{4!16}(\gamma_{\mu\nu\rho\sigma})_{\hat\alpha}{}^{\hat\beta}(\psi\gamma^{\mu\nu\rho\sigma}\xi)\,. \tag{A.17}$$

Given that the γ-matrices are symmetric, it is easy to show that

$$\begin{aligned} (\gamma_{\mu\nu})_{\hat\alpha}{}^{\hat\beta} &= -(\gamma_{\mu\nu})^{\hat\beta}{}_{\hat\alpha}\,, & (\gamma_{\mu\nu\rho\sigma})_{\hat\alpha}{}^{\hat\beta} &= (\gamma_{\mu\nu\rho\sigma})^{\hat\beta}{}_{\hat\alpha}\,, \\ (\gamma_{\mu\nu\rho})_{\hat\alpha\hat\beta} &= -(\gamma_{\mu\nu\rho})_{\hat\beta\hat\alpha}\,, & (\gamma_{\mu\nu\rho})^{\hat\alpha\hat\beta} &= -(\gamma_{\mu\nu\rho})^{\hat\beta\hat\alpha}\,, \\ (\gamma_{\mu\nu\rho\sigma\tau})_{\hat\alpha\hat\beta} &= (\gamma_{\mu\nu\rho\sigma\tau})_{\hat\beta\hat\alpha}\,, & (\gamma_{\mu\nu\rho\sigma\tau})^{\hat\alpha\hat\beta} &= (\gamma_{\mu\nu\rho\sigma\tau})^{\hat\beta\hat\alpha}\,. \end{aligned} \tag{A.18}$$

Then, using (A.16) in the case of two pure spinors $\lambda^{\hat\alpha}\lambda^{\hat\beta}$, we get

$$\lambda^{\hat\alpha}\lambda^{\hat\beta} = -\frac{1}{5!32}(\gamma^{\mu\nu\rho\sigma\tau})^{\hat\alpha\hat\beta}(\lambda\gamma_{\mu\nu\rho\sigma\tau}\lambda)\,, \tag{A.19}$$

since $\lambda\gamma_\mu\lambda = 0$ by definition and $\lambda\gamma_{\mu\nu\rho}\lambda = 0$ because $\lambda^{\hat\alpha}\lambda^{\hat\beta}$ is symmetric under $\hat\alpha \leftrightarrow \hat\beta$.

To conclude this section, let us mention the important identity

$$(\gamma^\mu)_{\hat\alpha(\hat\beta|}(\gamma_\mu)_{|\hat\gamma\hat\delta)} = 0\,, \tag{A.20}$$

where the convention for index symmetrization (denoted by the parentheses) is totally analogous to the one for antisymmetrization. Note this identity implies $(\lambda\gamma^\mu)_{\hat\alpha}(\lambda\gamma_\mu)_{\hat\beta} = 0$, which is very often used.

A.3 Dimensional reduction

Since in the main text we write expressions both in ten- and four-dimensional notation, it is important to clarify our notation and conventions. Breaking the SO(9, 1) Lorentz symmetry to SO(3, 1) × SO(6) ≃ SO(3, 1) × SU(4), an SO(9, 1) vector v^μ ($\mu = 0, \ldots, 9$) decomposes as

$$v^\mu \longmapsto (v^a, v^{[ij]}), \tag{A.21}$$

where v^a ($a = 0, \ldots, 3$) transforms under the representation **4** of SO(3, 1) and $v^{[ij]} = -v^{[ji]}$ ($i, j = 1, \ldots, 4$) transforms under the **6** of SU(4). The relation between the **6** of SU(4) and the **6** of SO(6) is given by the SO(6) Pauli matrices $(\rho_I)^{ij} = -(\rho_I)^{ji}$ ($I = 1, \ldots, 6$) in the following way:

$$v^{[ij]} = \frac{1}{2\mathrm{i}}(\rho_I)^{ij} v^{I+3}. \tag{A.22}$$

These matrices have the properties [36]

$$\begin{aligned}(\rho^I)^{ij}(\rho^J)_{jk} + (\rho^J)^{ij}(\rho^I)_{jk} &= 2\eta^{IJ}\delta^i_k\,,\\ (\rho^I)_{ij} &= \frac{1}{2}\varepsilon_{ijk\ell}(\rho^I)^{k\ell}\,,\\ (\rho^I)_{ij}(\rho_I)_{k\ell} &= -2\varepsilon_{ijk\ell}\,,\end{aligned} \tag{A.23}$$

where $\eta^{IJ} = \mathrm{diag}(1, 1, 1, 1, 1, 1)$ and $\varepsilon_{ijk\ell}$ is the SU(4)-invariant, totally antisymmetric tensor such that $\varepsilon_{1234} = 1$. Analogously, one can define the tensor $\varepsilon^{ijk\ell}$ such that $\varepsilon^{1234} = 1$. These satisfy the relation

$$\varepsilon_{ijk\ell}\varepsilon^{k\ell mn} = 4\delta^m_{[i}\delta^n_{j]}\,. \tag{A.24}$$

A left-handed Majorana-Weyl spinor $\xi^{\hat\alpha}$ ($\hat\alpha = 1, \ldots, 16$) transforming under the **16** of SO(9, 1) decomposes as

$$\xi^{\hat\alpha} \longmapsto (\xi^{\alpha i}, \bar\xi^{\dot\alpha}_j)\,, \tag{A.25}$$

where we use the standard two-component notation for chiral spinors and $\xi^{\alpha i}$ (resp. $\bar\xi^{\dot\alpha}_j$) transforms under the representation **4** (resp. $\bar{\mathbf{4}}$) of SU(4). Analogous conventions apply to right-handed Majorana-Weyl spinors of SO(9, 1).

We also need to know how to translate the SO(9, 1) Pauli matrices $(\gamma^\mu)_{\hat\alpha\hat\beta}$ and $(\gamma^\mu)^{\hat\alpha\hat\beta}$ to the language of SO(3, 1) × SU(4). Based on [37], we propose the following *ansatz* for the non-vanishing components:

$$\begin{aligned}(\gamma^a)_{(\alpha i)\binom{\dot\alpha}{j}} &= \delta^j_i(\sigma^a)_{\alpha\dot\alpha} = (\gamma^a)_{\binom{\dot\alpha}{j}(\alpha i)}\\ (\gamma^{[k\ell]})_{(\alpha i)(\beta j)} &= 2\varepsilon_{\alpha\beta}\delta^k_{[i}\delta^\ell_{j]}\\ (\gamma^{[k\ell]})_{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}} &= \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{ijk\ell}\end{aligned} \tag{A.26}$$

for $(\gamma^\mu)_{\hat\alpha\hat\beta}$ and

$$\begin{aligned}(\gamma^a)^{(\alpha i)\binom{\dot\alpha}{j}} &= \delta^i_j(\tilde\sigma^a)^{\dot\alpha\alpha} = (\gamma^a)^{\binom{\dot\alpha}{j}(\alpha i)} \\ (\gamma^{[k\ell]})^{(\alpha i)(\beta j)} &= \varepsilon^{\alpha\beta}\varepsilon^{ijk\ell} \\ (\gamma^{[k\ell]})^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}} &= 2\varepsilon^{\dot\alpha\dot\beta}\delta^k_{[i}\delta^\ell_{j]}\end{aligned} \tag{A.27}$$

for $(\gamma^\mu)^{\hat\alpha\hat\beta}$. It is straightforward to show that the above matrices satisfy the usual relation

$$(\gamma^\mu)_{\hat\alpha\hat\beta}(\gamma^\nu)^{\hat\beta\hat\gamma} + (\gamma^\nu)_{\hat\alpha\hat\beta}(\gamma^\mu)^{\hat\beta\hat\gamma} = -2\eta^{\mu\nu}\delta^{\hat\gamma}_{\hat\alpha}\,, \tag{A.28}$$

with $\eta^{[ij][k\ell]} := \frac{1}{2}\varepsilon^{ijk\ell}$.

As an example, we show how to obtain the dimensional reduction of the pure spinor constraints $\lambda\gamma^\mu\lambda = 0$ using (A.26). For $\lambda\gamma^a\lambda = 0$, we have

$$\lambda^{\hat\alpha}(\gamma^a)_{\hat\alpha\hat\beta}\lambda^{\hat\beta} = 0 \iff \lambda^{\alpha i}(\gamma^a)_{(\alpha i)\binom{\dot\alpha}{j}}\bar\lambda^{\dot\alpha}_j + \bar\lambda^{\dot\alpha}_j(\gamma^a)_{\binom{\dot\alpha}{j}(\alpha i)}\lambda^{\alpha i} = 2\lambda^{\alpha i}(\sigma^a)_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_i = 0\,,$$

whence

$$\lambda^{\alpha i}\bar\lambda^{\dot\alpha}_i = 0\,. \tag{A.29}$$

For $\lambda\gamma^{[ij]}\lambda = 0$, we have

$$\lambda^{\hat\alpha}(\gamma^{[ij]})_{\hat\alpha\hat\beta}\lambda^{\hat\beta} = 0 \iff \lambda^{\alpha k}(\gamma^{[ij]})_{(\alpha k)(\beta\ell)}\lambda^{\beta\ell} + \bar\lambda^{\dot\alpha}_k(\gamma^{[ij]})_{\binom{\dot\alpha}{k}\binom{\dot\beta}{\ell}}\bar\lambda^{\dot\beta}_\ell = 2(\lambda^i\lambda^j) - \varepsilon^{ijk\ell}(\bar\lambda_k\bar\lambda_\ell) = 0\,,$$

whence

$$(\lambda^i\lambda^j) = \frac{1}{2}\varepsilon^{ijk\ell}(\bar\lambda_k\bar\lambda_\ell)\,. \tag{A.30}$$

Appendix B

Harmonic Superspace

In this thesis we make use of a harmonic superspace composed by an $\mathcal{N}=4$, $d=4$ Minkowski superspace and the coset space SU(4)/S(U(2)×U(2)) [20, 21, 38]. In addition to the usual coordinates x^a, $\theta^{\alpha i}$ and $\bar{\theta}^{\dot{\alpha}}_i$, this superspace is parameterized by new bosonic variables u, called harmonic coordinates. In terms of indices, we write u as $u_{\mathcal{I}}{}^i = (u_A{}^i, \overline{u}_{A'}{}^i)$, and denote its inverse by $u_i{}^{\mathcal{I}} = (\overline{u}_i{}^A, u_i{}^{A'})$. The index $\mathcal{I}$ is transformed by the isotropy group $\mathrm{S(U(2)\times U(2))} \simeq \mathrm{SU(2)\times SU(2)\times U(1)}$ and splits naturally into $A = 1,2$ and $A' = 3,4$. The u's are defined such that $M := (u_A{}^i, \mathrm{i}\,\overline{u}_{A'}{}^i) \in \mathrm{SU}(4)$, and thus have the following properties:

$$\begin{gathered}\overline{\overline{u}_i{}^A} = u_A{}^i\,, \qquad \overline{u_i{}^{A'}} = \overline{u}_{A'}{}^i\,,\\ u_A{}^i\overline{u}_i{}^B = \delta_A^B\,, \qquad \overline{u}_{A'}{}^i u_i{}^{B'} = \delta_{A'}^{B'}\,, \qquad u_A{}^i u_i{}^{A'} = 0\,, \qquad \overline{u}_i{}^A u_A{}^j + u_i{}^{A'}\overline{u}_{A'}{}^j = \delta_i^j\,,\\ \varepsilon^{ijk\ell}\overline{u}_i{}^1\overline{u}_j{}^2 u_k{}^3 u_\ell{}^4 = -1\,.\end{gathered} \tag{B.1}$$

The bars on some of the u's indicate their charge under the $\mathrm{U}(1) \subset \mathrm{S(U(2)\times U(2))}$, which is opposite to that of the unbarred ones. More precisely, the harmonic U(1) charge of an object is defined to be the eigenvalue of the operator

$$D_{\mathrm{o}} := \frac{1}{2}\left[u_A{}^i\frac{\partial}{\partial u_A{}^i} - \overline{u}_{A'}{}^i\frac{\partial}{\partial \overline{u}_{A'}{}^i}\right] \tag{B.2}$$

when acting on that object, i.e. u (resp. $\overline{u}$) has U(1) charge $\frac{1}{2}$ (resp. $-\frac{1}{2}$).

The introduction of harmonic variables allows the definition of superfields which satisfy generalized chirality constraints. A superfield $\mathcal{F}$ which satisfies

$$u_A{}^i D_{\alpha i}\mathcal{F} = u_i{}^{A'}\bar{D}^i_{\dot{\alpha}}\mathcal{F} = 0\,, \tag{B.3}$$

where $D_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{\hat{\alpha}}} + (\gamma^a\theta)_{\hat{\alpha}}\partial_a$, is said to be G-analytic, while a superfield which satisfies

$$u_A{}^i\frac{\partial}{\partial\overline{u}_{A'}{}^i}\mathcal{F} = 0 \tag{B.4}$$

is said to be H-analytic. A superfield that is both G- and H-analytic is called an analytic superfield for short.

It is useful to define the SU(2)×SU(2)-invariant objects

$$(uu)^{ij} := \varepsilon^{AB} u_A{}^i u_B{}^j \qquad \text{and} \qquad (\overline{u}\overline{u})^{ij} := \varepsilon^{A'B'} \overline{u}_{A'}{}^i \overline{u}_{B'}{}^j \,, \tag{B.5}$$

where ε^{AB}, $\varepsilon^{A'B'}$ are completely analogous to $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$. Then, using (B.1), the following identities (among others) can be derived:

$$\begin{aligned}
(uu)_{ij} &:= \frac{1}{2}\varepsilon_{ijk\ell}(uu)^{k\ell} = \varepsilon_{A'B'}\, u_i{}^{A'} u_j{}^{B'} \,, && \text{(B.6a)}\\
(\overline{u}\overline{u})_{ij} &:= \frac{1}{2}\varepsilon_{ijk\ell}(\overline{u}\overline{u})^{k\ell} = \varepsilon_{AB}\, \overline{u}_i{}^{A} \overline{u}_j{}^{B} \,, && \text{(B.6b)}\\
(uu)_{ij}(\overline{u}\overline{u})^{jk} &= u_i{}^{A'} \overline{u}_{A'}{}^k \,, && \text{(B.6c)}\\
(uu)_{ij}(uu)^{jk} &= 0 \,, && \text{(B.6d)}\\
\varepsilon_{ijk\ell} &= -3\,(uu)_{[ij}(\overline{u}\overline{u})_{k]\ell} - 3\,(\overline{u}\overline{u})_{[ij}(uu)_{k]\ell} \,. && \text{(B.6e)}
\end{aligned}$$

We will show only the last one, as the others are relatively simple. First, note that the last equation in (B.1) can be written as

$$\varepsilon^{ijk\ell}\overline{u}_i{}^A \overline{u}_j{}^B u_k{}^{C'} u_\ell{}^{D'} = -\varepsilon^{AB}\varepsilon^{C'D'} \,. \tag{B.7}$$

Then, contracting both sides with $u_A{}^m$ and using $u_A{}^m \overline{u}_i{}^A = \delta_i^m - \overline{u}_{A'}{}^m u_i{}^{A'}$, we get

$$\varepsilon^{mjk\ell}\overline{u}_j{}^B u_k{}^{C'} u_\ell{}^{D'} = -\varepsilon^{AB}\varepsilon^{C'D'} u_A{}^m \,, \tag{B.8}$$

where we used $\varepsilon^{ijk\ell} u_i{}^{A'} u_k{}^{C'} u_\ell{}^{D'} = 0$, since $A', B', \ldots = 3$ to 4. Finally, contracting this equation with $\frac{1}{2}\varepsilon_{minp}\varepsilon_{BC}\varepsilon_{C'D'}\overline{u}_q{}^C$, we obtain

$$-3\,(\overline{u}\overline{u})_{q[i}(uu)_{np]} = \overline{u}_q{}^C u_C{}^m \varepsilon_{minp} \,, \tag{B.9}$$

where we used $\varepsilon_{minp}\varepsilon^{mjk\ell} = 6\,\delta^j_{[i}\delta^k_n\delta^\ell_{p]}$. Now, we could have begun this derivation by contracting (B.7) with $\overline{u}_{C'}{}^m$ instead of $u_A{}^m$. Then, carrying out steps analogous to those which led to (B.9), we would have obtained

$$-3\,(uu)_{q[i}(\overline{u}\overline{u})_{np]} = u_q{}^{C'} \overline{u}_{C'}{}^m \varepsilon_{minp} \,. \tag{B.10}$$

Thus, adding (B.9) and (B.10), we have

$$\left(\overline{u}_q{}^C u_C{}^m + u_q{}^{C'}\overline{u}_{C'}{}^m\right)\varepsilon_{minp} = \varepsilon_{qinp} = -3\,(\overline{u}\overline{u})_{q[i}(uu)_{np]} - 3\,(uu)_{q[i}(\overline{u}\overline{u})_{np]} \,, \tag{B.11}$$

which is equivalent to (B.6e).

In this thesis, we use the following conventions:

$$\begin{aligned}
D^4 &:= D^\alpha_i (uu)^{ij} D^\beta_j D_{\alpha k}(uu)^{k\ell} D_{\beta\ell} \,, && \text{(B.12a)}\\
D'^4 &:= D^\alpha_i (\overline{u}\overline{u})^{ij} D^\beta_j D_{\alpha k}(\overline{u}\overline{u})^{k\ell} D_{\beta\ell} \,, && \text{(B.12b)}\\
\bar{D}^4 &:= \bar{D}^i_{\dot\alpha}(\overline{u}\overline{u})_{ij}\bar{D}^j_{\dot\beta}\bar{D}^{\dot\alpha k}(\overline{u}\overline{u})_{k\ell}\bar{D}^{\dot\beta\ell} \,, && \text{(B.12c)}\\
\bar{D}'^4 &:= \bar{D}^i_{\dot\alpha}(uu)_{ij}\bar{D}^j_{\dot\beta}\bar{D}^{\dot\alpha k}(uu)_{k\ell}\bar{D}^{\dot\beta\ell} \,. && \text{(B.12d)}
\end{aligned}$$

Since there are only four independent $(\overline{u}D)_{\alpha A'}$'s and they anticommute, the "product" of four such derivatives can always be written in terms of D'^4. This implies that, for any G-analytic superfield G, one can write

$$D_{\alpha i}D_{\beta j}D_{\gamma k}D_{\delta\ell}G = \kappa_0 \left(\varepsilon_{(\alpha|\beta}\varepsilon_{|\gamma)\delta}(uu)_{ik}(uu)_{j\ell} + \underset{(\beta j)\leftarrow(\delta\ell)}{\overset{\nearrow(\gamma k)\searrow}{}}\right) D'^4 G\,, \tag{B.13}$$

where $\underset{(\beta j)\leftarrow(\delta\ell)}{\overset{\nearrow(\gamma k)\searrow}{}}$ means cyclic permutations (two more terms) and κ_0 is a constant to be determined. The form of the right-hand side of the above equation can be found by looking for an expression which respects the symmetries of the left-hand side, namely antisymmetry in the exchange of any two D's and vanishing U(1) charge. To determine the constant κ_0, one can contract both sides of (B.13) with $\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}(\overline{uu})^{ik}(\overline{uu})^{j\ell}$. The left-hand side then gives $-D'^4$, whereas the right-hand side gives $18\kappa_0 D'^4$. Hence, $\kappa_0 = -\frac{1}{18}$, and thus one can write

$$D_{\alpha i}D_{\beta j}D_{\gamma k}D_{\delta\ell}G = -\frac{1}{18} \left(\varepsilon_{(\alpha|\beta}\varepsilon_{|\gamma)\delta}(uu)_{ik}(uu)_{j\ell} + \underset{(\beta j)\leftarrow(\delta\ell)}{\overset{\nearrow(\gamma k)\searrow}{}}\right) D'^4 G \tag{B.14}$$

for any G-analytic superfield G. Of course, a completely analogous derivation holds in the case of four $\bar{D}^i_{\dot\alpha}$'s acting on G.

From (B.14), many particular cases can be obtained. For example, contracting both sides of it with $-\varepsilon^{\epsilon\beta}\varepsilon^{\gamma\delta}(\overline{uu})^{jk}$, we get

$$D_{\alpha i}D^{\epsilon}_{j}(\widetilde{D}^j D_\ell)G = -\frac{1}{4}\delta^{\epsilon}_{\alpha}(uu)_{i\ell}D'^4 G\,, \tag{B.15}$$

where we used the definition $\widetilde{D}^{\delta j} := (\overline{uu})^{jk}D^{\delta}_{k}$.

Integrals over the compact space SU(4)/S(U(2)×U(2)) are defined with the SU(4)-invariant Haar measure du, and have the following properties:

$$\int \mathrm{d}u\, 1 = 1\,, \tag{B.16a}$$

$$D_{\mathrm{o}} f(u,\overline{u}) \neq 0 \implies \int \mathrm{d}u\, f(u,\overline{u}) = 0\,, \tag{B.16b}$$

$$\int \mathrm{d}u\, {u_A}^i \frac{\partial}{\partial {\overline{u}_{A'}}^i} g(u,\overline{u}) = 0\,. \tag{B.16c}$$

Non-vanishing integrals, namely those whose integrand is U(1)-neutral and is not a total $\left({u_A}^i\frac{\partial}{\partial {\overline{u}_{A'}}^i}\right)$-derivative, can be performed by using symmetry arguments. For example, suppose we wanted to compute

$$\int \mathrm{d}u\,(uu)_{ij}(\overline{uu})_{k\ell}\,. \tag{B.17}$$

The result of the integration has to be SU(4)-invariant. Thus, taking the index structure into account, we conclude that it must be proportional to $\varepsilon_{ijk\ell}$, i.e.

$$\int \mathrm{d}u\,(uu)_{ij}(\overline{uu})_{k\ell} = c_0\,\varepsilon_{ijk\ell}\,, \tag{B.18}$$

where c_0 is some constant. To determine c_0, it suffices to contract both sides of the above equation with $\varepsilon^{ijk\ell}$. The left-hand side then gives $2\int \mathrm{d}u\,(uu)_{ij}(\overline{u}\overline{u})^{ij} = -4$, whereas the right-hand side gives $24\,c_0$. Hence, $c_0 = -\frac{1}{6}$ and thus

$$\int \mathrm{d}u\,(uu)_{ij}(\overline{u}\overline{u})_{k\ell} = -\frac{1}{6}\varepsilon_{ijk\ell}\,. \tag{B.19}$$

Any integral of this kind can be computed similarly, and the result always involves combinations of ε's and δ's, which are SU(4)-invariant tensors.

Finally, let us introduce the notion of harmonic conjugation. It acts in almost the same way as ordinary complex conjugation, except that it does not change the harmonic U(1) charge. More precisely, denoting this operation by a tilde, we have

$$\begin{aligned}
\widetilde{u_A{}^i} &= u_i{}^{A'}\,, & \widetilde{\overline{u}_i{}^A} &= \overline{u}_{A'}{}^i\,,\\
\widetilde{\overline{u}_{A'}{}^i} &= -\overline{u}_i{}^A\,, & \widetilde{u_i{}^{A'}} &= -u_A{}^i\,,\\
\widetilde{\varepsilon_{AB}} &= \varepsilon^{A'B'}\,, & \widetilde{\varepsilon^{A'B'}} &= \varepsilon_{AB}\,,\\
\widetilde{\varepsilon^{AB}} &= \varepsilon_{A'B'}\,, & \widetilde{\varepsilon_{A'B'}} &= \varepsilon^{AB}\,,
\end{aligned} \tag{B.20}$$

while for any other object it is equivalent to complex conjugation. Note that, while $\overline{(uu)_{ij}} = (\overline{u}\overline{u})^{ij}$, $\widetilde{(uu)_{ij}} = (uu)^{ij}$. One can show harmonic conjugation preserves G- and H-analyticity, which is not true in general for ordinary complex conjugation. The tildes over D's, Ω's and non-minimal pure-spinor variables that appear in some parts of the main text should not be confused with this operation.

Appendix C

SYM equations

The $\mathcal{N}=1$, $d=10$ super-Yang–Mills theory admits a formulation in superspace in terms of the on-shell superfields A_μ and $A_{\hat\alpha}$. Defining the supercovariant derivatives as

$$\begin{aligned} \nabla_\mu &:= \partial_\mu + A_\mu\,, \\ \nabla_{\hat\alpha} &:= D_{\hat\alpha} + A_{\hat\alpha}\,, \\ D_{\hat\alpha} &:= \frac{\partial}{\partial\theta^{\hat\alpha}} + (\gamma^\mu\theta)_{\hat\alpha}\partial_\mu\,, \end{aligned} \tag{C.1}$$

and the field-strength superfields as

$$\begin{aligned} F_{\hat\alpha\hat\beta} &:= \{\nabla_{\hat\alpha},\nabla_{\hat\beta}\} - 2(\gamma^\mu)_{\hat\alpha\hat\beta}\nabla_\mu\,, \\ F_{\hat\alpha\mu} &:= [\nabla_{\hat\alpha},\nabla_\mu]\,, \\ F_{\mu\nu} &:= [\nabla_\mu,\nabla_\nu]\,, \end{aligned} \tag{C.2}$$

it is easy to show that, in the linearized theory, the constraint $F_{\hat\alpha\hat\beta}=0$ implies

$$D_{(\hat\alpha}A_{\hat\beta)} = (\gamma^\mu)_{\hat\alpha\hat\beta}A_\mu\,, \tag{C.3}$$

which, using the trace of (A.28), can be written as

$$A_\mu = -\frac{1}{16}(\gamma_\mu)^{\hat\alpha\hat\beta}D_{\hat\alpha}A_{\hat\beta}\,. \tag{C.4}$$

Substituting (C.4) into (C.3), we get

$$D_{(\hat\alpha}A_{\hat\beta)} = -\frac{1}{16}(\gamma_\mu)_{\hat\alpha\hat\beta}(\gamma^\mu)^{\hat\gamma\hat\delta}D_{\hat\gamma}A_{\hat\delta}\,. \tag{C.5}$$

Note that this equation is equivalent to $(\gamma^{\mu_1\cdots\mu_5})^{\hat\alpha\hat\beta}D_{\hat\alpha}A_{\hat\beta}=0$. For $\hat\alpha=(\alpha i)$ and $\hat\beta=(\beta j)$, we have (recall $(\gamma_a)_{(\alpha i)(\beta j)}=0$)

$$\begin{aligned} D_{\alpha i}A_{\beta j} + D_{\beta j}A_{\alpha i} &= -\frac{1}{8}(\gamma_{[k\ell]})_{(\alpha i)(\beta j)}(\gamma^{[k\ell]})^{\hat\gamma\hat\delta}D_{\hat\gamma}A_{\hat\delta} \\ &= -\frac{1}{4}\varepsilon_{\alpha\beta}\left[2(D_{[i}A_{j]}) + \varepsilon_{ijk\ell}(\bar D^k\bar A^\ell)\right]\,, \end{aligned} \tag{C.6}$$

whence

$$(D_{[i}A_{j]}) = \frac{1}{2}\varepsilon_{ijk\ell}(\bar{D}^k\bar{A}^\ell)\,. \tag{C.7}$$

For $\hat{\alpha} = (\alpha i)$ and $\hat{\beta} = \binom{\dot{\beta}}{j}$, we have (recall $(\gamma_{[k\ell]})_{(\alpha i)\binom{\dot{\beta}}{j}} = 0$)

$$\begin{aligned} D_{\alpha i}\bar{A}^j_{\dot{\beta}} + \bar{D}^j_{\dot{\beta}}A_{\alpha i} &= -\frac{1}{8}(\gamma_a)_{(\alpha i)\binom{\dot{\beta}}{j}}(\gamma^a)^{\hat{\gamma}\hat{\delta}}D_{\hat{\gamma}}A_{\hat{\delta}} \\ &= \frac{1}{4}\delta^j_i\left(D_{\alpha k}\bar{A}^k_{\dot{\beta}} + \bar{D}^k_{\dot{\beta}}A_{\alpha k}\right). \end{aligned} \tag{C.8}$$

Because their $\theta = 0$ components are the same (the scalars ϕ_{ij}), we claim that

$$W_{ij} \equiv A_{[ij]} = -\frac{1}{16}(\gamma_{[ij]})^{\hat{\alpha}\hat{\beta}}D_{\hat{\alpha}}A_{\hat{\beta}}\,, \tag{C.9}$$

where W_{ij} is the Sohnius superfield of $\mathcal{N} = 4$ SYM [22] and we made use of (C.4). Here $A_{[ij]}$ and $A_{\hat{\beta}}$ should be understood as dimensionally reduced superfields, in the sense that they only depend on the four x^a. In four-dimensional notation, we get

$$\begin{aligned} W_{ij} &= -\frac{1}{32}\varepsilon_{ijk\ell}\left[(\gamma^{[k\ell]})^{(\alpha p)(\beta q)}D_{\alpha p}A_{\beta q} + (\gamma^{[k\ell]})^{\binom{\dot{\alpha}}{p}\binom{\dot{\beta}}{q}}\bar{D}^p_{\dot{\alpha}}\bar{A}^q_{\dot{\beta}}\right] \\ &= -\frac{1}{32}\varepsilon_{ijk\ell}\left[\varepsilon^{k\ell pq}(D_pA_q) + 2(\bar{D}^{[k}\bar{A}^{\ell]})\right] \\ &= -\frac{1}{4}(D_{[i}A_{j]})\,, \end{aligned} \tag{C.10}$$

where we made use of (C.7). Note that (C.6) then implies

$$D_{\alpha i}A_{\beta j} + D_{\beta j}A_{\alpha i} = 4\varepsilon_{\alpha\beta}W_{ij}\,. \tag{C.11}$$

Indeed, we can show that this superfield satisfies the same constraints as the Sohnius one. First, note that (C.7) implies $(W_{ij})^\dagger = \frac{1}{2}\varepsilon^{ijk\ell}W_{k\ell}$. Then, writing $W_{jk} = -\frac{1}{8}\varepsilon_{jk\ell m}\bar{D}^\ell_{\dot{\alpha}}\bar{A}^{\dot{\alpha}m}$, we have

$$\begin{aligned} D_{\alpha i}W_{jk} &= -\frac{1}{8}\varepsilon_{jk\ell m}D_{\alpha i}\bar{D}^\ell_{\dot{\alpha}}\bar{A}^{\dot{\alpha}m} \\ &= -\frac{1}{8}\varepsilon_{jk\ell m}\{D_{\alpha i},\bar{D}^\ell_{\dot{\alpha}}\}\bar{A}^{\dot{\alpha}m} + \frac{1}{8}\varepsilon_{jk\ell m}\bar{D}^\ell_{\dot{\alpha}}D_{\alpha i}\bar{A}^{\dot{\alpha}m} \\ &= -\frac{1}{4}\varepsilon_{jkim}(\sigma^a)_{\alpha\dot{\alpha}}\partial_a\bar{A}^{\dot{\alpha}m} + \underbrace{\frac{1}{8}\varepsilon_{jk\ell m}\bar{D}^\ell_{\dot{\alpha}}\bar{D}^{\dot{\alpha}m}A_{\alpha i}}_{=0} + \frac{1}{32}\varepsilon_{jk\ell i}\bar{D}^\ell_{\dot{\alpha}}\left(D_{\alpha p}\bar{A}^{\dot{\alpha}p} - \bar{D}^{\dot{\alpha}p}A_{\alpha p}\right), \end{aligned}$$

where we made use of (C.8). Therefore,

$$D_{\alpha i}W_{jk} = D_{\alpha[i}W_{jk]}\,. \tag{C.12}$$

Moreover, it can be shown that the Hermitian conjugate of the above equation implies

$$\bar{D}^k_{\dot{\alpha}}W_{ij} = \frac{2}{3}\delta^k_{[i|}\bar{D}^\ell_{\dot{\alpha}}W_{\ell|j]}\,. \tag{C.13}$$

Note that these constraints imply $(uu)^{ij}W_{ij}$ is an analytic superfield.

Studying the Bianchi identities for the field-strength superfields leads to the following (linearized) equations of motion [39]:

$$
\begin{aligned}
(\gamma_\mu W)_{\hat\alpha} &= \partial_\mu A_{\hat\alpha} - D_{\hat\alpha} A_\mu \,, && \text{(C.14a)}\\
D_{\hat\alpha} W^{\hat\beta} &= \frac{1}{2}(\gamma^{\mu\nu})_{\hat\alpha}{}^{\hat\beta} F_{\mu\nu} \,, && \text{(C.14b)}\\
D_{\hat\alpha} F_{\mu\nu} &= -2\,\partial_{[\mu}(\gamma_{\nu]} W)_{\hat\alpha} \,, && \text{(C.14c)}\\
(\gamma^\mu)_{\hat\alpha\hat\beta}\partial_\mu W^{\hat\beta} &= 0 \,, && \text{(C.14d)}\\
\partial^\mu F_{\mu\nu} &= 0 \,, && \text{(C.14e)}
\end{aligned}
$$

where $W^{\hat\alpha}$ is the superfield whose $\theta = 0$ component is the gluino $\xi^{\hat\alpha}$. These in turn imply (among other equations), by dimensional reduction,

$$
\begin{aligned}
D_{\alpha i} W_{jk} &= -\varepsilon_{ijk\ell} W^{\ell}_{\alpha} \,, && \text{(C.15a)}\\
D_{\alpha i} \bar W_{\dot\beta j} &= -2\,\partial_{\alpha\dot\beta} W_{ij} \,, && \text{(C.15b)}\\
D_{\alpha i} F_\beta{}^\gamma &= 0 \,, && \text{(C.15c)}\\
D_{\alpha i} F^{\dot\gamma}{}_{\dot\beta} &= -4\,\partial_{\alpha\dot\beta} \bar W^{\dot\gamma}_i \,, && \text{(C.15d)}\\
\partial_{\alpha\dot\beta} F^{\dot\beta}{}_{\dot\alpha} &= 0 \,, && \text{(C.15e)}
\end{aligned}
$$

as well as their Hermitian conjugates, where $\partial_{\alpha\dot\beta} := (\sigma^a)_{\alpha\dot\beta}\partial_a$, $F_\beta{}^\gamma := (\sigma^{ab})_\beta{}^\gamma F_{ab}$ and $F^{\dot\beta}{}_{\dot\alpha} := (\tilde\sigma^{ab})^{\dot\beta}{}_{\dot\alpha} F_{ab}$. Note that we have identified $A_{[ij]}$ with W_{ij} and assumed the superfields only depend on the four x^a.

Choosing the gauge $\theta^{\hat\alpha} A_{\hat\alpha} = 0$ as in [40], we can find expansions of the above superfields in powers of θ in the following manner. First, we contract both sides of (C.3) with $\theta^{\hat\alpha}$ to get

$$
\theta^{\hat\alpha} D_{\hat\alpha} A_{\hat\beta} + \theta^{\hat\alpha} D_{\hat\beta} A_{\hat\alpha} = \mathcal{D} A_{\hat\beta} - D_{\hat\beta}\underbrace{(\theta^{\hat\alpha} A_{\hat\alpha})}_{=0} + A_{\hat\beta} = (\mathcal{D}+1) A_{\hat\beta} = 2\,(\theta\gamma^\mu)_{\hat\beta} A_\mu \,, \quad \text{(C.16)}
$$

where we have defined $\mathcal{D} := \theta^{\hat\alpha} D_{\hat\alpha} = \theta^{\hat\alpha}\frac{\partial}{\partial\theta^{\hat\alpha}}$. Note this operator has the property that, if $S^{[n]}$ ($n = 0$ to 16) represents the term of order θ^n in the expansion of the superfield $S(x, \theta)$ in component fields, then $\mathcal{D} S^{[n]} = n S^{[n]}$. Hence, collecting the terms of the same θ-order, (C.16) implies $A^{[0]}_{\hat\alpha} = 0$ and

$$
A^{[n]}_{\hat\alpha} = \frac{2}{n+1}(\theta\gamma^\mu)_{\hat\alpha} A^{[n-1]}_\mu \,, \quad \text{(C.17a)}
$$

for $n = 1$ to 16. Similarly, (C.14a) and (C.14b) imply

$$
\begin{aligned}
A^{[n]}_\mu &= -\frac{1}{n}(\theta\gamma_\mu W^{[n-1]}) \,, && \text{(C.17b)}\\
W^{\hat\alpha[n]} &= \frac{1}{n}(\theta\gamma^{\mu\nu})^{\hat\alpha}\partial_\mu A^{[n-1]}_\nu \,. && \text{(C.17c)}
\end{aligned}
$$

Now, substituting (C.17c) into (C.17b), we obtain

$$A_\mu^{[n]} = -\frac{1}{n}\left(\theta\gamma_\mu W^{[n-1]}\right) = \frac{1}{n(n-1)}\left(\theta\gamma_\mu\gamma^{\nu\rho}\theta\right)\partial_\nu A_\rho^{[n-2]}\,, \tag{C.18}$$

where we have used $(\gamma^{\nu\rho})_{\hat\alpha}{}^{\hat\beta} = -(\gamma^{\nu\rho})^{\hat\beta}{}_{\hat\alpha}$. Thus, defining the operator $[\mathcal{O}]_\mu{}^\nu := (\theta\gamma_\mu{}^{\rho\nu}\theta)\partial_\rho$, we deduce

$$\begin{aligned} A_\mu^{[2k]} &= \frac{1}{(2k)!}[\mathcal{O}^k]_\mu{}^\nu a_\nu\,, \\ A_\mu^{[2k+1]} &= -\frac{1}{(2k+1)!}[\mathcal{O}^k]_\mu{}^\nu(\theta\gamma_\nu\xi)\,, \end{aligned} \tag{C.19}$$

where $a_\mu = A_\mu^{[0]}$, $\xi^{\hat\alpha} = W^{\hat\alpha[0]}$ and $[\mathcal{O}^0]_\mu{}^\nu \equiv [1]_\mu{}^\nu := \delta_\mu^\nu$.

With these building blocks, it is possible to construct the θ-expansion of any SYM superfield. For example, the Sohnius superfield can be expanded as

$$\begin{aligned} W_{ij} \equiv A_{[ij]} &= A_{[ij]}^{[0]} + A_{[ij]}^{[1]} + A_{[ij]}^{[2]} + \cdots \\ &= a_{[ij]} - (\theta\gamma_{[ij]}\xi) + \frac{1}{2}(\theta\gamma_{[ij]}\gamma^{ab}\theta)\partial_a a_b + \cdots \\ &= \phi_{ij} - \varepsilon_{ijk\ell}\theta^k\xi^\ell + 2\bar\theta_{[i}\bar\xi_{j]} + \frac{1}{4}\varepsilon_{ijk\ell}\theta^k\sigma^{ab}\theta^\ell f_{ab} - \frac{1}{2}\bar\theta_i\tilde\sigma^{ab}\bar\theta_j f_{ab} + \cdots\,, \end{aligned} \tag{C.20}$$

where $a_{[ij]} \equiv \phi_{ij}$, $f_{ab} = F_{ab}^{[0]}$ and we have omitted the term proportional to $\partial_a\phi_{ij}$ in $A_{[ij]}^{[2]}$.

Appendix D

Dimensionally reduced expressions

Although we do not explicitly use them in the main text, we derive here, for completeness, the dimensionally reduced forms of $\Omega_{(0)\hat{\alpha}\hat{\beta}}$ and $(\mathcal{T}D^5)^{\hat{\alpha}\hat{\beta}\hat{\gamma}}$, as they might be useful for the reader.

D.1 Reduced form of $\Omega_{(0)\hat{\alpha}\hat{\beta}}$

Recall the definition of $\Omega_{(0)\hat{\alpha}\hat{\beta}}$ given in (4.73):

$$\Omega_{(0)\hat{\alpha}\hat{\beta}} := (uu)^{ij}(\gamma^\mu \widetilde{D})_{\hat{\alpha}}(\gamma^\nu \widetilde{D})_{\hat{\beta}}(\widetilde{D}\gamma_{\mu\nu[ij]}\widetilde{D})\,. \tag{D.1}$$

Using the *ansatz* for the dimensional reduction of the γ-matrices given in (A.26) and (A.27), we find

$$\Omega_{(0)(\alpha i)(\beta j)} = 4\,\varepsilon_{\alpha\beta}(\overline{uu})_{ij}D'^4 - 48\,(\overline{uu})_{im}(\overline{uu})_{jn}(\overline{uu})^{k\ell}D_{\alpha k}D_{\beta\ell}(\bar{D}^m\bar{D}^n) - 4\,\varepsilon_{\alpha\beta}(\overline{uu})_{ij}\bar{D}^4\,, \tag{D.2a}$$

$$\begin{aligned}\Omega_{(0)\binom{\dot\alpha}{i}(\beta j)} = \Omega_{(0)(\beta j)\binom{\dot\alpha}{i}} &= 32\,(\overline{uu})^{k\ell}(\overline{uu})^{im}(\overline{uu})_{jn}(D_kD_m)D_{\beta\ell}\bar{D}^n_{\dot\alpha}\\ &\quad + 32\,(\overline{uu})_{k\ell}(\overline{uu})_{jm}(\overline{uu})^{in}D_{\beta n}\bar{D}^\ell_{\dot\alpha}(\bar{D}^k\bar{D}^m)\,,\end{aligned} \tag{D.2b}$$

$$\Omega_{(0)\binom{\dot\alpha}{i}\binom{\dot\beta}{j}} = 4\,\varepsilon_{\dot\alpha\dot\beta}(\overline{uu})^{ij}\bar{D}^4 + 48\,(\overline{uu})^{im}(\overline{uu})^{jn}(\overline{uu})_{k\ell}\bar{D}^k_{\dot\alpha}\bar{D}^\ell_{\dot\beta}(D_mD_n) - 4\,\varepsilon_{\dot\alpha\dot\beta}(\overline{uu})^{ij}D'^4\,. \tag{D.2c}$$

Note that these coefficients can also be obtained (up to an overall factor) by imposing that $\widetilde{\Omega}_{(0)}G \propto \lambda^{\hat\alpha}\lambda^{\hat\beta}\Omega_{(0)\hat\alpha\hat\beta}G$ be annihilated by $\lambda^{\hat\alpha}D_{\hat\alpha} = \lambda^{\alpha i}D_{\alpha i} + \bar\lambda_{\dot\alpha i}\bar{D}^{\dot\alpha i}$ for any G-analytic superfield G and using the fact that $\Omega_{(0)\hat\alpha\hat\beta}$ is γ-traceless.*

D.2 Reduced form of $(\mathcal{T}D^5)^{\hat\alpha\hat\beta\hat\gamma}$

In principle, the dimensional reduction of $\mathcal{T}D^5$ (cf. (4.69)) could also be obtained directly by using the formulas (A.26) and (A.27), but that would be a very long

*The γ-tracelessness condition might not be obvious from (D.1), but it is not difficult to show $(\gamma^\mu)^{\hat\alpha\hat\beta}\Omega_{(0)\hat\alpha\hat\beta} = 0$ by direct calculation using γ-matrix identities.

and tedious task. Fortunately, there is an easier way of obtaining this result, as we describe in the following.

We begin by noting that, if $\Lambda_{\hat\alpha}$ is the derivative of $\lambda^{\hat\alpha}$, such that $\Lambda\gamma^\mu\Lambda = 0$, then

$$(\Lambda\gamma^\mu D)(\Lambda\gamma^\nu D)(\Lambda\gamma^\rho D)(D\gamma_{\mu\nu\rho}D) = \Lambda_{\hat\alpha_1}\Lambda_{\hat\alpha_2}\Lambda_{\hat\alpha_3}(\mathcal{T}D^5)^{\hat\alpha_1\hat\alpha_2\hat\alpha_3}\,. \tag{D.3}$$

The left-hand side of the above equation is not so difficult to compute. Up to an overall factor, we find

$$\begin{aligned}(\Lambda\gamma^\mu D)(\Lambda\gamma^\nu D)(\Lambda\gamma^\rho D)(D\gamma_{\mu\nu\rho}D) \;=\;& \varepsilon^{mnj\ell}(\bar\Lambda^i\bar\Lambda^k)(\Lambda_m D_n)(D_i D_j)(D_k D_\ell)\\ &+ 4\,(\bar\Lambda^j\bar\Lambda^k)(\bar\Lambda^\ell\bar D^i)(D_i D_j)(D_k D_\ell)\\ &+ 3\,\varepsilon^{mnj\ell}(\bar\Lambda^i\bar D^k)(\Lambda_k D_\ell)(\Lambda_m D_n)(D_i D_j)\\ &+ 4\,(\bar\Lambda^\ell\bar\Lambda^k)(\Lambda_j D_\ell)(D_i D_k)(\bar D^i\bar D^j)\\ &+ 12\,(\bar\Lambda^i\bar D^k)(\bar\Lambda^j\bar D^\ell)(\Lambda_k D_\ell)(D_i D_j)\\ &+ 2\,\varepsilon^{mn\ell k}(\Lambda_i D_k)(\Lambda_j D_\ell)(\Lambda_m D_n)(\bar D^i\bar D^j)\\ &+ \text{H.c.}\,,\end{aligned} \tag{D.4}$$

where "H.c." stands for the "Hermitian conjugate". One can check that the expression obtained from the above by substituting Λ for λ and D for θ is annihilated by $\lambda^{\hat\alpha}D_{\hat\alpha}$.

Comparing (D.4) and (D.3), one can deduce the components of $\mathcal{T}D^5$. For example, considering the expansion of the right-hand side of (D.3),

$$\begin{aligned}&\Lambda_{\alpha i}\Lambda_{\beta j}\Lambda_{\gamma k}(\mathcal{T}D^5)^{(\alpha i)(\beta j)(\gamma k)} + 3\,\Lambda_{\alpha i}\Lambda_{\beta j}\bar\Lambda^k_{\dot\gamma}(\mathcal{T}D^5)^{(\alpha i)(\beta j)\binom{\dot\gamma}{k}}\\ &+ 3\,\Lambda_{\alpha i}\bar\Lambda^j_{\dot\beta}\bar\Lambda^k_{\dot\gamma}(\mathcal{T}D^5)^{(\alpha i)\binom{\dot\beta}{j}\binom{\dot\gamma}{k}} + \bar\Lambda^i_{\dot\alpha}\bar\Lambda^j_{\dot\beta}\bar\Lambda^k_{\dot\gamma}(\mathcal{T}D^5)^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}\binom{\dot\gamma}{k}}\,,\end{aligned} \tag{D.5}$$

it is not difficult to see that there are two independent possible contributions for the term appearing in the first line of (D.4), since $(\Lambda_i\Lambda_j) = \frac{1}{2}\varepsilon_{ijk\ell}(\bar\Lambda^k\bar\Lambda^\ell)$. Namely,

$$(\mathcal{T}D^5)^{(\alpha i)(\beta j)(\gamma k)}\Big|_{D^5\bar D^0} = \kappa_1\,\varepsilon^{\alpha\beta}\varepsilon^{ijmn}\varepsilon^{k\ell pq}D^\gamma_\ell(D_m D_p)(D_n D_q) + {}_{(\alpha i)}\overset{\nearrow(\beta j)\searrow}{\longleftarrow}{}_{(\gamma k)} \tag{D.6}$$

and

$$(\mathcal{T}D^5)^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}(\gamma k)}\Big|_{D^5\bar D^0} = \kappa_2\,\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{k\ell pq}D^\gamma_\ell(D_i D_p)(D_j D_q)\,, \tag{D.7}$$

where κ_1 and κ_2 are constants to be determined, $\big|_{D^5\bar D^0}$ refers to terms containing 5 D's and no $\bar D$ and ${}_{(\alpha i)}\overset{\nearrow(\beta j)\searrow}{\longleftarrow}{}_{(\gamma k)}$ means cyclic permutations. Substituting (D.6) and (D.7) into (D.5) and then comparing with (D.4), we get

$$3\kappa_2 - 6\kappa_1 = 1\,. \tag{D.8}$$

Furthermore, the γ-tracelessness condition $(\gamma^{[k\ell]})_{\hat\alpha\hat\beta}(\mathcal{T}D^5)^{\hat\alpha\hat\beta\hat\gamma} = 0$ gives

$$6\kappa_1 + 2\kappa_2 = 0\,. \tag{D.9}$$

Hence, $\kappa_1 = -\frac{1}{15}$ and $\kappa_2 = \frac{1}{5}$.

All the other components of $\mathcal{T}D^5$ can be obtained in a similar way and the result is listed in the following:

$$
\begin{aligned}
(\mathcal{T}D^5)^{(\alpha i)(\beta j)(\gamma k)} \;=\; & -\frac{1}{15}\varepsilon^{\alpha\beta}\varepsilon^{ijmn}\varepsilon^{k\ell pq}D^\gamma_\ell(D_mD_p)(D_nD_q) + {}_{(\alpha i)}\overset{\nearrow(\beta j)\searrow}{\longleftarrow}{}_{(\gamma k)} \\
& -\frac{2}{3}\varepsilon^{\alpha\beta}\varepsilon^{ijpq}D^\gamma_p(D_\ell D_q)(\bar D^\ell\bar D^k) + {}_{(\alpha i)}\overset{\nearrow(\beta j)\searrow}{\longleftarrow}{}_{(\gamma k)} \\
& + 2\,D^\alpha_\ell D^\beta_m D^\gamma_n(\bar D^{(i}\bar D^j)\varepsilon^{k)\ell mn} \\
& -\frac{8}{15}\varepsilon^{\alpha\beta}D^\gamma_\ell(\bar D^\ell\bar D^{[i})(\bar D^{j]}\bar D^k) + {}_{(\alpha i)}\overset{\nearrow(\beta j)\searrow}{\longleftarrow}{}_{(\gamma k)}\,,
\end{aligned}
\tag{D.10a}
$$

$$
\begin{aligned}
(\mathcal{T}D^5)^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}(\gamma k)} \;=\; & \frac{1}{5}\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{k\ell pq}D^\gamma_\ell(D_iD_p)(D_jD_q) \\
& -4\,\bar D^{\dot\alpha[k|}\bar D^{\dot\beta|\ell]}D^\gamma_\ell(D_iD_j) \\
& +\frac{8}{5}\delta^k_{(i}(D_{j)}D_\ell)D^\gamma_m\bar D^{\dot\alpha[\ell|}\bar D^{\dot\beta|m]} \\
& +\frac{2}{5}\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{ijpq}D^\gamma_\ell(\bar D^\ell\bar D^p)(\bar D^q\bar D^k) \\
& +\varepsilon_{\ell mn(i}D^\gamma_{j)}\bar D^{\dot\alpha\ell}\bar D^{\dot\beta m}(\bar D^n\bar D^k) \\
& -\frac{1}{5}\delta^k_{(i}\varepsilon_{j)mnp}D^\gamma_\ell\bar D^{\dot\alpha m}\bar D^{\dot\beta p}(\bar D^n\bar D^\ell)\,,
\end{aligned}
\tag{D.10b}
$$

$$
(\mathcal{T}D^5)^{\binom{\dot\alpha}{i}(\beta j)(\gamma k)} = \overline{(\mathcal{T}D^5)^{(\alpha i)\binom{\dot\beta}{j}\binom{\dot\gamma}{k}}}\,, \qquad (\mathcal{T}D^5)^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}\binom{\dot\gamma}{k}} = \overline{(\mathcal{T}D^5)^{(\alpha i)(\beta j)(\gamma k)}}\,.
\tag{D.10c}
$$

Appendix E

The psu(2, 2|4) algebra

The psu(2, 2|4) algebra is a supersymmetric extension of the four-dimensional conformal algebra so(2, 4) $\simeq$ su(2, 2), and it is also the algebra of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superspace isometry group. It contains the generators $P_a, M_{ab}, q_{\alpha i}, \bar{q}^{i}_{\dot{\alpha}} = (q_{\alpha i})^\dagger$ of the super Poincaré group, $U_i^{\ j} = (U_j^{\ i})^\dagger$ of SU(4), K_a of conformal boosts, Δ of dilatations and the conformal supersymmetry generators s^i_α, $\bar{s}_{\dot{\alpha} i} = (s^i_\alpha)^\dagger$. The complete structure of the algebra is given by:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= 2\,(\eta_{c[b}M_{a]d} + \eta_{d[a}M_{b]c})\,, & [P_a, K_b] &= 2\,(\eta_{ab}\Delta + M_{ab})\,, \\
[M_{ab}, P_c] &= 2\,\eta_{c[b}P_{a]}\,, & [M_{ab}, K_c] &= 2\,\eta_{c[b}K_{a]}\,, \\
[\Delta, P_a] &= P_a\,, & [\Delta, K_a] &= -K_a\,, \\
[\Delta, q_{\alpha i}] &= \tfrac{1}{2} q_{\alpha i}\,, & [\Delta, s^i_\alpha] &= -\tfrac{1}{2} s^i_\alpha\,, \\
[M_{ab}, q_{\alpha i}] &= \tfrac{1}{2}(\sigma_{ab})_\alpha{}^\beta q_{\beta i}\,, & [M_{ab}, s^i_\alpha] &= \tfrac{1}{2}(\sigma_{ab})_\alpha{}^\beta s^i_\beta\,, \\
[q_{\alpha i}, K_a] &= \mathrm{i}\,(\sigma_a)_{\alpha\dot{\alpha}}\bar{s}_i^{\dot{\alpha}}\,, & [s^i_\alpha, P_a] &= \mathrm{i}\,(\sigma_a)_{\alpha\dot{\alpha}}\bar{q}^{\dot{\alpha} i}\,, \\
\{q_{\alpha i}, \bar{q}^{j}_{\dot{\alpha}}\} &= 2\mathrm{i}\,\delta_i^j(\sigma^a)_{\alpha\dot{\alpha}}P_a\,, & \{s^i_\alpha, \bar{s}_{\dot{\alpha} j}\} &= 2\mathrm{i}\,\delta^i_j(\sigma^a)_{\alpha\dot{\alpha}}K_a\,,
\end{aligned}
$$

$$
\begin{aligned}
\{q_{\alpha i}, s^{\beta j}\} &= \delta_i^j(\sigma^{ab})_\alpha{}^\beta M_{ab} + 2\,\delta_\alpha^\beta(2U_i^{\ j} - \delta_i^j\Delta)\,, \\
[U^i_{\ j}, q_{\alpha k}] &= \delta^i_k q_{\alpha j} - \tfrac{1}{4}\delta^i_j q_{\alpha k}\,, \\
[U^i_{\ j}, s^{\alpha k}] &= -\delta^k_j s^{\alpha i} + \tfrac{1}{4}\delta^i_j s^{\alpha k}\,, \\
[U^i_{\ j}, U^k_{\ \ell}] &= \delta^i_\ell U^k_{\ j} - \delta^k_j U^i_{\ \ell}\,,
\end{aligned}
$$

as well as the relations obtained from these by Hermitian conjugation. All the other (anti)commutators vanish. Note that, except for the SU(4) generators, in our conventions the bosonic generators are anti-Hermitian.

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