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Sobolev orthogonal polynomials following from coherent pairs of measures of the second kind on the real line

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Orientador: Prof. Dr. Alagacone Sri Ranga

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To my family.

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Joy is the simplest form of gratitude.

Karl Barth

RESUMO

O principal objetivo desta tese é estudar os polinômios ortogonais com respeito a uma classe de produtos internos do tipo Sobolev que envolve pares coerentes de medidas de segundo tipo na reta real. As fórmulas de conexão entre a sequência de polinômios ortogonais mônicos com respeito ao produto interno de Sobolev e a sequência de polinômios ortogonais mônicos com respeito a uma das medidas que aparecem no produto interno são amplamente analisadas. Além disso, mostramos que os zeros dos polinômios ortogonais de Sobolev são os autovalores de uma matriz dada através de uma simples modificação de uma conhecida matriz de Jacobi associada a uma das medidas do produto interno de Sobolev. Finalmente, estudamos um exemplo de par coerente de medidas de segundo tipo na reta real no qual umas das medidas é a medida de Jacobi, e possibilita um estudo detalhado dos polinômios e coeficientes de conexão associados.

Palavras-chave: Polinômios ortogonais na reta real. Polinômios ortogonais de Sobolev. Pares coerentes de segundo tipo. Sequências encadeadas positivas.

ABSTRACT

The main objective in this thesis is to study the orthogonal polynomials with respect to a class of Sobolev-type inner products which follows from coherent pairs of positive measures of the second kind on the real line. The connection formulas involving the sequence of monic orthogonal polynomials with respect to the Sobolev inner product and the sequence of monic orthogonal polynomials with respect to one of the measures that appear in the inner product are thoroughly analyzed. It is also shown that the zeros of the Sobolev orthogonal polynomials are the eigenvalues of a matrix which is a simple modification of a well known Jacobi matrix associated with one of the measures in the Sobolev inner product. Finally, we deal with a special example of a coherent pair of positive measures of the second kind on the real line where one of the measures is the Jacobi measure and it provides a much more detailed analysis of the polynomials and the associated connection coefficients.

Keywords: Orthogonal polynomials on the real line. Sobolev orthogonal polynomials. Coherent pairs of the second kind. Positive chain sequences.

Contents

Introduction

The theory of orthogonal polynomials which has gained an increasing attention over the past decades plays a wide role in mathematics, physics and engineering. For example, orthogonal polynomials appear in many problems regarding quadrature rules, signal processing, operator and spectral theory.

Let $\mathbf{v}$ be a linear functional defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients and consider $\mathbb{P}^*$ the space of all linear functionals defined on $\mathbb{P}$. We denote by $\langle \mathbf{v}, p \rangle$ the action of the linear functional $\mathbf{v} \in \mathbb{P}^*$ on the polynomial $p \in \mathbb{P}$. In what follows, we will refer to $\mathbf{v} \in \mathbb{P}^*$ as *moment functional*.

A sequence of monic polynomials $\{\mathcal{P}_n\}_{n\geq 0}$ on $\mathbb{P}$ is said to be a sequence of *monic orthogonal polynomials* (MOP) with respect to the moment functional $\mathbf{v}$ if $deg(\mathcal{P}_n) = n$ and

$$\langle \mathbf{v}, \mathcal{P}_m \mathcal{P}_n \rangle = \hbar_n \delta_{m,n}, \quad \hbar_n \neq 0, \quad m, n = 0, 1, 2, \ldots .$$

A moment functional $\mathbf{v}$ is called quasi-definite if and only if there exists $\{\mathcal{P}_n\}_{n\geq 0}$ a sequence of MOP with respect to $\mathbf{v}$. In this case, the sequence of MOP $\{\mathcal{P}_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\mathcal{P}_{n+1}(x) = (x - \mathfrak{c}_{n+1})\mathcal{P}_n(x) - \mathfrak{d}_{n+1}\mathcal{P}_{n-1}(x), \quad n \geq 1,$$

with $\mathcal{P}_0(x) = 1$, $\mathcal{P}_1(x) = x - \mathfrak{c}_1$ and $\mathfrak{d}_{n+1} \neq 0$ for $n \geq 1$. Moreover, if $\mathfrak{c}_n \in \mathbb{R}$ and $\mathfrak{d}_{n+1} > 0$ for all $n \geq 1$ then $\mathbf{v}$ is said to be positive definite and, in addition, has an integral representation

$$\langle \mathbf{v}, p \rangle = \int_{\mathbb{R}} p(x) d\nu(x), \quad p \in \mathbb{P},$$

where ν is a positive measure with infinite support on $\mathbb{R}$. We will refer to the sequence of MOP with respect to a positive definite moment functional $\mathbf{v}$ as the sequence of MOP with respect to a positive measure ν.

In this work we are interested in orthogonal polynomials with respect to the Sobolev-type inner product $\langle \cdot, \cdot \rangle_{\mathfrak{S}}$ given by

$$\langle f, g \rangle_{\mathfrak{S}} = \int_{\mathbb{R}} f(x) g(x) d\nu_0(x) + s \int_{\mathbb{R}} f'(x) g'(x) d\nu_1(x), \quad s > 0,$$

where $\{\nu_0, \nu_1\}$ is a pair of positive measures supported on the real line. We will denote by $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{S}}$, which we call sequence of *monic Sobolev orthogonal polynomials* (MSOP) with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{S}}$.

A pair of positive measures $\{\nu_0, \nu_1\}$ supported on the real line is said to be a coherent pair of positive measures on the real line if the corresponding sequences of MOP

$\{\mathcal{P}_n(\nu_0;x)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1;x)\}_{n\geq 0}$ satisfy

$$\mathcal{P}_n(\nu_1;x) = \frac{1}{n+1}\left[\mathcal{P}'_{n+1}(\nu_0;x) - \sigma_n \mathcal{P}'_n(\nu_0;x)\right], \quad \sigma_n \neq 0, \quad n \geq 1. \tag{1}$$

This concept of coherence between pairs of positive measures on the real line was introduced in 1991 by Iserles, Koch, Nørsett and Sanz-Serna [13] in the framework of approximation theory. The authors of [13] proved that when $\{\nu_0, \nu_1\}$ is a coherent pair of positive measures on the real line the corresponding sequence of MSOP $\{\mathcal{S}_n(\nu_0,\nu_1,s;x)\}_{n\geq 0}$ with respect to the inner product $\langle \cdot\,,\cdot\rangle_{\mathfrak{S}}$ satisfies the connection formulas

$$\begin{aligned} &\mathcal{S}_{n+1}(\nu_0,\nu_1,s;x) - \gamma_n \mathcal{S}_n(\nu_0,\nu_1,s;x) = \mathcal{P}_{n+1}(\nu_0;x) - \sigma_n \mathcal{P}_n(\nu_0;x), \\ &\mathcal{S}'_{n+1}(\nu_0,\nu_1,s;x) - \gamma_n \mathcal{S}'_n(\nu_0,\nu_1,s;x) = (n+1)\mathcal{P}_n(\nu_1;x), \end{aligned} \qquad n \geq 1.$$

These formulas proved to be an important tool in studies regarding the analytic properties of the corresponding Sobolev orthogonal polynomials, such as the location of their zeros and some related asymptotic relations.

A complete classification of pairs of positive measures supported on the real line that satisfy the coherence property (1) was due to Meijer [28]. He proved in 1997 that if $\{\nu_0, \nu_1\}$ is a coherent pair of measures on the real line, then one of the measures must be classical (either Jacobi or Laguerre) and the other one is a rational perturbation of it.

The concept of coherence was further extended to measures on the unit circle in [4], where the authors introduced the coherence property for pairs of signed measures supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. By referring to [4], a pair $\{\rho_0, \rho_1\}$ of positive measures supported on the unit circle is said to be a coherent pair of positive measures on the unit circle if the corresponding sequences of MOP $\{\Phi_n(\rho_0;z)\}_{n\geq 0}$ and $\{\Phi_n(\rho_1;z)\}_{n\geq 0}$ satisfy

$$\Phi_n(\rho_1;z) = \frac{1}{n+1}\left[\Phi'_{n+1}(\rho_0;z) - \sigma_n \Phi'_n(\rho_0;z)\right], \quad \sigma_n \neq 0, \quad n \geq 1.$$

As it was shown in [4], if $\{\rho_0, \rho_1\}$ is a coherent pair of positive measures on the unit circle then the following can be stated:

- If ρ_0 is the Lebesgue measure ($d\rho_0(z) = \frac{dz}{2\pi i z}$), then the measure ρ_1 is such that

$$d\rho_1(z) = \frac{d\rho_0(z)}{|z-\alpha|^2},$$

with $|\alpha| < 1$. This means ρ_1 belongs to the Bernstein-Szegő class.

- If ρ_1 is the Lebesgue measure, then the measure ρ_0 is such that

$$d\rho_0(z) = |z-\alpha|^2 d\rho_1(z).$$

Also, they prove that the only Bernstein-Szegő measure ρ_0, for which $\{\rho_0, \rho_1\}$ is a coherent pair, is the Lebesgue measure. A full classification of all coherent pairs of measures supported on the unit circle is not given and remains as an open problem.

More recently, in Sri Ranga [34] the author presented an example of a family of pairs of measures $\{\rho_0, \rho_1\}$ supported on the unit circle such that there holds the relation

$$\Phi_n(\rho_1;z) - \tau_n \Phi_{n-1}(\rho_1;z) = \frac{1}{n+1}\Phi'_{n+1}(\rho_0;z), \quad \tau_n \neq 0, \qquad n \geq 1. \tag{2}$$

Motivated by this example in [34], pairs of measures on the unit circle satisfying (2) were further investigated in [21]. The authors of [21] referred to such pairs of measures as coherent pairs of measures of the second kind on the unit circle.

In the paper [11], we established a characterization of pairs of positive measures $\{\nu_0, \nu_1\}$ on the real line for which $\{\mathcal{P}_n(\nu_0; x)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1; x)\}_{n\geq 0}$, respectively the corresponding sequences of MOP, satisfy

$$\mathcal{P}_n(\nu_1; x) - \tau_n \mathcal{P}_{n-1}(\nu_1; x) = \frac{1}{n+1}\mathcal{P}'_{n+1}(\nu_0; x), \quad \tau_n \neq 0, \quad n \geq 1. \tag{3}$$

Following [21], we also referred to the pair of measures $\{\nu_0, \nu_1\}$ on the real line satisfying (3) as *coherent pair of positive measures of the second kind on the real line* (CPPM2K on the real line, for short). In fact, we were able to solve this problem by dealing with a more general approach concerning the characterization of pairs of quasi-definite moment functionals.

As introduced in [11], a pair of moment functionals $\{\mathbf{v}_0, \mathbf{v}_1\}$ is said to be a coherent pair of moment functionals of the second kind, if the corresponding sequences of MOP $\{P_n^{(0)}\}_{n\geq 0}$ and $\{P_n^{(1)}\}_{n\geq 0}$ satisfy

$$\frac{1}{n+1} P_{n+1}^{(0)\,\prime}(x) = P_n^{(1)}(x) - \tau_n P_{n-1}^{(1)}(x), \quad \tau_n \neq 0, \quad n \geq 1.$$

The results in [11] provided a complete study of these pairs of moment functionals and the corresponding sequences of orthogonal polynomials. For example, it was shown that $\{\mathbf{v}_0, \mathbf{v}_1\}$ is a coherent pair of moment functionals of the second kind if and only if there exists an admissible pair of polynomials (A_3, A_2) with $\deg(A_3) \leq 3$ and $\deg(A_2) = 2$, such that

$$\mathcal{D}\mathbf{v}_1 = A_2 \mathbf{v}_0 \quad \text{and} \quad \mathbf{v}_1 = A_3 \mathbf{v}_0.$$

Here, $\mathcal{D}\mathbf{v}$ denotes the distributional derivative of $\mathbf{v}$ defined by $\langle \mathcal{D}\mathbf{v}, p \rangle = -\langle \mathbf{v}, p' \rangle$ for $p \in \mathbb{P}$. We recall that the admissibility condition of (A_3, A_2) holds if $\mathfrak{y}_2 + n\mathfrak{t}_3 \neq 0$ for $n \geq 1$, where $\mathfrak{y}_2$ is the leading coefficient of A_2 and $\mathfrak{t}_3$ is coefficient of degree 3 in A_3.

The motivation for such a study on CPPM2Ks on the real line is that this provides a convenient approach to the analysis of the orthogonal polynomials with respect to the Sobolev inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$. In fact, when $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line, the corresponding sequence of MSOP $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$ exists and satisfies the connection formulas

$$\left.\begin{aligned} &\mathcal{S}_{n+1}(\nu_0, \nu_1, s; x) - \gamma_n \mathcal{S}_n(\nu_0, \nu_1, s; x) = \mathcal{P}_{n+1}(\nu_0; x), \\ &\mathcal{S}'_{n+1}(\nu_0, \nu_1, s; x) - \gamma_n \mathcal{S}'_n(\nu_0, \nu_1, s; x) = (n+1)\left[\mathcal{P}_n(\nu_1; x) - \tau_n \mathcal{P}_{n-1}(\nu_1; x)\right], \end{aligned}\right\} \quad n \geq 1, \tag{4}$$

with $\mathcal{S}_1(\nu_0, \nu_1, s; x) = \mathcal{P}_1(\nu_0; x)$. These simple connection formulas allow us to study the analytic properties of the polynomials $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ with great ease.

The main objective in this thesis is to provide a detailed study of the orthogonal polynomials $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$ when $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line. This includes an extensive analysis of the connection coefficients γ_n that appear in (4). It is also shown that the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ are the eigenvalues of a matrix which is a simple modification of the $n \times n$ Jacobi matrix associated with the sequence $\{\mathcal{P}_n(\nu_0; x)\}_{n\geq 0}$.

We also look at a special example of a CPPM2K on the real line which provides much more detailed information regarding its corresponding orthogonal polynomials. Precisely, the example that we consider is such that

$$\text{CPPM2K-J:}\quad \{\nu_0, \nu_1\} = \{\tilde{\mu}^{(\alpha,\beta,q,\epsilon)},\, \mu^{(\alpha+1,\beta+1)}\},$$

where the measures $\mu^{(\alpha,\beta)}$ and $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$, which we consider to be probability measures, are defined as follows

(i) $\mu^{(\alpha,\beta)}$ is the Jacobi probability measure on $[-1,1]$ given by

$$d\mu^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}dx.$$

Its corresponding MOP are the monic Jacobi classical orthogonal polynomials which are defined for $\alpha, \beta > -1$.

(ii) $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ is the probability measure such that

$$d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) = (1-\epsilon)\frac{\tilde{\eta}^{(\alpha,\beta,q)}}{x-q}d\mu^{(\alpha,\beta)}(x) + \epsilon\, \boldsymbol{\delta}_q,$$

with $|q| \geq 1$ and $0 \leq \epsilon < 1$, where $\boldsymbol{\delta}_q$ denotes the Dirac delta function located at q. As a consequence,

$$\int_{[-1,1]\cup\{q\}} f(x)d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) = (1-\epsilon)\int_{-1}^{1} f(x)\frac{\tilde{\eta}^{(\alpha,\beta,q)}}{x-q}d\mu^{(\alpha,\beta)}(x) + \epsilon f(q).$$

Notice that the measure $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ is obtained through a rational modification of $\mu^{(\alpha,\beta)}$ jointly with the addition of a Dirac mass point at $x = q$. Furthermore, $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ is known in the literature as a *Geronimus* perturbation of $\mu^{(\alpha,\beta)}$. As stated in [11], the pair CPPM2K-J constitutes an example of a CPPM2K on the real line in which one of the measures is classical and the other one is a rational perturbation of it.

Another set of important results that we have presented in this thesis emerged during our attempts to study the asymptotic behavior of the connection coefficients γ_n that follow from the pair CPPM2K-J. These results found in Chapter 2 provide a novel way to represent the parameter sequences of a traditional positive chain sequence in the theory of orthogonal polynomials. Such results, which have been published in [19], also play a key role in the remaining part of the thesis since they provided another means to examine the orthogonal polynomials following from the pair CPPM2K-J.

This work is organized as follows. Chapter 1 summarizes the basic concepts about special functions, positive chain sequences, moment functionals and orthogonal polynomials to be used in the sequel.

In Chapter 2 we provide a representation of all the parameter sequences of a positive chain sequence that has been of importance in the theory of orthogonal polynomials on the real line. We also explore some of the consequences and applications that follow from this representation.

Chapter 3 is devoted to the study of the Sobolev orthogonal polynomials $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$ when $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line. We also provide a detailed study of the connection coefficients γ_n appearing in (4). Finally, in Chapter 4 we deal with the Sobolev orthogonal polynomials that follow from the pair CPPM2K-J. The main results in these chapters were published in [18].

As mentioned previously, some of the main results contained in this thesis are also presented in the following texts, which we have also listed within the bibliography at the end of this thesis:

[18] G. A. Marcato, F. Marcellán, A. Sri Ranga, Yen Chi Lun, Coherent pairs of measures of the second kind on the real line and Sobolev orthogonal polynomials. An application to a Jacobi case. Submitted.

[19] G. A. Marcato, A. Sri Ranga, Yen Chi Lun, Parameters of a positive chain sequence associated with orthogonal polynomials, *Proc. Amer. Math. Soc.*, **150** (2022), 2553-2567.

We have also contributed to the following paper which contains some results used in the development of this thesis.

[11] M. Hancco Suni, G. A. Marcato, F. Marcellán, A. Sri Ranga, Coherent pairs of moment functionals of the second kind and associated orthogonal polynomials and Sobolev orthogonal polynomials, *J. Math. Anal. Appl.*, **525** (2023), 127118, DOI: https://doi.org/10.1016/j.jmaa.2023.127118.

1 Preliminaries

In this chapter we give an overview of the basic knowledge required to the development of this work. We only point out results which are important for understanding the material contained in this thesis. This requires some basic background in the theory of orthogonal polynomial sequences.

We start by giving a brief introduction to special functions and positive chain sequences. After that, we present some of the fundamental concepts and results on the theory of orthogonal polynomials on the real line.

1.1 Special Functions

In this section we give a short overview on special functions and hypergeometric series. We refer to Andrews, Askey and Roy [1].

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we define the *Pochhammer symbol* $(a)_n$ by

$$(a)_0 = 1, \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1.$$

It is straightforward to see that

(a) $(1)_n = n!$ for $n \in \mathbb{N}$;

(b) Let $n, m \in \mathbb{N}$. Then $(-m)_n = 0$ if $n > m$. A useful relation is

$$\frac{(a)_n}{(a)_m} = \begin{cases} (a+m)_{n-m} & \text{if} \quad n \geq m, \\ \dfrac{1}{(a+n)_{m-n}} & \text{if} \quad m \geq n. \end{cases}$$

The *Gamma function* $\Gamma(z)$ is defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^{z-1}}{(z)_n},$$

for all complex numbers $z \neq 0, \pm1, \pm2, \ldots$. Follows directly from this definition that

$$\Gamma(z+1) = z\Gamma(z).$$

Also, one can prove that $\Gamma(n+1) = n!$ and $\Gamma(z+n) = (z)_n\Gamma(z)$ for $n \in \mathbb{N}$.

The generalized hypergeometric series is a series

$$\sum_{n=0}^{\infty} c_n$$

for which $c_0 = 1$ and the ratio of successive terms can be written as

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)}{(n+b_1)(n+b_2)\cdots(n+b_q)(n+1)} z.$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} c_n = {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z \right) &= {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},\end{aligned}$$

with $a_i \in \mathbb{C}$, $i = 1, 2, \ldots, p$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2\}$, $i = 1, 2, \ldots, q$. By the ratio test ${}_pF_q$ converges absolutely for all z if $p \le q$ and for $|z| < 1$ if $p = q + 1$.

The *Gauss hypergeometric function* or simply the *hypergeometric function*, denoted by ${}_2F_1(a, b;\, c; z)$, is defined as

$${}_2F_1\left(a, b;\ c;\ z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where a, b and c are complex parameters.

1.2 Positive Chain Sequences

In this section we provide a brief introduction to positive chain sequences. We only present results which are important for understanding the material contained in this thesis. For a detailed discussion on the theory of positive chain sequences we cite the book [6] of Chihara. For the theory of general chain sequences (in a slightly more general setting) we cite Wall [37].

Definition 1.1. We say that a sequence of real numbers $\{\alpha_n\}_{n\ge1}$ is a *positive chain sequence* if there exists a second sequence $\{g_n\}_{n\ge0}$ such that

(i) $0 \le g_0 < 1, \quad 0 < g_n < 1, \quad n \ge 1;$

(ii) $\alpha_n = (1 - g_{n-1}) g_n, \quad n \ge 1.$

The sequence $\{g_n\}_{n\ge0}$ is called a *parameter sequence of* $\{\alpha_n\}_{n\ge1}$.

We remark that a parameter sequence needs not be unique.

Theorem 1.2. *Let $\{\alpha_n\}_{n\ge1}$ be a positive chain sequence and let both $\{g_n\}_{n\ge0}$ and $\{h_n\}_{n\ge0}$ be parameter sequences of $\{\alpha_n\}_{n\ge1}$. Then $g_n < h_n$ for $n \ge 1$ if and only if $g_0 < h_0$.*

Definition 1.3. Let $\{\alpha_n\}_{n\ge1}$ denote a positive chain sequence. A parameter sequence $\{m_n\}_{n\ge0}$ of $\{\alpha_n\}_{n\ge1}$ is called its *minimal parameter sequence* if $m_0 = 0$.

Observe that every positive chain sequence $\{\alpha_n\}_{n\ge1}$ has a minimal parameter sequence $\{m_n\}_{n\ge0}$ which can be obtained by setting $m_0 = 0$ and

$$m_n = \frac{\alpha_n}{1 - m_{n-1}}, \quad n \ge 1.$$

If the minimal parameter sequence $\{m_n\}_{n\geq 0}$ is the only parameter sequence of $\{\alpha_n\}_{n\geq 1}$, we say that $\{\alpha_n\}_{n\geq 1}$ is uniquely determined. By Theorem 1.2, if $\{g_n\}_{n\geq 0}$ is any other parameter sequence of $\{\alpha_n\}_{n\geq 1}$ then

$$m_n < g_n, \quad n = 0, 1, 2, \ldots .$$

When the positive chain sequence $\{\alpha_n\}_{n\geq 1}$ is not uniquely determined we can also talk about its maximal parameter sequence $\{M_n\}_{n\geq 0}$.

Definition 1.4. Let $\{\alpha_n\}_{n\geq 1}$ denote a positive chain sequence. A parameter sequence $\{M_n\}_{n\geq 0}$ of $\{\alpha_n\}_{n\geq 1}$ is called its *maximal parameter sequence* if $\{g_n\}_{n\geq 0}$ is any other parameter sequence of $\{\alpha_n\}_{n\geq 1}$ then $g_n < M_n$ for $n \geq 0$.

Now we summarize some basic results on the theory of positive chain sequences which are necessary for the development of the subsequent chapters.

Theorem 1.5. *Let $\{\alpha_n\}_{n\geq 1}$ be a positive chain sequence and let $\{m_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ be, respectively, its minimal and maximal parameter sequences. Let $\{\beta_n\}_{n\geq 1}$ be a positive chain sequence with a parameter sequence $\{h_n\}_{n\geq 0}$. If $\alpha_n \leq \beta_n$ for $n \geq 1$, then*

$$m_n \leq h_n \leq M_n, \quad n \geq 0.$$

Moreover, if we have in addition $\alpha_{N_0} < \beta_{N_0}$ for some $N_0 \geq 1$, then

$$m_n < h_n \quad \textit{for} \quad n \geq N_0 \qquad \textit{and} \qquad h_j < M_j \quad \textit{for} \quad j = 0, 1, \ldots, N_0 - 1.$$

In particular, if $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ are positive chain sequences such that $\alpha_n < \beta_n$ for $n \geq 1$, then $\{\alpha_n\}_{n\geq 1}$ has multiple parameter sequences.

Theorem 1.6 (Comparison Test)**.** *Let $\{\alpha_n\}_{n\geq 1}$ denote a positive chain sequence and let $\{c_n\}_{n\geq 1}$ be a sequence. If $0 < c_n \leq \alpha_n$ for $n \geq 1$, then $\{c_n\}_{n\geq 1}$ is also a positive chain sequence.*

Theorem 1.7. *Let $\{\alpha_n\}_{n\geq 1}$ be a positive chain sequence and let $\{m_n\}_{n\geq 0}$ be its minimal parameter sequence. If $\{g_n\}_{n\geq 0}$ is any non-maximal parameter sequence of $\{\alpha_n\}_{n\geq 1}$ then*

$$\lim_{n\to\infty} \frac{m_n}{g_n} = 1.$$

Theorem 1.8. *Let $\{\alpha_n\}_{n\geq 1}$ be a positive chain sequence such that*

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

Then $0 \leq \alpha \leq 1/4$. Moreover, if $\{\alpha_n\}_{n\geq 1}$ is uniquely determined, then

$$\lim_{n\to\infty} m_n = \frac{1}{2}[1 + \sqrt{1 - 4\alpha}].$$

Otherwise, if $\{\alpha_n\}_{n\geq 1}$ has multiple parameters sequences, then

$$\lim_{n\to\infty} m_n = \frac{1}{2}[1 - \sqrt{1 - 4\alpha}] \qquad \textit{and} \qquad \lim_{n\to\infty} M_n = \frac{1}{2}[1 + \sqrt{1 - 4\alpha}].$$

1.3 Orthogonal Polynomials on the Real Line

In this section we give some of the basic results on the theory of orthogonal polynomials on the real line to be used throughout this thesis. These results can be found in Chihara [6], Ismail [14] and Szegő [35]. We utilize here an algebraic approach to the study of linear functionals defined on the space of polynomials which was introduced by P. Maroni in [22] and played a central role in a variety of studies concerning orthogonal polynomials since it provides a general perspective to the study of the topic (see also [24, 26, 27]).

Let $\mathbf{v}$ be a linear functional defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $\mathbb{P}^*$ denote its algebraic dual space, i.e., the linear space of all linear functionals defined on $\mathbb{P}$. If $\mathbf{v} \in \mathbb{P}^*$ and $p \in \mathbb{P}$, then we denote by $\langle \mathbf{v}, p \rangle$ the action of the linear functional $\mathbf{v}$ on the polynomial $p \in \mathbb{P}$.

Definition 1.9. Given $\mathbf{v} \in \mathbb{P}^*$, we denote by

$$(\mathbf{v})_n = \langle \mathbf{v}, x^n \rangle, \quad n = 0, 1, 2, \ldots,$$

the *moment* of order n of $\mathbf{v}$.

Since the moments play a key role in the study of these linear functionals, it is also customary to call such linear functionals as *moment functionals*. Clearly, if $\mathbf{u}, \mathbf{v} \in \mathbb{P}^*$ are such that $(\mathbf{u})_n = (\mathbf{v})_n$ for all $n \geq 0$, then $\mathbf{u} = \mathbf{v}$.

Now we present the definitions of some basic operations in the space $\mathbb{P}^*$.

Definition 1.10. Let $\mathbf{v} \in \mathbb{P}^*$, $\pi \in \mathbb{P}$ and $q \in \mathbb{C}$.

(i) The *left multiplication of* $\mathbf{v}$ *by* π, denoted by $\pi\mathbf{v}$, is the linear functional on $\mathbb{P}^*$ defined by

$$\langle \pi\mathbf{v}, p \rangle = \langle \mathbf{v}, \pi p \rangle, \quad p \in \mathbb{P}.$$

(ii) The *distributional derivative of* $\mathbf{v}$, denoted by $\mathcal{D}\mathbf{v}$, is the linear functional on $\mathbb{P}^*$ defined by

$$\langle \mathcal{D}\mathbf{v}, p \rangle = -\langle \mathbf{v}, p' \rangle, \quad p \in \mathbb{P},$$

and satisfies

$$\mathcal{D}(\pi\mathbf{v}) = \pi'\mathbf{v} + \pi\mathcal{D}\mathbf{v}.$$

(iii) We define the *division of* $\mathbf{v}$ *by* $(x - q)$ as

$$\left\langle \frac{1}{x-q}\mathbf{v}, p \right\rangle = \left\langle \mathbf{v}, \frac{p(x) - p(q)}{x - q} \right\rangle, \quad p \in \mathbb{P}.$$

(iv) The linear functional $\boldsymbol{\delta}_q$ given by

$$\langle \boldsymbol{\delta}_q, p \rangle = p(q), \quad p \in \mathbb{P},$$

is said to be the *Dirac delta* linear functional supported at q.

It is straightforward to verify that

$$(x - q)\left[\frac{1}{x-q}\mathbf{v}\right] = \mathbf{v} \qquad \text{and} \qquad \frac{1}{x-q}\left[(x-q)\,\mathbf{v}\right] = \mathbf{v} - (\mathbf{v})_0\,\boldsymbol{\delta}_q.$$

Throughout this thesis $\delta_{m,n}$ will denote the Kronecker's delta symbol

$$\delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Let us introduce the basic notion of orthogonality which will be used along this thesis.

Definition 1.11. A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ on $\mathbb{P}$ is said to be the sequence of *monic orthogonal polynomials* (MOP for short) with respect to the moment functional $\mathbf{v}$ if

(i) $deg(P_n) = n, \quad n = 0, 1, 2, \ldots;$

(ii) $\langle \mathbf{v}, P_m P_n \rangle = \hbar_n \delta_{m,n}$, with $\hbar_n = \langle \mathbf{v}, P_n^2 \rangle \neq 0$, $m, n = 0, 1, 2, \ldots$.

The next theorem is a direct consequence of Definition 1.11.

Theorem 1.12. *Let $\mathbf{v}$ be a moment functional and let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials. Then the following statements are equivalent*

(a) $\{P_n\}_{n\geq 0}$ *is the sequence of MOP with respect to* $\mathbf{v}$;

(b) $\langle \mathbf{v}, \pi P_n \rangle = 0$ *for all* $\pi \in \mathbb{P}$ *such that* $\deg(\pi) < n$*, while* $\langle \mathbf{v}, \pi P_n \rangle \neq 0$ *if* $\deg(\pi) = n$;

(c) $\langle \mathbf{v}, x^m P_n \rangle = \hbar_n \delta_{m,n}$*, where* $\hbar_n = \langle \mathbf{v}, P_n^2 \rangle \neq 0$ *for* $m = 0, 1, \ldots, n$.

In order to study the existence of a sequence of MOP with respect to a moment functional $\mathbf{v}$ let us introduce the *Hankel* matrices

$$\mathbf{H}_n = \left[(\mathbf{v})_{i+j} \right]_{i,j=0}^{n}, \quad n = 0, 1, 2, \ldots.$$

Theorem 1.13. *Let $\mathbf{v}$ be a moment functional and let $\{\mathbf{H}_n\}_{n\geq 0}$ be its corresponding sequence of Hankel matrices. Then there exists a sequence of MOP with respect to $\mathbf{v}$ if and only if*

$$\det(\mathbf{H}_n) \neq 0, \quad n \geq 0.$$

In this case $\mathbf{v}$ is said to be quasi-definite.

Some important properties in relation to the concept of orthogonality emerge in the so-called positive definite case.

Definition 1.14. A moment functional $\mathbf{v}$ is said to be *positive definite* if its moments are real and

$$\det(\mathbf{H}_n) > 0, \quad n \geq 0.$$

When $\mathbf{v}$ is positive definite we also have $\langle \mathbf{v}, \pi \rangle > 0$ for every nonzero and non-negative real polynomial π.

Definition 1.15. Let $\mathbf{v}$ be a positive definite moment functional and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOP. The sequence of *orthonormal polynomials* with respect to $\mathbf{v}$, which we denote by $\{p_n\}_{n\geq 0}$, is given by the normalization

$$p_n(x) = \frac{P_n(x)}{\sqrt{\hbar_n}}, \quad n \geq 0,$$

where we assume $p_n(x)$ to be a polynomial with positive leading coefficient. Consequently, the following orthogonality relation holds

$$\langle \mathbf{v}, p_m p_n \rangle = \delta_{m,n}, \quad m, n = 0, 1, 2, \ldots.$$

As we shall see in the sequel, every positive definite moment functional can be represented as a Stieltjes integral with respect to a positive measure supported on an infinite subset of the real line.

Definition 1.16. Consider $\nu \geq 0$ a non-decreasing bounded real function defined on $[a, b]$ with $-\infty \leq a < b \leq +\infty$. The set

$$\operatorname{supp}(\nu) = \{\xi \in [a, b] : \nu(\xi + \varepsilon) - \nu(\xi - \varepsilon) > 0, \quad \text{for all} \quad \varepsilon > 0\}$$

is called the *support* of ν.

Definition 1.17. Let $\nu \geq 0$ be a bounded non-decreasing function defined on $[a, b]$ with infinite support. Then ν is called a *positive measure* on $[a, b]$ if the moments defined by the Stieltjes integral

$$\nu_n = \int_a^b x^n d\nu(x), \quad n = 0, 1, 2, \ldots,$$

all exist. Moreover, ν is said to be a *probability measure* if $\nu_0 = 1$.

An absolutely continuous measure ν has the form

$$d\nu(x) = \omega(x)dx,$$

where $\omega \geq 0$ is a nonzero bounded real function on (a, b) known as *weight function.*

It is well known that if $\mathbf{v}$ is a positive definite moment functional, there exists a positive measure ν with infinite support $E \subseteq \mathbb{R}$ such that $\mathbf{v}$ has an integral representation

$$\langle \mathbf{v}, p \rangle = \int_E p(x)\, d\nu(x), \quad p \in \mathbb{P}.$$

Remark 1.18. Let $\mathbf{v}$ be a positive definite moment functional represented by the positive measure ν and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOP. We can also refer to $\{P_n\}_{n\geq 0}$ as the sequence of MOP with respect to the positive measure ν. It is important to emphasize that in the following chapters we will limit our scope to the study of sequences of MOP with respect to positive measures supported on the real line.

One of the most important characteristics of orthogonal polynomials on the real line is the so-called *three-term recurrence relation* (TTRR, for short) in which three consecutive polynomials are connected by a simple relation.

Theorem 1.19. *Let $\mathbf{v}$ be a quasi-definite moment functional and let $\{P_n\}_{n\geq 0}$ be its corresponding sequence of MOP. Then the polynomials P_n satisfy the three-term recurrence relation*

$$P_{n+1}(x) = (x - \mathfrak{c}_{n+1})P_n(x) - \mathfrak{d}_{n+1}P_{n-1}(x), \quad n \geq 1, \tag{1.1}$$

with $P_0(x) = 1$ and $P_1(x) = x - \mathfrak{c}_1$, where the coefficients $\mathfrak{c}_n$ and $\mathfrak{d}_{n+1}$ are given by

$$\mathfrak{c}_n = \frac{1}{\hbar_{n-1}} \left\langle \mathbf{v},\, x P_{n-1}^2 \right\rangle \quad \textit{and} \quad \mathfrak{d}_{n+1} = \frac{\hbar_n}{\hbar_{n-1}} \neq 0, \quad n \geq 1.$$

The TTRR (1.1) can be written in matrix form,

$$x\, \mathbf{p}_n(x) = \mathbf{J}_n\, \mathbf{p}_n(x) + P_n(x)\, \mathbf{e}_n, \quad n \geq 2, \tag{1.2}$$

where $\mathbf{e}_n$ is the n-th column of the $n \times n$ identity matrix,

$$\mathbf{p}_n(x) = \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix}, \qquad \mathbf{J}_n = \begin{bmatrix} \mathfrak{c}_1 & 1 & 0 & \cdots & 0 & 0 \\ \mathfrak{d}_2 & \mathfrak{c}_2 & 1 & \cdots & 0 & 0 \\ 0 & \mathfrak{d}_3 & \mathfrak{c}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathfrak{c}_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & \mathfrak{d}_n & \mathfrak{c}_n \end{bmatrix}.$$

We refer to $\mathbf{J}_n$ as the $n \times n$ *monic Jacobi matrix* associated with the sequence $\{P_n\}_{n\geq 0}$. It is straightforward to see that the zeros of P_n are the eigenvalues of $\mathbf{J}_n$.

The converse of the previous theorem is valid and it is known in the literature as Favard's Theorem.

Theorem 1.20 (Favard)**.** *Let $\{\mathfrak{c}_n\}_{n\geq 1}$ and $\{\mathfrak{d}_n\}_{n\geq 2}$ be two arbitrary sequences of complex numbers, and let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials defined by the TTRR* (1.1)*. Then there exists a unique moment functional $\mathbf{v}$ such that*

$$\langle \mathbf{v}, 1\rangle \neq 0 \quad \text{and} \quad \langle \mathbf{v}, P_m P_n\rangle = 0 \quad \text{if} \quad n \neq m.$$

Moreover, $\mathbf{v}$ is quasi-definite and $\{P_n\}_{n\geq 0}$ is its corresponding sequence of MOP if and only if $\mathfrak{d}_{n+1} \neq 0$ for $n \geq 1$, while $\mathbf{v}$ is positive definite with $\{P_n\}_{n\geq 0}$ as its corresponding sequence of MOP if and only if $\mathfrak{c}_n \in \mathbb{R}$ and $\mathfrak{d}_{n+1} > 0$ for each $n \geq 1$.

Another important fact is that sequences of MOP satisfy the so-called Christoffel-Darboux identity.

Theorem 1.21 (Christoffel-Darboux Formula)**.** *Let $\{P_n\}_{n\geq 0}$ be a sequence of MOP satisfying the TTRR* (1.1) *with $\mathfrak{d}_{n+1} \neq 0$ for $n \geq 1$. Then*

$$\sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{\mathcal{h}_j} = \frac{1}{\mathcal{h}_n} \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x-y}, \quad n \geq 0.$$

Moreover, if $\mathbf{v}$ is positive definite then

$$\sum_{j=0}^{n} \frac{P_j^2(x)}{\mathcal{h}_j} = \frac{1}{\mathcal{h}_n}[P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)] > 0, \quad n \geq 0.$$

With $\{\mathfrak{c}_n\}_{n\geq 1}$ and $\{\mathfrak{d}_n\}_{n\geq 2}$ given as in (1.1), we can consider $\{P_n^*\}_{n\geq 0}$ as the sequence of monic *associated polynomials* of $\{P_n\}_{n\geq 0}$ given by the TTRR

$$P_{n+1}^*(x) = (x - \mathfrak{c}_{n+1})P_n^*(x) - \mathfrak{d}_{n+1}P_{n-1}^*(x), \quad n \geq 1,$$

where $P_0^*(x) = 0$ and $P_1^*(x) = 1$. Notice that $\deg(P_n^*) = n-1$. Another important representation of $\{P_n^*\}_{n\geq 0}$ is

$$P_n^*(x) = \frac{1}{(\mathbf{v})_0}\left\langle \mathbf{v}, \frac{P_n(y) - P_n(x)}{y-x}\right\rangle, \quad n \geq 0.$$

Definition 1.22. Let $\mathbf{v}$ be a positive definite moment functional. The *support* of $\mathbf{v}$ is the largest interval $(a,b) \subseteq \mathbb{R}$ where $\langle \mathbf{v}, \pi\rangle > 0$ for every real polynomial π which is non-negative on (a,b) and does not vanish identically on (a,b).

The next result gives information about the zeros of MOP with respect to positive definite moment functionals.

Theorem 1.23. *Let $(a, b) \subseteq \mathbb{R}$ be the support of the positive definite moment functional* $\mathbf{v}$ *and let $\{P_n\}_{n\geq 0}$ be the sequence of MOP with respect to* $\mathbf{v}$. *Then*

(a) *All zeros of P_n are real, simple and lie inside (a, b);*

(b) *The polynomials P_n and P_{n+1} have no common zeros;*

(c) *Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ denote the zeros of P_n. Then,*

$$x_{n+1,j} < x_{n,j} < x_{n+1,j+1}, \quad j = 1, 2, \ldots, n.$$

This property is called the interlacing property.

We now turn our attention to the so-called semiclassical moment functionals. Firstly, let us define the admissibility condition of a pair of polynomials.

Definition 1.24. Let ϕ and ψ be two nonzero polynomials such that $\deg(\phi) = M \geq 0$ and $\deg(\psi) = N \geq 1$ with leading coefficients λ_M^ϕ and λ_N^ψ, respectively. Then, (ϕ, ψ) is said to be an *admissible pair* of polynomials if either $N \neq M - 1$ or if $N = M - 1$, then $n\lambda_{N+1}^\phi + \lambda_N^\psi \neq 0$, $n \geq 0$.

Definition 1.25. A quasi-definite moment functional $\mathbf{v} \in \mathbb{P}^*$ is said to be *semiclassical* if it satisfies

$$\mathcal{D}(\phi\mathbf{v}) = \psi\mathbf{v}, \tag{1.3}$$

where (ϕ, ψ) is an admissible pair of polynomials.

We remark that the pair of polynomials (ϕ, ψ) satisfying (1.3) is not unique. Also, the equation (1.3) is known in the literature as *Pearson* equation (see [23], [26], [32]).

To a semiclassical moment functional $\mathbf{v}$ one can associate the class of $\mathbf{v}$ as the non-negative integer number given by

$$\min\Big\{\max\{deg(\phi) - 2, deg(\psi) - 1\} : \mathcal{D}(\phi\mathbf{v}) = \psi\mathbf{v} \text{ and } (\phi, \psi) \text{ is admissible}\Big\}.$$

When $\mathbf{v}$ is semiclassical of class zero, the so-called *Classical Orthogonal Polynomials* appear (up to a change of variables). These are the families of Hermite, Laguerre, Jacobi and Bessel polynomials. In other words, the classical MOP are the only monic orthogonal polynomials such that their corresponding moment functional $\mathbf{v}$ satisfies the Pearson equation (1.3) with $deg(\phi) \leq 2$ and $deg(\psi) = 1$. In this case, we say that $\mathbf{v}$ is *classical*. In addition, a classical moment functional $\mathbf{v}$ has an integral representation

$$\langle \mathbf{v}, p \rangle = \int_E p(x)\,\omega(x)dx, \quad p \in \mathbb{P},$$

where $\omega(x)$ is a weight function and/or a measure with support set E.

Remark 1.26. Differently from the other three families of Classical MOP, the Bessel polynomials are orthogonal with respect to a quasi-definite linear functional.

With the notation introduced in this section, the following table outlines the principal parameters for the classical families of MOP.

P_n	Hermite	Laguerre	Jacobi	Bessel
ϕ	1	x	$1-x^2$	x^2
ψ	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$	$(\alpha+2)x+2$
ω	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1-x)^\beta$	$x^\alpha e^{-2/x}$
E	$\mathbb{R}$	$(0,+\infty)$	$[-1,1]$	$\mathbb{T}=\{z\in\mathbb{C}:\lvert z\rvert=1\}$
$\mathfrak{c}_n$	0	$\alpha+2n-1$	$\frac{\beta^2-\alpha^2}{(\alpha+\beta+2n-2)(\alpha+\beta+2n)}$	$-\frac{2\alpha}{(\alpha+2n-2)(\alpha+2n)}$
$\mathfrak{d}_{n+1}$	$\frac{n}{2}$	$n(\alpha+n)$	$\frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+2n+1)}$	$-\frac{4n(\alpha+n)}{(\alpha+2n-1)(\alpha+2n)^2(\alpha+\beta+2n+1)}$
		$\alpha>-1$	$\alpha,\beta>-1$	$\alpha\notin\{0,-1,-2,\ldots\}$

Table 1.1: The Classical families of MOP

Finally, we present some well known theorems that we will need in the subsequent chapters when analyzing the zeros of certain Sobolev orthogonal polynomials.

Theorem 1.27 ([33, p. 365]). *Let $\{P_n\}_{n\geq 0}$ denote the sequence of MOP with respect to the weight function $\omega(x)$ on (a,b). Any polynomial $f(z)$ of exact degree n can be written in terms of $\{P_j(z)\}_{j=0}^n$ as*

$$f(z)=c_0P_0(z)+c_1P_1(z)+\cdots+c_nP_n(z),\quad c_n\neq 0. \tag{1.4}$$

Then all the zeros of $f(z)$ lie in the strip

$$|\operatorname{Im}(z)|<1+\max_{0\leq j\leq n-1}\left\{\left|\frac{c_j}{c_n}\right|\right\}.$$

Theorem 1.28 ([10, p. 62]). *Let $\zeta_1,\zeta_2,\ldots,\zeta_n$ denote the zeros of the polynomial $f(z)$ given in (1.4) and let $\mathscr{h}_n=\int_a^b P_n^2(x)\omega(x)dx$. Then*

$$\sum_{j=1}^{n}|\operatorname{Im}(\zeta_j)|\leq\frac{1}{\sqrt{\mathscr{h}_{n-1}}}\Big(\sum_{j=0}^{n-1}\mathscr{h}_j\left|\frac{c_j}{c_n}\right|^2\Big)^{1/2}$$

with the equality if and only if $c_0=c_1=\cdots=c_{n-2}=0$ and $\operatorname{Re}(c_{n-1}/c_n)=0$.

1.4 Jacobi Polynomials

In this section we recall some of the basic properties of the Jacobi classical orthogonal polynomials. The results given here, which are required for our study, can be easily found among the basic literature (see, for example, Andrews, Askey and Roy [1], Chihara [6], Ismail [14] and Szegő [35]) associated with Jacobi polynomials. The notation introduced here will be used along this thesis

For $\alpha, \beta > -1$, let us denote by $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ the sequence of monic Jacobi orthogonal polynomials given by

$$\mathcal{P}_n^{(\alpha,\beta)}(x) = \frac{2^n n!\,(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}\, P_n^{(\alpha,\beta)}(x), \quad n \geq 0,$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n,\ \alpha+\beta+n+1 \\ \alpha+1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right)$$

is the n-th degree Jacobi polynomial as usually defined. These polynomials can also be defined via the Rodrigues' formula as follows

$$\begin{aligned}\mathcal{P}_n^{(\alpha,\beta)}(x) = (-1)^n \frac{(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}}&(1-x)^{-\alpha}(1+x)^{-\beta} \\ \times \frac{d^n}{dx^n}&\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right].\end{aligned}$$

It is known that, with the probability measure $\mu^{(\alpha,\beta)}$ on $[-1,1]$ given by

$$d\mu^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1, \tag{1.5}$$

there holds the orthogonality relation

$$\int_{-1}^{1} \mathcal{P}_m^{(\alpha,\beta)}(x)\,\mathcal{P}_n^{(\alpha,\beta)}(x)\,d\mu^{(\alpha,\beta)}(x) = \hbar_n^{(\alpha,\beta)}\delta_{m,n}, \quad \text{for} \quad m,n = 0,1,2,\ldots, \tag{1.6}$$

with

$$\hbar_n^{(\alpha,\beta)} = \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}(\alpha+\beta+2)_{2n}}\, 2^{2n} n!, \quad n \geq 0.$$

The monic Jacobi polynomials $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ satisfy the TTRR

$$\mathcal{P}_{n+1}^{(\alpha,\beta)}(x) = (x - c_{n+1}^{(\alpha,\beta)})\mathcal{P}_n^{(\alpha,\beta)}(x)(x) - d_{n+1}^{(\alpha,\beta)}\mathcal{P}_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1,$$

with $\mathcal{P}_0^{(\alpha,\beta)}(x) = 1$ and $\mathcal{P}_1^{(\alpha,\beta)}(x) = x - c_1^{(\alpha,\beta)}$, where the coefficients $c_n^{(\alpha,\beta)}$ and $d_{n+1}^{(\alpha,\beta)}$ are given by

$$\left.\begin{aligned} c_n^{(\alpha,\beta)} &= \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n-2)(\alpha+\beta+2n)}, \\ d_{n+1}^{(\alpha,\beta)} &= \frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}, \end{aligned}\right\} \quad n \geq 1. \tag{1.7}$$

The n zeros of $\mathcal{P}_n^{(\alpha,\beta)}$ are simple and lie inside $(-1,1)$. Moreover, for $x=1$ we have

$$\mathcal{P}_0^{(\alpha,\beta)}(1) = 1 \quad \text{and} \quad \mathcal{P}_n^{(\alpha,\beta)}(1) = \frac{2^n(\alpha+1)_n(\alpha+\beta+2)_{n-1}}{(\alpha+\beta+2)_{2n-1}}, \quad n \geq 1. \tag{1.8}$$

Finally, the derivatives of $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ satisfy

$$\mathcal{P}_n^{(\alpha,\beta)\,\prime}(x) = n\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1. \tag{1.9}$$

2 Companion Orthogonal Polynomials and Positive Chain Sequences

The purpose of this chapter is to provide a representation of all the parameter sequences of a positive chain sequence that has been very important in the theory of orthogonal polynomials on the real line. We also explore some of the consequences and applications that follow from this representation. An example of a class of positive chain sequences with explicit formulas for all the parameter sequences is given. As mentioned, the results presented here were published in [19].

2.1 Introduction

Here we let ν denote a positive measure supported on the real line. Throughout this chapter we will assume ν is such that if $[a, b]$ is the smallest closed interval containing the support of ν then $\mathbb{R} \setminus [a, b]$ is non-empty. We will also assume ν is such that its moments given by

$$\nu_n = \int_a^b x^n d\nu(x), \quad n = 0, 1, 2, \ldots,$$

all exist. Let $\{\mathcal{Q}_n\}_{n\geq 0}$ denote the sequence of MOP with respect to ν and consider again the sequence of positive constants $\{\hbar_n\}_{n\geq 0}$ such that

$$\int_a^b \mathcal{Q}_m(x)\mathcal{Q}_n(x)d\nu(x) = \hbar_n \delta_{m,n}, \quad m, n = 0, 1, 2, \ldots.$$

Let us consider the TTRR satisfied by $\{\mathcal{Q}_n\}_{n\geq 0}$. Then we write

$$\mathcal{Q}_{n+1}(x) = (x - \mathfrak{c}_{n+1})\mathcal{Q}_n(x) - \mathfrak{d}_{n+1}\mathcal{Q}_{n-1}(x), \quad n \geq 1, \tag{2.1}$$

with $\mathcal{Q}_0(x) = 1$ and $\mathcal{Q}_1(x) = x - \mathfrak{c}_1$, where the coefficients $\mathfrak{c}_n$ and $\mathfrak{d}_n$ are given by

$$\mathfrak{c}_n = \frac{1}{\hbar_{n-1}} \int_a^b x\mathcal{Q}_{n-1}^2(x)d\nu(x) \quad \text{and} \quad \mathfrak{d}_{n+1} = \frac{\hbar_n}{\hbar_{n-1}} > 0 \quad \text{for} \quad n \geq 1.$$

It is straightforward to see that $\mathfrak{c}_n \in (a, b)$ for $n \geq 1$.

Now we consider the most traditional positive chain sequence that follows from the TTRR (2.1). Here, for $q \in \mathbb{R} \setminus (a, b)$, we define the positive chain sequence $\{\alpha_n^{(q)}\}_{n\geq 1}$ by

$$\alpha_n^{(q)} = \frac{\mathfrak{d}_{n+1}}{(q - \mathfrak{c}_n)(q - \mathfrak{c}_{n+1})}, \quad n \geq 1, \tag{2.2}$$

where $\{\mathfrak{c}_n\}_{n\geq 1}$ and $\{\mathfrak{d}_{n+1}\}_{n\geq 1}$ are as in (2.1). This positive chain sequence was first introduced in Wall and Wetzel [38] in connection with the continued fraction (J-fraction)

$$\frac{1}{\left| z-\mathfrak{c}_1\right.} - \frac{\mathfrak{d}_2}{\left| z-\mathfrak{c}_2\right.} - \cdots - \frac{\mathfrak{d}_n}{\left| z-\mathfrak{c}_n\right.} - \cdots$$

and it was further investigated in Chihara [5]. In particular, the following result concerning its minimal parameter sequence has been shown (see [5] and [6]).

Theorem 2.1 ([6, p. 110]). *Let $\{\alpha_n^{(q)}\}_{n\geq 1}$ be the positive chain sequence given by (2.2). Then its minimal parameter sequence $\{m_n^{(q)}\}_{n\geq 0}$ can be explicitly represented by*

$$m_0^{(q)} = 0 \quad \textit{and} \quad m_n^{(q)} = 1 - \frac{\mathcal{Q}_{n+1}(q)}{(q-\mathfrak{c}_{n+1})\mathcal{Q}_n(q)} = \frac{\mathfrak{d}_{n+1}\mathcal{Q}_{n-1}(q)}{(q-\mathfrak{c}_{n+1})\mathcal{Q}_n(q)}, \quad n \geq 1. \tag{2.3}$$

Proof. From (2.1) we can deduce

$$\frac{\mathfrak{d}_{n+1}}{(q-\mathfrak{c}_n)(q-\mathfrak{c}_{n+1})} = \frac{\mathcal{Q}_n(q)}{(q-\mathfrak{c}_n)\mathcal{Q}_{n-1}(q)}\left[1 - \frac{\mathcal{Q}_{n+1}(q)}{(q-\mathfrak{c}_{n+1})\mathcal{Q}_n(q)}\right], \quad n \geq 1.$$

Thus, $\alpha_n^{(q)} = (1-m_{n-1}^{(q)})m_n^{(q)}$ for $n \geq 1$ with $m_n^{(q)}$ given by (2.3). From the positiveness of $\alpha_n^{(q)}$ and $m_n^{(q)}(\epsilon)$ we see that $0 < m_n^{(q)}(\epsilon) < 1$ for $n \geq 1$. This proves the theorem. ■

The aim of this chapter is to provide a characterization of all the parameter sequences of the positive chain sequence $\{\alpha_n^{(q)}\}_{n\geq 1}$ and explore some of its consequences. To be more precise, with the use of a companion measure $\tilde{\nu}^{(q,\epsilon)}$ which is a rational perturbation of ν, we obtain a representation of all the parameter sequences of $\{\alpha_n^{(q)}\}_{n\geq 1}$.

The outline of the chapter is as follows. In Section 2.2 we provide our main result and its proof. In Section 2.3, we explore some results which are obtained as consequences of the result provided in the former section. Finally, in Section 2.4 we give an example of a family of positive chain sequences with explicit expressions for all the parameter sequences.

2.2 The Main Result

In order to state our main result let us introduce the companion measure $\tilde{\nu}^{(q,\epsilon)}$ and its corresponding sequence of MOP. Precisely, for $q \in \mathbb{R} \setminus (a,b)$ and $0 \leq \epsilon < 1$ we define the positive measure $\tilde{\nu}^{(q,\epsilon)}$ as

$$d\tilde{\nu}^{(q,\epsilon)}(x) = (1-\epsilon)\frac{\tilde{\kappa}^{(q)}}{q-x}d\nu(x) + \epsilon\,\boldsymbol{\delta}_q. \tag{2.4}$$

As already mentioned in the introduction, the measure $\tilde{\nu}^{(q,\epsilon)}$ constitutes a Geronimus perturbation of ν (see Maroni [26] for an algebraic approach).

We set the constant $\tilde{\kappa}^{(q)}$ in (2.4) as

$$\frac{1}{\tilde{\kappa}^{(q)}} = \int_a^b \frac{1}{q-x}d\nu(x). \tag{2.5}$$

Consequently, $\tilde{\nu}^{(q,\epsilon)}$ becomes a probability measure for any value of ϵ. In addition, for $q \in \mathbb{R}\backslash[a,b]$ it is easy to see that $\tilde{\kappa}^{(q)} \neq 0$. However, if $q = a$ or $q = b$ this is not necessarily true. Thus, except where otherwise indicated, in this chapter we always assume $\tilde{\kappa}^{(q)} \neq 0$.

Let $\{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}\}_{n\geq 0}$ be the sequence of MOP with respect to the probability measure $\tilde{\nu}^{(q,\epsilon)}$ and consider the sequence of positive constants $\{\tilde{\hbar}_n^{(q,\epsilon)}\}_{n\geq 0}$ such that

$$\int_{[a,b]\cup\{q\}} \widetilde{\mathcal{Q}}_m^{(q,\epsilon)}(x)\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x)d\tilde{\nu}^{(q,\epsilon)}(x) = \tilde{\hbar}_n^{(q,\epsilon)}\delta_{m,n}, \quad m,n = 0,1,2,\ldots.$$

Since $\tilde{\nu}^{(q,\epsilon)}$ is a probability measure we can write $\tilde{\hbar}_0^{(q,\epsilon)} = 1$. Moreover,

$$\begin{aligned}\tilde{\hbar}_n^{(q,\epsilon)} &= \int_{[a,b]\cup\{q\}} \left[\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x)\right]^2 d\tilde{\nu}^{(q,\epsilon)}(x) \\ &= (1-\epsilon)\int_a^b \left[\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x)\right]^2 \frac{\tilde{\kappa}^{(q)}}{q-x}d\nu(x) \ + \ \epsilon\left[\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)\right]^2, \quad n \geq 1.\end{aligned}$$

Notice also that

$$\int_{[a,b]\cup\{q\}} (x-q)\pi(x)d\tilde{\nu}^{(q,\epsilon)}(x) = -(1-\epsilon)\tilde{\kappa}^{(q)}\int_a^b \pi(x)d\nu(x), \tag{2.6}$$

for every real polynomial π.

Finally, let $\{\tilde{\mathfrak{c}}_n^{(q,\epsilon)}\}_{n\geq 1}$ and $\{\tilde{\mathfrak{d}}_{n+1}^{(q,\epsilon)}\}_{n\geq 1}$ be the coefficients in the TTRR

$$\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x) = (x - \tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x) - \tilde{\mathfrak{d}}_{n+1}^{(q,\epsilon)}\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(x), \quad n \geq 1, \tag{2.7}$$

where $\widetilde{\mathcal{Q}}_1^{(q,\epsilon)}(x) = x - \tilde{\mathfrak{c}}_1^{(q,\epsilon)}$ and $\widetilde{\mathcal{Q}}_0^{(q,\epsilon)}(x) = 1$.

The main result in this chapter is the following theorem.

Theorem 2.2. *Given the positive measure ν as stated prior to* (2.1) *let $\{\alpha_n^{(q)}\}_{n\geq 1}$ be the positive chain sequence given by* (2.2)*. Let $\{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}\}_{n\geq 0}$ be the sequence of MOP with respect to the probability measure $\tilde{\nu}^{(q,\epsilon)}$ given by* (2.4) *and let*

$$g_n^{(q)}(\epsilon) = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}, \quad n \geq 0,$$

where $\mathfrak{c}_{n+1}$ are as in (2.1)*. Then for any $\epsilon \in [0,1)$, the sequence $\{g_n^{(q)}(\epsilon)\}_{n\geq 0}$ is a parameter sequence of $\{\alpha_n^{(q)}\}_{n\geq 1}$. Moreover, the following can be stated:*

(i) *If* $0 \leq \epsilon_1 < \epsilon_2 < 1$ *then* $g_n^{(q)}(\epsilon_1) > g_n^{(q)}(\epsilon_2) > m_n^{(q)}$ *for* $n \geq 0$;

(ii) $\{g_n^{(q)}(0)\}_{n\geq 0}$ *is the maximal parameter sequence of* $\{\alpha_n^{(q)}\}_{n\geq 1}$;

(iii) $\lim_{\epsilon\to 1} g_n^{(q)}(\epsilon) = m_n^{(q)}$ *for* $n \geq 0$.

We point out that the idea of connecting companion orthogonal polynomial sequences using positive chain sequences is not new. For example, this idea has been used in [7] to orthogonal polynomials on the unit circle. Likewise, the authors of [5] and [36] use a similar strategy by shifting the interval $[a,b]$ to the interval $[0, b-a]$ and then making the connection via a sequence of orthogonal polynomials associated with a symmetric measure.

Before providing the proof of the Theorem 2.2 we will need some preliminary results. In particular, we will make use of Theorem 2.4 which gives another method to describe all the parameter sequences of the positive chain sequence $\{\alpha_n^{(q)}\}_{n\geq 1}$.

We start by stating the well known Christoffel's formulas.

Lemma 2.3 (Christoffel's formulas). *Consider the positive measures ν and $\tilde{\nu}^{(q,\epsilon)}$ as stated prior to (2.4) and let $\{\mathcal{Q}_n\}_{n\geq 0}$ and $\{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}\}_{n\geq 0}$ be, respectively, their corresponding sequences of MOP. Then*

$$(x-q)\mathcal{Q}_n(x) = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x) - \tilde{\chi}_n^{(q,\epsilon)}\,\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x), \quad n \geq 0, \tag{2.8}$$

where

$$\tilde{\chi}_n^{(q,\epsilon)} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} = (1-\epsilon)\frac{\tilde{\kappa}^{(q)}\,\hbar_n}{\tilde{\hbar}_n^{(q,\epsilon)}}, \quad n \geq 0. \tag{2.9}$$

Proof. Clearly (2.8) holds for $n=0$. From $\int_{[a,b]\cup\{q\}} \widetilde{\mathcal{Q}}_1^{(q,\epsilon)}(x)d\tilde{\nu}^{(q,\epsilon)}(x) = 0$ and (2.6),

$$0 = \int_{[a,b]\cup\{q\}} [x-q+q-\tilde{\mathfrak{c}}_1^{(q,\epsilon)}]d\tilde{\nu}^{(q,\epsilon)}(x) = -(1-\epsilon)\tilde{\kappa}^{(q)}\hbar_0 + q - \tilde{\mathfrak{c}}_1^{(q,\epsilon)}.$$

Then (2.9) holds for $n=0$. For $n\geq 1$ if we write

$$(x-q)\mathcal{Q}_n(x) = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x) + \sum_{j=0}^{n}\lambda_{n,j}\widetilde{\mathcal{Q}}_j^{(q,\epsilon)}(x), \tag{2.10}$$

then multiplying both sides by $\mathcal{Q}_j$ and integrating with respect to $\tilde{\nu}^{(q,\epsilon)}$ we obtain

$$\begin{aligned}\lambda_{n,j} &= \frac{1}{\tilde{\hbar}_j^{(q,\epsilon)}}\int_{[a,b]\cup\{q\}}(x-q)\mathcal{Q}_n(x)\widetilde{\mathcal{Q}}_j^{(q,\epsilon)}(x)d\tilde{\nu}^{(q,\epsilon)}(x)\\ &= -(1-\epsilon)\frac{\tilde{\kappa}^{(q)}}{\tilde{\hbar}_j^{(q,\epsilon)}}\int_a^b \mathcal{Q}_n(x)\widetilde{\mathcal{Q}}_j^{(q,\epsilon)}(x)d\nu(x).\end{aligned}$$

Hence $\lambda_{n,j}=0$ for $j=0,1,\ldots,n-1$ and (2.10) becomes

$$(x-q)\mathcal{Q}_n(x) = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x) + \lambda_{n,n}\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x), \tag{2.11}$$

with

$$\lambda_{n,n} = -(1-\epsilon)\frac{\tilde{\kappa}^{(q)}\,\hbar_n}{\tilde{\hbar}_n^{(q,\epsilon)}}.$$

The evaluation of (2.11) at $x=q$ gives the desired result. ∎

Another important fact is that the polynomial $\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}$ can be written in terms of the polynomials $\mathcal{Q}_{n+1}$ and $\mathcal{Q}_n$ using a simple connection formula. To be more precise, by expressing $\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}$ as a linear combination of the polynomials $\{\mathcal{Q}_j\}_{j=0}^{n+1}$ one can deduce

$$\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x) = \mathcal{Q}_{n+1}(x) - \tilde{\mathfrak{b}}_n^{(q,\epsilon)}\,\mathcal{Q}_n(x), \quad n\geq 0, \tag{2.12}$$

where the connecting coefficients $\tilde{\mathfrak{b}}_n^{(q,\epsilon)}$ are given by

$$\tilde{\mathfrak{b}}_n^{(q,\epsilon)} = \frac{\tilde{\hbar}_{n+1}^{(q,\epsilon)}}{(1-\epsilon)\tilde{\kappa}^{(q)}\,\hbar_n}, \quad n\geq 0. \tag{2.13}$$

The coefficients $\mathfrak{c}_n$, $\mathfrak{d}_n$, $\tilde{\mathfrak{c}}_n^{(q,\epsilon)}$, $\tilde{\mathfrak{d}}_n^{(q,\epsilon)}$ and $\tilde{\mathfrak{b}}_n^{(q,\epsilon)}$ are known to satisfy the formulas (see [2], [20], [25])

$$\begin{aligned}&\frac{\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)}}{\tilde{\mathfrak{b}}_n^{(q,\epsilon)}} + \tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)} + \tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)} = q = \frac{\mathfrak{d}_{n+1}}{\tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)}} + \mathfrak{c}_{n+1} + \tilde{\mathfrak{b}}_n^{(q,\epsilon)}, \\ &\frac{\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)}}{\tilde{\mathfrak{b}}_n^{(q,\epsilon)}} = \frac{\mathfrak{d}_{n+1}}{\tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)}} \quad \text{and} \quad \tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)} + \tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)} = \mathfrak{c}_{n+1} + \tilde{\mathfrak{b}}_n^{(q,\epsilon)},\end{aligned} \qquad n \geq 1, \tag{2.14}$$

with the initial conditions

$$\tilde{\mathfrak{c}}_1^{(q,\epsilon)} = \mathfrak{c}_1 + \tilde{\mathfrak{b}}_0^{(q,\epsilon)} \qquad \text{and} \qquad \frac{\tilde{\mathfrak{d}}_2^{(q,\epsilon)}}{\tilde{\mathfrak{b}}_0^{(q,\epsilon)}} + \tilde{\mathfrak{c}}_1^{(q,\epsilon)} = q.$$

By writing $\mathfrak{d}_{n+1} = \hbar_n/\hbar_{n-1}$ with $\hbar_n$ given by (2.9) and then using (2.13) we get

$$\mathfrak{d}_{n+1} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} \frac{\tilde{\hbar}_n^{(q,\epsilon)}}{(1-\epsilon)\tilde{\kappa}^{(q)}\hbar_{n-1}} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} \tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)}, \quad n \geq 1. \tag{2.15}$$

Combining this with (2.14) one finds

$$\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} \tilde{\mathfrak{b}}_n^{(q,\epsilon)}, \quad n \geq 0. \tag{2.16}$$

With $\{\mathfrak{c}_n\}_{n\geq 1}$ and $\{\mathfrak{d}_{n+1}\}_{n\geq 1}$ given as in the TTRR (2.1) let $\{\mathcal{Q}_n^*\}_{n\geq 0}$ be the sequence of monic associated polynomials of $\{\mathcal{Q}_n\}_{n\geq 0}$ given by the TTRR

$$\mathcal{Q}_{n+1}^*(x) = (x - \mathfrak{c}_{n+1})\mathcal{Q}_n^*(x) - \mathfrak{d}_{n+1}\mathcal{Q}_{n-1}^*(x), \quad n \geq 1, \tag{2.17}$$

with $\mathcal{Q}_0^*(x) = 0$ and $\mathcal{Q}_1^*(x) = 1$. We remind that the polynomials $\{\mathcal{Q}_n^*\}_{n\geq 0}$ have the integral representation

$$\mathcal{Q}_n^*(x) = \frac{1}{\hbar_0}\int_a^b \frac{\mathcal{Q}_n(x) - \mathcal{Q}_n(y)}{x-y} d\nu(y), \quad n \geq 0.$$

Along this chapter, we will denote by $\mathcal{F}(z)$ the function defined by

$$\mathcal{F}(z) = \frac{z - \mathfrak{c}_1}{\hbar_0}\int_a^b \frac{1}{z-x} d\nu(x), \quad z \in \mathbb{R} \setminus (a,b). \tag{2.18}$$

The next theorem provides another way to represent all the parameter sequences of $\{\alpha_n^{(q)}\}_{n\geq 1}$. In fact, this theorem will be useful to verify that $\{g_n^{(q)}(0)\}_{n\geq 0}$ in Theorem 2.2 is the maximal parameter sequence of $\{\alpha_n^{(q)}\}_{n\geq 1}$.

Theorem 2.4 ([3]). *For any $t \in [0, 1/\mathcal{F}(q)]$ the sequence $\{\hat{g}_n^{(q)}(t)\}_{n\geq 0}$ defined by*

$$\hat{g}_n^{(q)}(t) = 1 - \frac{\mathcal{Q}_{n+1}(q) - t(q-\mathfrak{c}_1)\mathcal{Q}_{n+1}^*(q)}{(q - \mathfrak{c}_{n+1})\Big[\mathcal{Q}_n(q) - t(q-\mathfrak{c}_1)\mathcal{Q}_n^*(q)\Big]}, \quad n \geq 0,$$

is a parameter sequence of the positive chain sequence $\{\alpha_n^{(q)}\}_{n\geq 1}$ given by (2.2). Moreover, $\{\hat{g}_n^{(q)}(1/\mathcal{F}(q))\}_{n\geq 0}$ is its maximal parameter sequence.

With these results we are now able to consider the proof of Theorem 2.2.

2.2.1 Proof of the Main Result

Proof of Theorem 2.2. We begin by considering the positive chain sequence formulas that follow from the TTRR (2.7). We write

$$\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}\left[1-\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\right]=\frac{\tilde{\mathfrak{d}}_{n+1}^{(q,\epsilon)}}{(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})(q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})},\quad n\geq 1.$$

Hence, from (2.16)

$$\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}\left[1-\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\right]=\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}\,\frac{\tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)}}{(q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})},$$

for $n\geq 1$. Thus,

$$\frac{\tilde{\mathfrak{b}}_{n-1}^{(q,\epsilon)}}{q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)}}=1-\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_{n+1}^{(q,\epsilon)})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)},\quad n\geq 0,\tag{2.19}$$

where we set $\tilde{\mathfrak{b}}_{-1}^{(q,\epsilon)}=0$.

Furthermore, combining (2.14) with (2.15),

$$\mathfrak{d}_{n+1}=\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\left[\tilde{\mathfrak{c}}_n^{(q,\epsilon)}+\tilde{\mathfrak{b}}_{n-2}^{(q,\epsilon)}-\mathfrak{c}_n\right],\quad n\geq 1,$$

where again $\tilde{\mathfrak{b}}_{-1}^{(q,\epsilon)}=0$. Hence from (2.19), by writing

$$\tilde{\mathfrak{c}}_n^{(q,\epsilon)}+\tilde{\mathfrak{b}}_{n-2}^{(q,\epsilon)}-\mathfrak{c}_n=-(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})+\tilde{\mathfrak{b}}_{n-2}^{(q,\epsilon)}+(q-\mathfrak{c}_n),$$

we find

$$\begin{aligned}\mathfrak{d}_{n+1}&=\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\left[-(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})+(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})\left(1-\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{(q-\tilde{\mathfrak{c}}_n^{(q,\epsilon)})\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}\right)+(q-\mathfrak{c}_n)\right]\\&=\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\left[-\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}+(q-\mathfrak{c}_n)\right],\quad n\geq 1.\end{aligned}$$

Therefore,

$$\alpha_n^{(q)}=\frac{\mathfrak{d}_{n+1}}{(q-\mathfrak{c}_n)(q-\mathfrak{c}_{n+1})}=\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\left[-\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{(q-\mathfrak{c}_n)\widetilde{\mathcal{Q}}_{n-1}^{(q,\epsilon)}(q)}+1\right],\quad n\geq 1.$$

From this if we set

$$g_n^{(q)}(\epsilon)=\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)},\quad n\geq 0,$$

then

$$[1-g_{n-1}^{(q)}(\epsilon)]g_n^{(q)}(\epsilon)=\alpha_n^{(q)},\quad n\geq 1.$$

Thus, from the positiveness of $g_n^{(q)}(\epsilon)$ and $\alpha_n^{(q)}$ we see that $\{g_n^{(q)}(\epsilon)\}_{n\geq 0}$ is a parameter sequence of $\{\alpha_n^{(q)}\}_{n\geq 1}$. This proves the first part of the theorem.

Taking $n = 0$ in (2.9) gives

$$g_0^{(q)}(\epsilon) = \frac{\widetilde{\mathcal{Q}}_1^{(q,\epsilon)}(q)}{(q - \mathfrak{c}_1)\widetilde{\mathcal{Q}}_0^{(q,\epsilon)}(q)} = (1-\epsilon)\frac{\tilde{\kappa}^{(q)}\hbar_0}{q - \mathfrak{c}_1}.$$

From this if $0 \leq \epsilon_1 < \epsilon_2 < 1$, then

$$g_0^{(q)}(0) \geq g_0^{(q)}(\epsilon_1) > g_0^{(q)}(\epsilon_2) > 0.$$

Thus, from Theorem 1.2

$$g_n^{(q)}(0) \geq g_n^{(q)}(\epsilon_1) > g_n^{(q)}(\epsilon_2) > 0 \quad \text{for} \quad n \geq 0.$$

This proves assertion (i).

To prove assertion (ii) we compare the parameter sequence $\{g_n^{(q)}(\epsilon)\}_{n\geq 0}$ with the parameter sequence $\{\hat{g}_n^{(q)}(t)\}_{n\geq 0}$ given by Theorem 2.4. From (2.5) and (2.18)

$$g_0^{(q)}(\epsilon) = (1-\epsilon)\frac{\tilde{\kappa}^{(q)}\hbar_0}{q - \mathfrak{c}_1} = \frac{1-\epsilon}{\mathcal{F}(q)}. \tag{2.20}$$

Since the initial parameter of a parameter sequence uniquely determines the remaining parameters, by setting $\hat{g}_0^{(q)}(t) = g_0^{(q)}(\epsilon)$ one finds that if

$$t = t(\epsilon) = \frac{1-\epsilon}{\mathcal{F}(q)},$$

then

$$\hat{g}_n^{(q)}\Big((1-\epsilon)/\mathcal{F}(q)\Big) = g_n^{(q)}(\epsilon), \quad n \geq 0, \tag{2.21}$$

for any $0 \leq \epsilon < 1$. In particular, from (2.21) we have $\hat{g}_n^{(q)}(1/\mathcal{F}(q)) = g_n^{(q)}(0)$. Thus, from Theorem 2.4 we can verify that $\{g_n^{(q)}(0)\}_{n\geq 0}$ is the maximal parameter sequence of $\{\alpha_n^{(q)}\}_{n\geq 1}$. This proves assertion (ii).

Now to prove assertion (iii) we first remark that when $\epsilon = 1$ the measure $\tilde{\nu}$ becomes a trivial measure with just a single mass point. Thus, we are not able to represent the minimal parameter sequence $\{m_n^{(q)}\}_{n\geq 0}$ using the result given in Theorem 2.2. On the other hand, since $g_0^{(q)}(\epsilon) \to 0$ as $\epsilon \to 1$ there follows

$$\lim_{\epsilon\to 1} g_n^{(q)}(\epsilon) = m_n^{(q)}, \quad n \geq 0.$$

This completes the proof of the theorem. ■

2.3 Some Properties and Applications

This section is devoted to the study of some results which follow as consequences of Theorem 2.2. We shall establish some connection formulas that will play a key role in the following chapters. Finally, we study aspects about the monotonicity and asymptotic behavior of the parameters $g_n^{(q)}(\epsilon)$ and the connection coefficients $\tilde{\mathfrak{b}}_n^{(q,\epsilon)}$.

The next result allows us to determine when $\{\alpha_n^{(q)}\}_{n\geq 1}$ is a multiple or a single parameter positive chain sequence.

Theorem 2.5.

(i) *Let* $q \in \mathbb{R}\setminus[a,b]$. *Then* $\tilde{\kappa}^{(q)} \neq 0$ *and* $\{\alpha_n^{(q)}\}_{n\geq 1}$ *is a multiple parameter positive chain sequence.*

(ii) *Let* $q = a$ *or* $q = b$. *Then* $\{\alpha_n^{(q)}\}_{n\geq 1}$ *is a multiple parameter positive chain sequence if* $\tilde{\kappa}^{(q)} \neq 0$ *and it is a single parameter positive chain sequence if* $\tilde{\kappa}^{(q)} = 0$.

Proof. Let us first assume $\tilde{\kappa}^{(q)} \neq 0$. From Theorem 2.2 and taking into account (2.20), we can see that

$$g_0^{(q)}(\epsilon) = (1-\epsilon)\hbar_0 \frac{\tilde{\kappa}^{(q)}}{q - \mathfrak{c}_1} > 0$$

uniquely determines a parameter sequence of $\{\alpha_n^{(q)}\}_{n\geq 1}$ for each ϵ in the range $[0,1)$. However, if either $q = a$ or $q = b$ and $\tilde{\kappa}^{(q)} = 0$, the minimal parameter sequence $\{m_n^{(q)}\}_{n\geq 0}$, which can be given by (2.3), exists and is the only parameter sequence of $\{\alpha_n\}_{n\geq 1}$. ■

2.3.1 Connection Formulas

We now look at some consequences of the results contained in Theorem 2.2. Firstly, from Theorem 2.4 and from (2.21)

$$g_n^{(q)}(\epsilon) = 1 - \frac{\mathcal{Q}_{n+1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n+1}(q)}{(q-\mathfrak{c}_{n+1})\Big[\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)\Big]}, \quad n \geq 0. \tag{2.22}$$

Hence, from Theorem 2.2 and using $g_n^{(q)}(\epsilon) = \alpha_n^{(q)}/[1 - g_{n-1}^{(q)}(\epsilon)]$ we deduce

$$\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} = g_n^{(q)}(\epsilon) = \frac{\mathfrak{d}_{n+1}}{q-\mathfrak{c}_{n+1}} \frac{\mathcal{Q}_{n-1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n-1}(q)}{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)} \tag{2.23}$$

for $n \geq 1$. Combining these with (2.9) we find the alternative formulas to the coefficients

$$\tilde{\chi}_n^{(q,\epsilon)} = (q-\mathfrak{c}_{n+1})g_n^{(q)}(\epsilon) = \mathfrak{d}_{n+1}\frac{\mathcal{Q}_{n-1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n-1}(q)}{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)}, \quad n \geq 1, \tag{2.24}$$

which appear in the Christoffel's formulas (2.8).

On the other hand, from (2.23) one gets

$$\mathfrak{d}_{n+1}\frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)} = \frac{\mathfrak{d}_{n+1}}{(q-\mathfrak{c}_{n+1})g_n^{(q)}(\epsilon)} = \frac{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)}{\mathcal{Q}_{n-1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n-1}(q)}, \quad n \geq 1. \tag{2.25}$$

Therefore, taking into account (2.15), we get the following formulas for the connection coefficients $\tilde{\mathfrak{b}}_n^{(q,\epsilon)}$ that appear in (2.7)

$$\tilde{\mathfrak{b}}_n^{(q,\epsilon)} = (q-\mathfrak{c}_{n+1})[1 - g_n^{(q)}(\epsilon)] = \frac{\mathcal{Q}_{n+1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n+1}(q)}{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)}, \quad n \geq 0. \tag{2.26}$$

With these formulas we can state the following theorem.

Theorem 2.6.

$$\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q) = (1-\epsilon)\tilde{\kappa}^{(q)}\hbar_0 \frac{\mathfrak{d}_2\mathfrak{d}_3\cdots\mathfrak{d}_{n+1}}{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_n^*(q)}, \quad n \geq 1,$$

with $\widetilde{\mathcal{Q}}_1^{(q,\epsilon)}(q) = (1-\epsilon)\tilde{\kappa}^{(q)}\hbar_0$.

Proof. From (2.9) we get the expression for $\widetilde{\mathcal{Q}}_1^{(q,\epsilon)}(q)$ and

$$\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q) = \prod_{j=0}^{n} \frac{\widetilde{\mathcal{Q}}_{j+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_{j}^{(q,\epsilon)}(q)} = (1-\epsilon)\tilde{\kappa}^{(q)}\hbar_0 \prod_{j=1}^{n} \tilde{\chi}_j^{(q,\epsilon)}, \quad n \geq 1.$$

Thus the desired result follows from (2.24). ■

The next theorem states that the rational functions

$$\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x)\mathcal{Q}_n^*(x) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(x)\mathcal{Q}_{n-1}^*(x)}{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(x)\mathcal{Q}_n(x) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(x)\mathcal{Q}_{n-1}(x)}, \quad n \geq 1,$$

are invariant under n when evaluated at $x = q$.

Theorem 2.7.

$$\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_n^*(q) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}^*(q)}{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_n(q) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}(q)} = \frac{1}{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}}, \quad n \geq 1.$$

Proof. Multiplying (2.25) by $\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}^*(q)$ gives

$$\mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}^*(q) = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\frac{\mathcal{Q}_n(q)\mathcal{Q}_{n-1}^*(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_n^*(q)\mathcal{Q}_{n-1}^*(q)}{\mathcal{Q}_{n-1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n-1}^*(q)}, \quad n \geq 1.$$

From this

$$\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_n^*(q) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}^*(q) = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\frac{\mathcal{Q}_{n-1}(q)\mathcal{Q}_n^*(q) - \mathcal{Q}_n(q)\mathcal{Q}_{n-1}^*(q)}{\mathcal{Q}_{n-1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n-1}^*(q)}$$

for $n \geq 1$. Similarly, by using (2.25) to compute $\mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}(q)$, one finds

$$\frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_n(q) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}(q)}{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}} = \widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)\mathcal{Q}_n^*(q) - \mathfrak{d}_{n+1}\widetilde{\mathcal{Q}}_{n}^{(q,\epsilon)}(q)\mathcal{Q}_{n-1}^*(q)$$

for $n \geq 1$. This proves the desired result. ■

Now let $\{\widetilde{\mathcal{O}}_n^{(q,\epsilon)}\}_{n\geq 0}$ denote the sequence of orthonormal polynomials with respect to the measure $\tilde{\nu}^{(q,\epsilon)}$. Then we write

$$\widetilde{\mathcal{O}}_n^{(q,\epsilon)}(x) = \frac{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x)}{\sqrt{\tilde{\hbar}_n^{(q,\epsilon)}}}, \quad n \geq 0.$$

Since $\tilde{\hbar}_0^{(q,\epsilon)} = 1$ and $\tilde{\hbar}_n^{(q,\epsilon)} = \tilde{\hbar}_0^{(q,\epsilon)}\tilde{\mathfrak{d}}_2^{(q,\epsilon)}\tilde{\mathfrak{d}}_3^{(q,\epsilon)}\cdots\tilde{\mathfrak{d}}_{n+1}^{(q,\epsilon)}$ for $n \geq 1$, we get

$$[\widetilde{\mathcal{O}}_n^{(q,\epsilon)}(x)]^2 = \frac{[\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(x)]^2}{\tilde{\mathfrak{d}}_2^{(q,\epsilon)}\tilde{\mathfrak{d}}_3^{(q,\epsilon)}\cdots\tilde{\mathfrak{d}}_{n+1}^{(q,\epsilon)}}, \quad n \geq 1, \tag{2.27}$$

with $[\widetilde{\mathcal{O}}_0^{(q,\epsilon)}(x)]^2 = 1$.

The following lemma will be useful in the proof of Theorem 2.9.

Lemma 2.8.

$$[\widetilde{\mathcal{O}}_{n+1}^{(q,\epsilon)}(q)]^2 = \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathfrak{d}_2\mathfrak{d}_3\cdots\mathfrak{d}_{n+1}}{[\mathcal{Q}_n(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_n^*(q)][\mathcal{Q}_{n+1}(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n+1}^*(q)]}, \quad n \geq 1,$$

with

$$[\widetilde{\mathcal{O}}_1^{(q,\epsilon)}(q)]^2 = \frac{1-\epsilon}{\mathcal{F}(q)-(1-\epsilon)} = \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_1^*(q)}{\mathcal{Q}_1(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_1^*(q)}.$$

Proof. Firstly, we have from (2.16) and (2.26)

$$\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}\,\tilde{\mathfrak{b}}_n^{(q,\epsilon)} = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}(q-\mathfrak{c}_{n+1})[1-g_n^{(q)}(\epsilon)], \quad n \geq 0.$$

Then

$$\frac{1}{1-g_n^{(q)}(\epsilon)} = \frac{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)}\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)}, \quad n \geq 0.$$

Multiplying both sides by $g_n^{(q)}(\epsilon)$ and then using (2.27) yields

$$\frac{g_n^{(q)}(\epsilon)}{1-g_n^{(q)}(\epsilon)} = \frac{[\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)]^2}{\tilde{\mathfrak{d}}_{n+2}^{(q,\epsilon)}[\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)]^2} = \frac{[\widetilde{\mathcal{O}}_{n+1}^{(q,\epsilon)}(q)]^2}{[\widetilde{\mathcal{O}}_n^{(q,\epsilon)}(q)]^2}, \quad n \geq 0. \tag{2.28}$$

On the other hand, from (2.22) and (2.23),

$$\frac{g_n^{(q)}(\epsilon)}{1-g_n^{(q)}(\epsilon)} = \mathfrak{d}_{n+1}\frac{\mathcal{Q}_{n-1}(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n-1}^*(q)}{\mathcal{Q}_{n+1}(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n+1}^*(q)}, \quad n \geq 1. \tag{2.29}$$

Setting $n=0$ in (2.20) we find

$$\frac{g_0^{(q)}(\epsilon)}{1-g_0^{(q)}(\epsilon)} = \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_1^*(q)}{\mathcal{Q}_1(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_1^*(q)}.$$

Thus, comparing the product

$$[\widetilde{\mathcal{O}}_{n+1}^{(q,\epsilon)}(q)]^2 = \prod_{j=0}^{n}\frac{g_j^{(q)}(\epsilon)}{1-g_j^{(q)}(\epsilon)}$$

in (2.28) and (2.29) gives the required result. ∎

The next theorem gives an expression for the sum $\sum_{n=1}^{N}[\widetilde{\mathcal{O}}_n^{(q,\epsilon)}(q)]^2$.

Theorem 2.9.

$$\sum_{n=1}^{N}\prod_{j=0}^{n-1}\frac{g_j^{(q)}(\epsilon)}{1-g_j^{(q)}(\epsilon)} = \sum_{n=1}^{N}[\widetilde{\mathcal{O}}_n^{(q,\epsilon)}(q)]^2 = \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_N^*(q)}{\mathcal{Q}_N(q)-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_N^*(q)}, \quad N \geq 1.$$

Proof. Using the TTRRs (2.1) and (2.17), we can deduce

$$\begin{aligned}\mathcal{Q}_n(q)\mathcal{Q}^*_{n+1}(q) - \mathcal{Q}_{n+1}(q)\mathcal{Q}^*_n(q) &= -\mathfrak{d}_{n+1}\left[\mathcal{Q}_n(q)\mathcal{Q}^*_{n-1}(q) - \mathcal{Q}_{n-1}(q)\mathcal{Q}^*_n(q)\right] \\ &= \mathfrak{d}_2\mathfrak{d}_3\ldots\mathfrak{d}_{n+1}, \quad n \geq 1.\end{aligned} \tag{2.30}$$

Then we can write

$$\begin{aligned}(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathfrak{d}_2\mathfrak{d}_3\cdots\mathfrak{d}_{n+1} = (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_n(q)\mathcal{Q}^*_{n+1}(q) - [(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}]^2\mathcal{Q}^*_n(q)\mathcal{Q}^*_{n+1}(q) \\ + [(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}]^2\mathcal{Q}^*_n(q)\mathcal{Q}^*_{n+1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}_{n+1}(q)\mathcal{Q}^*_n(q)\end{aligned}$$

for $n \geq 1$. Therefore, substituting this in Lemma 2.8 we get

$$[\widetilde{\mathcal{O}}^{(q,\epsilon)}_{n+1}(q)]^2 = \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n+1}(q)}{\mathcal{Q}_{n+1}(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_{n+1}(q)} - \frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)}{\mathcal{Q}_n(q) - (1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_n(q)},$$

for $n \geq 1$. Thus, the summation $\sum_{n=1}^{N}[\widetilde{\mathcal{O}}^{(q,\epsilon)}_n(q)]^2$ gives the required result. ∎

The value of $\sum_{n=1}^{\infty}[\widetilde{\mathcal{O}}^{(q,\epsilon)}_n(q)]^2$ gives a characterization of the mass point of $\tilde{\nu}^{(q,\epsilon)}$ at $x = q$ (see [14, Thm. 2.5.6]). By Theorem 2.9

$$\sum_{n=1}^{\infty}[\widetilde{\mathcal{O}}^{(q,\epsilon)}_n(q)]^2 = \lim_{N\to\infty}\frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_N(q)/\mathcal{Q}_N(q)}{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{Q}^*_N(q)/\mathcal{Q}_N(q)}.$$

Since ν is a positive measure whose moments all exist, it is known that for $q \in \mathbb{R}\setminus[a,b]$ (see [14, Thm. 2.6.2])

$$\lim_{N\to\infty}\frac{\mathcal{Q}^*_N(q)}{\mathcal{Q}_N(q)} = \frac{1}{\hbar_0\tilde{\kappa}^{(q)}}. \tag{2.31}$$

Therefore,

$$\sum_{n=1}^{\infty}[\widetilde{\mathcal{O}}^{(q,\epsilon)}_n(q)]^2 = \frac{1-\epsilon}{\epsilon}.$$

2.3.2 Asymptotic Results

The following theorem provides some results regarding the monotonicity and asymptotic behavior of $\tilde{\chi}^{(q,\epsilon)}_n = \widetilde{\mathcal{Q}}^{(q,\epsilon)}_{n+1}(q)/\widetilde{\mathcal{Q}}^{(q,\epsilon)}_n(q)$ with respect to ϵ.

Theorem 2.10.

(a) *Let* $0 \leq \epsilon_1 < \epsilon_2 < 1$. *Then*

$$\left|\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon_1)}_{n+1}(q)}{\widetilde{\mathcal{Q}}^{(q,\epsilon_1)}_n(q)}\right| > \left|\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon_2)}_{n+1}(q)}{\widetilde{\mathcal{Q}}^{(q,\epsilon_2)}_n(q)}\right| > \mathfrak{d}_{n+1}\left|\frac{\mathcal{Q}_{n-1}(q)}{\mathcal{Q}_n(q)}\right|, \quad n \geq 1.$$

Moreover, $|\tilde{\mathfrak{b}}^{(q,\epsilon_1)}_n| < |\tilde{\mathfrak{b}}^{(q,\epsilon_2)}_n|$ *for* $n \geq 0$.

(b) *For* $0 < \epsilon < 1$ *and* $n \geq 1$,

$$(1-\epsilon)\frac{|\tilde{\kappa}^{(q)}|\hbar_n}{[\mathcal{Q}_n(q)]^2} < \left|\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon)}_{n+1}(q)}{\widetilde{\mathcal{Q}}^{(q,\epsilon)}_n(q)} - \mathfrak{d}_{n+1}\frac{\mathcal{Q}_{n-1}(q)}{\mathcal{Q}_n(q)}\right| < (1-\epsilon)\frac{|\tilde{\kappa}^{(q)}|\hbar_n}{\mathcal{Q}_n(q)[\mathcal{Q}_n(q) - \tilde{\kappa}^{(q)}\hbar_0\mathcal{Q}^*_n(q)]}.$$

(c) $\displaystyle\lim_{\epsilon\to 1}\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon)}_{n+1}(q)}{\widetilde{\mathcal{Q}}^{(q,\epsilon)}_{n}(q)}=\mathfrak{d}_{n+1}\frac{\mathcal{Q}_{n-1}(q)}{\mathcal{Q}_n(q)},\quad n\ge 1.$

Proof. From $g^{(q)}_n(\epsilon_1)>g^{(q)}_n(\epsilon_2)>m^{(q)}_n$ for $n\ge 1$,

$$\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon_1)}_{n+1}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}^{(q,\epsilon_1)}_{n}(q)}>\frac{\widetilde{\mathcal{Q}}^{(q,\epsilon_2)}_{n+1}(q)}{(q-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}^{(q,\epsilon_2)}_{n}(q)}>m^{(q)}_n=\mathfrak{d}_{n+1}\frac{\mathcal{Q}_{n-1}(q)}{(q-\mathfrak{c}_{n+1})\mathcal{Q}_n(q)},\quad n\ge 1.$$

From this the first part of (a) follows. The latter part of (a) follows from (2.26).

Now to prove assertion (b), we have from (2.23)

$$g^{(q)}_n(\epsilon)=\left[\frac{\mathfrak{d}_{n+1}\mathcal{Q}_{n-1}(q)}{(q-\mathfrak{c}_{n+1})\mathcal{Q}_n(q)}\right]\frac{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n-1}(q)}{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n}(q)},\quad n\ge 1,$$

where $\mathcal{W}_n(q)=\mathcal{Q}^*_n(q)/\mathcal{Q}_n(q)$ for $n\ge 0$. Then from (2.3)

$$\frac{g^{(q)}_n(\epsilon)}{m^{(q)}_n}=\frac{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n-1}(q)}{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n}(q)},\quad n\ge 1.$$

Thus, one writes

$$\frac{g^{(q)}_n(\epsilon)}{m^{(q)}_n}=1+\frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}[\mathcal{W}_n(q)-\mathcal{W}_{n-1}(q)]}{1-(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n}(q)},\quad n\ge 1.$$

On the other hand, from (2.30) we have for $q\in\mathbb{R}\setminus(a,b)$

$$\mathcal{W}_{n+1}(q)-\mathcal{W}_n(q)=\frac{\mathfrak{d}_2\mathfrak{d}_3\ldots\mathfrak{d}_{n+1}}{\mathcal{Q}_{n+1}(q)\mathcal{Q}_n(q)},\quad n\ge 1.$$

Hence, for $q\in\mathbb{R}\setminus(a,b)$,

$$\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n+1}(q)-\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_{n}(q)>0,\quad n\ge 1.$$

Formula (2.20) yields $\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_1(q)=g^{(q)}_0(\epsilon)/(1-\epsilon)>0$.

Furthermore, from (2.31) we have for $q\in\mathbb{R}\setminus[a,b]$,

$$\lim_{n\to\infty}\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_n(q)=1.$$

Thus, for $q\in\mathbb{R}\setminus(a,b)$, we can write

$$0<\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_1(q)<\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_2(q)<\cdots<\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_n(q)<1.$$

Consequently, for $0<\epsilon<1$,

$$1+(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}[\mathcal{W}_n(q)-\mathcal{W}_{n-1}(q)]<\frac{g^{(q)}_n(\epsilon)}{m^{(q)}_n}<1+\frac{(1-\epsilon)\hbar_0\tilde{\kappa}^{(q)}[\mathcal{W}_n(q)-\mathcal{W}_{n-1}(q)]}{1-\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_n(q)}.$$

Since $\hbar_n=\hbar_0\mathfrak{d}_2\cdots\mathfrak{d}_{n+1}$ for $n\ge 1$, from (2.3) we get

$$m^{(q)}_n\hbar_0\tilde{\kappa}^{(q)}[\mathcal{W}_n(q)-\mathcal{W}_{n-1}(q)]=\frac{\tilde{\kappa}^{(q)}\hbar_n}{(q-\mathfrak{c}_{n+1})[\mathcal{Q}_n(q)]^2},\quad n\ge 1.$$

Therefore,

$$\frac{(1-\epsilon)\tilde{\kappa}^{(q)}\hbar_n}{(q-\mathfrak{c}_{n+1})[\mathcal{Q}_n(q)]^2}<g^{(q)}_n-m^{(q)}_n<\frac{(1-\epsilon)\tilde{\kappa}^{(q)}\hbar_n}{(q-\mathfrak{c}_{n+1})[\mathcal{Q}_n(q)]^2[1-\hbar_0\tilde{\kappa}^{(q)}\mathcal{W}_n(q)]}.$$

Thus, from (2.3) assertion (b) holds. The assertion (c) immediately follows by letting $\epsilon\to 1$ in (b). This completes the proof of the Theorem. ■

The next theorem gives the behavior of $g_n^{(q)}(\epsilon)$ as $|q| \to \infty$.

Theorem 2.11. *If* $0 \le \epsilon < 1$, *then*

$$\lim_{q\to\infty} g_0^{(q)}(\epsilon) = 1 - \epsilon.$$

Moreover, the following statements hold:

(i) *Let* $n \ge 1$. *Then*

$$\lim_{q\to\infty} g_n^{(q)}(0) = 1.$$

Precisely,

$$g_n^{(q)}(0) = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,0)}(q)}{(q - \mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,0)}(q)} = 1 - \mathfrak{d}_{n+2}\, q^{-2} + O\left(q^{-3}\right) \quad \textit{as} \quad |q| \to \infty.$$

(ii) *Let* $n \ge 1$. *If* $0 < \epsilon < 1$, *then*

$$\lim_{q\to\infty} g_n^{(q)}(\epsilon) = 0.$$

Precisely,

$$g_n^{(q)}(\epsilon) = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(q,\epsilon)}(q)}{(q - \mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(q,\epsilon)}(q)} = \mathfrak{d}_{n+1}\, q^{-2} + O\left(q^{-3}\right) \quad \textit{as} \quad |q| \to \infty.$$

Proof. Since

$$\frac{1}{q-x} = \sum_{j=0}^{\infty} \frac{x^j}{q^{j+1}}$$

we can write

$$\frac{1}{\tilde{\kappa}^{(q)}} = \int_a^b \frac{1}{q-x} d\nu(x) = \sum_{j=0}^{\infty} \frac{\nu_j}{q^{j+1}},$$

where $\nu_0 = \hbar_0$ and $\nu_j = \int_a^b x^j d\nu(x)$ for $j \ge 1$. Thus, from (2.20)

$$g_0^{(q)}(\epsilon) = (1-\epsilon)\frac{\tilde{\kappa}^{(q)}\hbar_0}{q - \mathfrak{c}_1} = \frac{(1-\epsilon)\hbar_0}{[\nu_0 + \sum_{j=1}^{\infty}(\nu_j/q^j)][1 - \mathfrak{c}_1/q]} \longrightarrow (1-\epsilon) \quad \text{as} \quad |q| \to \infty.$$

On the other hand, from (2.22)

$$1 - g_n^{(q)}(\epsilon) = \frac{\mathcal{Q}_{n+1}(q)}{(q - \mathfrak{c}_{n+1})\mathcal{Q}_n(q)} \, \frac{(1/\tilde{\kappa}^{(q)}) - (1-\epsilon)\hbar_0 \mathcal{Q}_{n+1}^*(q)/\mathcal{Q}_{n+1}(q)}{(1/\tilde{\kappa}^{(q)}) - (1-\epsilon)\hbar_0 \mathcal{Q}_n^*(q)/\mathcal{Q}_n(q)}, \quad n \ge 1. \tag{2.32}$$

Now we analyze the second term in the product (2.32). Note that $q^n/\mathcal{Q}_n(q) = 1 + O\left(q^{-1}\right)$. Then

$$\begin{aligned}
\hbar_0 \frac{\mathcal{Q}_n^*(q)}{\mathcal{Q}_n(q)} &= \frac{1}{\mathcal{Q}_n(q)} \int_a^b \frac{\mathcal{Q}_n(q) - \mathcal{Q}_n(x)}{q-x} d\nu(x) \\
&= \frac{1}{\tilde{\kappa}^{(q)}} - \frac{1}{\mathcal{Q}_n(q)} \int_a^b \frac{\mathcal{Q}_n(x)}{q-x} d\nu(x) \\
&= \frac{1}{\tilde{\kappa}^{(q)}} - \frac{q^n}{\mathcal{Q}_n(q)} \sum_{j=0}^{\infty} \frac{1}{q^{2n+j+1}} \int_a^b x^{n+j} \mathcal{Q}_n(x) d\nu(x) \\
&= \frac{1}{\tilde{\kappa}^{(q)}} - \sum_{j=0}^{\infty} \frac{\nu_j^{(n)}}{q^{2n+j+1}}.
\end{aligned}$$

With the observation $\nu_0^{(n)} = \hbar_n$ we find

$$\frac{\dfrac{1}{\tilde{\kappa}^{(q)}} - (1-\epsilon)\hbar_0 \dfrac{\mathcal{Q}^*_{n+1}(q)}{\mathcal{Q}_{n+1}(q)}}{\dfrac{1}{\tilde{\kappa}^{(q)}} - (1-\epsilon)\hbar_0 \dfrac{\mathcal{Q}^*_{n}(q)}{\mathcal{Q}_{n}(q)}} = \frac{\epsilon \displaystyle\sum_{j=0}^{\infty} \frac{\nu_j}{q^{j+1}} + (1-\epsilon) \sum_{j=0}^{\infty} \frac{\nu_j^{(n+1)}}{q^{2n+j+3}}}{\epsilon \displaystyle\sum_{j=0}^{\infty} \frac{\nu_j}{q^{j+1}} + (1-\epsilon) \sum_{j=0}^{\infty} \frac{\nu_j^{(n)}}{q^{2n+j+1}}}.$$

From this, the second term in the product (2.32) is

$$\frac{\dfrac{1}{\tilde{\kappa}^{(q)}} - (1-\epsilon)\hbar_0 \dfrac{\mathcal{Q}^*_{n+1}(q)}{\mathcal{Q}_{n+1}(q)}}{\dfrac{1}{\tilde{\kappa}^{(q)}} - (1-\epsilon)\hbar_0 \dfrac{\mathcal{Q}^*_{n}(q)}{\mathcal{Q}_{n}(q)}} = \begin{cases} \mathfrak{d}_{n+2}\, q^{-2} + O\left(q^{-3}\right) & \text{if} \quad \epsilon = 0, \\ 1 + O\left(q^{-2n}\right) & \text{if} \quad \epsilon \neq 0. \end{cases}$$

For the first term in the product (2.32) we have

$$\frac{\mathcal{Q}_{n+1}(q)}{(q - \mathfrak{c}_{n+1})\mathcal{Q}_n(q)} = \frac{\mathcal{Q}_{n+1}(q)}{\mathcal{Q}_{n+1}(q) + \mathfrak{d}_{n+1}\mathcal{Q}_{n-1}(q)} = 1 - \mathfrak{d}_{n+1} q^{-2} + O\left(q^{-3}\right).$$

Multiplying these two terms gives the desired results. ■

We now look at some consequences of Theorem 2.11. Firstly, let us observe that by using $\mathfrak{c}_1 = \nu_1/\nu_0$ and $\hbar_1 = \nu_2 - \mathfrak{c}_1\nu_1$ we obtain

$$\frac{\tilde{\kappa}^{(q)}}{q} = \left[\sum_{j=0}^{\infty} \frac{\nu_j}{q^j}\right]^{-1} = \frac{1}{\hbar_0} - \frac{\mathfrak{c}_1}{\hbar_0} q^{-1} - \frac{\mathfrak{d}_2}{\hbar_0} q^{-2} + O\left(q^{-3}\right).$$

Then

$$\hbar_0 \tilde{\kappa}^{(q)} = q - \mathfrak{c}_1 - \mathfrak{d}_2 q^{-1} + O\left(q^{-2}\right). \tag{2.33}$$

The next results give the behavior of $\tilde{\mathfrak{b}}_n^{(q,\epsilon)}$ and $\tilde{\hbar}_n^{(q,\epsilon)}$ as $|q|$ tends to ∞.

Corollary 2.12. *If* $0 \le \epsilon < 1$, *then*

$$\tilde{\mathfrak{b}}_0^{(q,\epsilon)} = \epsilon q - \epsilon \mathfrak{c}_1 + (1-\epsilon)\mathfrak{d}_2 q^{-1} + O\left(q^{-2}\right) \quad \textit{as} \quad |q| \to \infty.$$

Let $n \ge 1$. *Then*

$$\left.\begin{aligned} \tilde{\mathfrak{b}}_n^{(q,0)} &= \mathfrak{d}_{n+2} q^{-1} + O\left(q^{-2}\right), \\ \tilde{\mathfrak{b}}_n^{(q,\epsilon)} &= q - \mathfrak{c}_{n+1} - \mathfrak{d}_{n+1} q^{-1} + O\left(q^{-2}\right), \quad 0 < \epsilon < 1, \end{aligned}\right\} \quad \textit{as} \quad |q| \to \infty.$$

Proof. By combining the initial conditions in (2.14) and Theorem 2.6 one finds

$$\begin{aligned} \tilde{\mathfrak{b}}_0^{(q,\epsilon)} = \tilde{\mathfrak{c}}_1^{(q,\epsilon)} - \mathfrak{c}_1 &= q - \tilde{\mathcal{Q}}_1^{(q,\epsilon)}(q) - \mathfrak{c}_1 \\ &= q - \mathfrak{c}_1 - (1-\epsilon)\hbar_0 \tilde{\kappa}^{(q)}. \end{aligned}$$

Hence, the result concerning $\tilde{\mathfrak{b}}_0^{(q,\epsilon)}$ follows from (2.33). Taking into account (2.26) together with Theorem 2.11 gives the remaining results. ■

Corollary 2.13. *If* $0 \leq \epsilon < 1$ *then*

$$\tilde{\hbar}_1^{(q,\epsilon)} = (1-\epsilon)\Big[\epsilon q^2 - 2\mathfrak{c}_1 \epsilon q + (1-2\epsilon)\mathfrak{d}_2 + \mathfrak{c}_1^2 \epsilon + O\left(q^{-1}\right)\Big] \quad as \quad |q| \to \infty.$$

Let $n \geq 1$. *Then*

$$\tilde{\hbar}_{n+1}^{(q,0)} = \frac{\hbar_{n+1}}{\hbar_0} + O\left(q^{-1}\right),$$

$$\tilde{\hbar}_{n+1}^{(q,\epsilon)} = \frac{(1-\epsilon)\hbar_n}{\hbar_0}\Big[q^2 - (\mathfrak{c}_1 + \mathfrak{c}_{n+1})q + \mathfrak{c}_1\mathfrak{c}_{n+1} - \mathfrak{d}_2 - \mathfrak{d}_{n+1} + O\Big(q^{-1}\Big)\Big], \quad 0 < \epsilon < 1,$$

as $|q| \to \infty$.

Proof. From (2.13) we can write

$$\tilde{\hbar}_{n+1}^{(q,\epsilon)} = (1-\epsilon)\hbar_n\, \tilde{\kappa}^{(q)}\, \tilde{\mathfrak{b}}_n^{(q,\epsilon)}, \quad n \geq 0.$$

Thus the results follow from Corollary 2.12 and (2.33). ■

2.3.3 Nevai Class: Asymptotics

Following [30], a positive measure ν is said to belong to the Nevai class $M[\mathfrak{c}, 2\sqrt{\mathfrak{d}}]$ if its corresponding sequence of MOP is such that $\mathfrak{c}_n \to \mathfrak{c}$ and $\mathfrak{d}_n \to \mathfrak{d}$. Then, in this case, from (2.2)

$$\lim_{n\to\infty} \alpha_n^{(q)} = \alpha^{(q)} = \frac{\mathfrak{d}}{(q-\mathfrak{c})^2}.$$

We recall from Theorem 2.5 that $\{\alpha_n^{(q)}\}_{n\geq 1}$ is a multiple parameter positive chain sequence with our assumption $\tilde{\kappa}^{(q)} \neq 0$. Thus, from Theorems 1.7 and 1.8 we obtain that $\alpha^{(q)} \leq 1/4$ and

$$\lim_{n\to\infty} g_n^{(q)}(0) = \frac{1}{2}\Big[1 + \sqrt{1-4\alpha^{(q)}}\Big],$$

$$\lim_{n\to\infty} g_n^{(q)}(\epsilon) = \lim_{n\to\infty} m_n^{(q)} = \frac{1}{2}\Big[1 - \sqrt{1-4\alpha^{(q)}}\Big], \quad 0 < \epsilon < 1.$$

On the other hand, from (2.26) we see that $|\tilde{\mathfrak{b}}_n^{(q,\epsilon)}|$ are bounded for any fixed $q \in \mathbb{R}\setminus(a,b)$. Thus, (2.26) combined with the convergence of both $g_n^{(q)}(\epsilon)$ and $\mathfrak{c}_n$ guarantees the existence of the limit

$$\tilde{\mathfrak{b}}^{(q,\epsilon)} = \lim_{n\to\infty} \tilde{\mathfrak{b}}_n^{(q,\epsilon)}.$$

Hence, from (2.14) we can conclude that $\tilde{\mathfrak{c}}_n^{(q,\epsilon)} \to \mathfrak{c}$ and $\tilde{\mathfrak{d}}_n^{(q,\epsilon)} \to \mathfrak{d}$. Then we can use these to confirm the well known fact that the measure $\tilde{\nu}^{(q,\epsilon)}$ also belongs to the Nevai class $M[\mathfrak{c}, 2\sqrt{\mathfrak{d}}]$ (see [8], [30]).

2.4 An Example

In this section we consider an application of Theorem 2.2 to obtain an example of a positive chain sequence whose all the parameter sequences can be explicitly given. Here, we assume that the positive measure ν is the Jacobi measure

$$d\nu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha, \beta > -1,$$

where $[a, b] = [-1, 1]$. Hence, the monic orthogonal polynomials $\mathcal{Q}_n$ are the monic Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}$ defined in Section 1.4.

Our aim here is to analyze the associated positive chain sequence $\{\alpha_n^{(q)}\}_{n\geq 1}$ in the case $q = 1$, which we denote by $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$. In other words,

$$a_n^{(1,\alpha,\beta)} = \frac{d_{n+1}^{(\alpha,\beta)}}{(1-c_n^{(\alpha,\beta)})(1-c_{n+1}^{(\alpha,\beta)})}, \quad n \geq 1, \tag{2.34}$$

where $\{c_n^{(\alpha,\beta)}\}_{n\geq 1}$ and $\{d_{n+1}^{(\alpha,\beta)}\}_{n\geq 1}$ are the coefficients appearing in (1.7).

Now, from (1.8) we write

$$\mathcal{Q}_0(1) = 1 \quad \text{and} \quad \mathcal{Q}_n(1) = \frac{2^n(\alpha+1)_n(\alpha+\beta+2)_{n-1}}{(\alpha+\beta+2)_{2n-1}}, \quad n \geq 1,$$

and hence the minimal parameter sequence $\{m_n^{(1,\alpha,\beta)}\}_{n\geq 0}$ of the positive chain sequence $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$ is such that $m_0^{(1,\alpha,\beta)} = 0$ and

$$m_n^{(1,\alpha,\beta)} = \frac{d_{n+1}^{(\alpha,\beta)}\mathcal{Q}_{n-1}(1)}{(1-c_{n+1}^{(\alpha,\beta)})\mathcal{Q}_n(1)} = \frac{1}{1-c_{n+1}^{(\alpha,\beta)}}\frac{2n(\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}, \quad n \geq 1,$$

where

$$1 - c_{n+1}^{(\alpha,\beta)} = \frac{2(\alpha-1)(\alpha+\beta)+4(n+1)(\alpha+\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)} \ > \ 0, \quad n \geq 0. \tag{2.35}$$

In order to make use of Theorem 2.2 to find all the remaining parameter sequences of $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$ it is necessary that $\tilde{\kappa}^{(1)} = \tilde{\kappa}^{(1,\alpha,\beta)}$ given by

$$\frac{1}{\tilde{\kappa}^{(1,\alpha,\beta)}} = \int_{-1}^{1}\frac{1}{1-x}(1-x)^{\alpha}(1+x)^{\beta}dx$$

exists. Thus, we must assume $\alpha > 0$ and we can state the following.

Theorem 2.14. *Let $\alpha > 0$ and $\beta > -1$. For any $\epsilon \in [0, 1]$, let*

$$g_0^{(1,\alpha,\beta)}(\epsilon) = \frac{1}{1-c_1^{(\alpha,\beta)}}\frac{2\alpha}{\alpha+\beta+1}(1-\epsilon) \qquad \textit{and}$$

$$g_n^{(1,\alpha,\beta)}(\epsilon) = \frac{1}{1-c_{n+1}^{(\alpha,\beta)}}\frac{2n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}$$
$$\times \frac{\left[(1-\epsilon)(n-1)!\,(\beta+1)_{n-1}+\epsilon(\alpha+1)_{n-1}(\alpha+\beta+1)_{n-1}\right]}{\left[(1-\epsilon)n!\,(\beta+1)_n+\epsilon(\alpha+1)_n(\alpha+\beta+1)_n\right]}, \quad n \geq 1.$$

Then $\{g_n^{(1,\alpha,\beta)}(\epsilon)\}_{n\geq 0}$ is a parameter sequence of the positive chain sequence $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$ given by (2.34).

Moreover, the following can be stated

(a) $\{g_n^{(1,\alpha,\beta)}(0)\}_{n\geq 0}$ *is the maximal parameter sequence of* $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$;

(b) $\{g_n^{(1,\alpha,\beta)}(1)\}_{n\geq 0}$ *coincides with the minimal parameter sequence* $\{m_n^{(1,\alpha,\beta)}\}_{n\geq 0}$.

Proof. By Theorem 2.2, for any $\epsilon \in [0,1)$, the sequence $\{g_n^{(1,\alpha,\beta)}(\epsilon)\}_{n\geq 0}$ is a parameter sequence of $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$ if

$$g_n^{(1,\alpha,\beta)}(\epsilon) = \frac{\widetilde{\mathcal{Q}}_{n+1}^{(1,\epsilon)}(1)}{(1-\mathfrak{c}_{n+1})\widetilde{\mathcal{Q}}_n^{(1,\epsilon)}(1)}, \quad n\geq 0,$$

where $\{\widetilde{\mathcal{Q}}_n^{(1,\epsilon)}\}_{n\geq 0}$ is the sequence of MOP with respect to the probability measure

$$\begin{aligned} d\tilde{\nu}^{(1,\epsilon)}(x) &= (1-\epsilon)\frac{\tilde{\kappa}^{(1,\alpha,\beta)}}{1-x}d\nu(x) \;+\; \epsilon\,\boldsymbol{\delta}_1 \\ &= (1-\epsilon)\tilde{\kappa}^{(1,\alpha,\beta)}(1-x)^{\alpha-1}(1+x)^{\beta}dx \;+\; \epsilon\,\boldsymbol{\delta}_1. \end{aligned}$$

Observe that

$$\frac{1}{\tilde{\kappa}^{(1,\alpha,\beta)}} = \int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta}dx = \frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Also, $\widetilde{\mathcal{Q}}_n^{(1,0)}(x)$ are the monic Jacobi polynomials $\mathcal{P}_n^{(\alpha-1,\beta)}(x)$. Thus, from (1.8)

$$\widetilde{\mathcal{Q}}_0^{(1,0)}(1) = 1 \quad\text{and}\quad \widetilde{\mathcal{Q}}_n^{(1,0)}(1) = \frac{2^n(\alpha)_n(\alpha+\beta+1)_{n-1}}{(\alpha+\beta+1)_{2n-1}}, \quad n\geq 1.$$

Therefore, the maximal parameter sequence of $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$ is

$$g_0^{(1,\alpha,\beta)}(0) = \frac{1}{1-c_1^{(\alpha,\beta)}}\frac{2\alpha}{\alpha+\beta+1} \quad\text{and}$$

$$g_n^{(1,\alpha,\beta)}(0) = \frac{1}{1-c_{n+1}^{(\alpha,\beta)}}\frac{2(\alpha+n)(\alpha+\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}, \quad n\geq 1.$$

Now, to find all the other parameter sequences of $\{a_n^{(1,\alpha,\beta)}\}_{n\geq 1}$, we identify the polynomials $\widetilde{\mathcal{Q}}_n^{(1,\epsilon)}(x)$ with the so-called Koornwinder polynomials. We remind that the study of the orthogonal polynomials with respect to the measure

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}dx \;+\; M\boldsymbol{\delta}_{-1} \;+\; N\boldsymbol{\delta}_1, \tag{2.36}$$

is attributed to Koornwinder [17]. If we denote by $\widehat{Q}_n^{(\alpha,\beta,M,N)}(x)$ the corresponding monic n-th degree orthogonal polynomial, then with the value of $\tilde{\kappa}^{(1,\alpha,\beta)}$ given above we can see that

$$\widetilde{\mathcal{Q}}_n^{(1,\epsilon)}(x) = \widehat{Q}_n^{(\alpha-1,\beta,0,\epsilon/(1-\epsilon))}(x), \quad n\geq 1.$$

Hence, using some results found in [15] and [17], we obtain

$$\widetilde{\mathcal{Q}}_n^{(1,\epsilon)}(1) = \frac{(1-\epsilon)2^n(n-1)!(\alpha)_n(\beta+1)_{n-1}(\alpha+\beta+1)_{n-1}}{(\alpha+\beta+1)_{2n-1}\Big[(1-\epsilon)(n-1)!(\beta+1)_{n-1}+\epsilon(\alpha+1)_{n-1}(\alpha+\beta+1)_{n-1}\Big]},$$

for $n\geq 1$. Then the results of the theorem follow from (2.35). ∎

Now we consider the special case in which $\alpha = \beta = \lambda + 1/2 > 0$. Here $c_n^{(\alpha,\beta)} = 0$ for $n \geq 1$ and we can state the following corollary.

Corollary 2.15. *For $\lambda > -1/2$, let $\{a_n^{(\lambda)}\}_{n\geq 1}$ be the positive chain sequence given by*

$$a_n^{(\lambda)} = \frac{n(2\lambda + n + 1)}{4(\lambda + n)(\lambda + n + 1)}, \quad n \geq 1.$$

For any $\epsilon \in [0, 1]$, let

$$g_0^{(\lambda)}(\epsilon) = \frac{2\lambda + 1}{2\lambda + 2}(1 - \epsilon) \quad \textit{and}$$

$$g_n^{(\lambda)}(\epsilon) = \frac{n(2\lambda + n + 1)}{2(\lambda + n + 1)} \frac{\big[(1 - \epsilon)(n - 1)! + \epsilon\,(2\lambda + 2)_{n-1}\big]}{\big[(1 - \epsilon)n! + \epsilon\,(2\lambda + 2)_n\big]}, \quad n \geq 1.$$

Then $\{g_n^{(\lambda)}(\epsilon)\}_{n\geq 0}$ is a parameter sequence of the positive chain sequence $\{a_n^{(\lambda)}\}_{n\geq 1}$.

We remark that the minimal parameter sequence $\{g_n^{(\lambda)}(1)\}_{n\geq 0}$ and the maximal parameter sequence $\{g_n^{(\lambda)}(0)\}_{n\geq 0}$ of $\{a_n^{(\lambda)}\}_{n\geq 1}$ are well known and have appeared, for example, in [7].

3 Sobolev Orthogonal Polynomials from CPPM2Ks on the Real Line

The aim of this chapter is to consider the orthogonal polynomials $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ with respect to the positive definite Sobolev-type inner product

$$\langle f, g\rangle_{\mathfrak{S}} = \int f(x)g(x)d\nu_0(x) + s\int f'(x)g'(x)d\nu_1(x), \quad s > 0,$$

where the pair of positive measures $\{\nu_0, \nu_1\}$ is a *coherent pair of positive measures of the second kind* (CPPM2K) on the real line. We study some properties of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ and establish the connection formulas they satisfy with the MOP corresponding to the measure ν_0. It is also shown that the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ are the eigenvalues of a matrix which is a single line modification of the well known $n \times n$ monic Jacobi matrix associated with the measure ν_0. The main results in this chapter were published in [18].

3.1 Introduction

Let $\{\nu_0, \nu_1\}$ be a pair of positive measures supported on the real line. Following [11], we say that $\{\nu_0, \nu_1\}$ is a *coherent pair of positive measures of the second kind on the real line* (CPPM2K on the real line, for short) if the corresponding sequences of MOP $\{\mathcal{P}_n(\nu_0; x)\}_{n\geq 0}$ and $\{\mathcal{P}_n(\nu_1; x)\}_{n\geq 0}$ satisfy

$$\frac{1}{n+1}\mathcal{P}'_{n+1}(\nu_0; x) = \mathcal{P}_n(\nu_1; x) - \tau_n \mathcal{P}_{n-1}(\nu_1; x), \quad \tau_n \neq 0, \quad n \geq 1. \tag{3.1}$$

It was shown in [11] that a pair of positive measures $\{\nu_0, \nu_1\}$ supported on the real line is a CPPM2K on the real line if and only if there exists an admissible pair of polynomials (A_3, A_2), with $\deg(A_3) \leq 3$ and $\deg(A_2) = 2$, such that

$$\begin{aligned} -\int p'(x)d\nu_1(x) &= \int p(x)A_2(x)d\nu_0(x) \quad \text{and} \\ \int p(x)d\nu_1(x) &= \int p(x)A_3(x)d\nu_0(x), \end{aligned} \tag{3.2}$$

for every real polynomial p. Recall that the admissibility condition of (A_3, A_2) holds if $\mathfrak{y}_2 + n\mathfrak{t}_3 \neq 0$ for $n \geq 1$, where $\mathfrak{y}_2$ is the leading coefficient of A_2 and $\mathfrak{t}_3$ is coefficient of degree 3 in A_3. The characterization formula (3.2) shows that ν_0 and ν_1 are semiclassical positive measures of class at most one.

Moreover, the constants τ_n in the coherence formula (3.1) satisfy

$$\tau_n = \tau_n(\nu_0, \nu_1) = \frac{\hbar_{n+1}^{\nu_0}}{(n+1)\hbar_{n-1}^{\nu_1}}[\mathfrak{y}_2 + (n-1)\mathfrak{t}_3], \quad n \geq 1,$$

where

$$\begin{aligned} \hbar_n^{\nu_0} &= \int \mathcal{P}_n(\nu_0; x)\mathcal{P}_n(\nu_0; x)d\nu_0(x), \\ \hbar_n^{\nu_1} &= \int \mathcal{P}_n(\nu_1; x)\mathcal{P}_n(\nu_1; x)d\nu_1(x), \end{aligned} \qquad n \geq 0.$$

We define the Sobolev-type inner product $\langle \cdot\, , \cdot \rangle_{\mathfrak{S}}$ as

$$\langle f, g \rangle_{\mathfrak{S}} = \int f(x)g(x)d\nu_0(x) \ + \ s\int f'(x)g'(x)d\nu_1(x), \quad s > 0, \tag{3.3}$$

where $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line. Let us denote by $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ the sequence of *monic Sobolev orthogonal polynomials* (MSOP for short) with respect to the inner product $\langle \cdot\, , \cdot \rangle_{\mathfrak{S}}$. Clearly, the positive definiteness of the inner product $\langle \cdot\, , \cdot \rangle_{\mathfrak{S}}$ assures the existence of the sequence $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$.

Finally, we remark that the inner product $\langle \cdot\, , \cdot \rangle_{\mathfrak{S}}$ can be rewritten as

$$\langle f, g \rangle_{\mathfrak{S}} = \langle f, g \rangle_{\nu_0} + s\langle f', g' \rangle_{\nu_1}, \quad s > 0,$$

where $\langle f, g \rangle_{\nu_0} = \int f(x)g(x)d\nu_0(x)$ and $\langle f, g \rangle_{\nu_1} = \int f(x)g(x)d\nu_1(x)$.

The outline of the chapter is as follows. In Section 3.2 we deduce a preliminary result concerning $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ as a linear combination of $\{\mathcal{P}_j(\nu_0; x)\}_{j=0}^n$. In Section 3.3 we prove that two consecutive Sobolev polynomials can be written in terms of a orthogonal polynomial corresponding to the measure ν_0 using a simple connection formula. Further results about the connection coefficients are also explored. In Section 3.4 we deal with the connection coefficients in the special case they are assumed to be rational functions.

Finally, in Section 3.5 we show that the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ are the eigenvalues of a matrix which is a simple modification of the well known $n \times n$ Jacobi matrix associated with the measure ν_0. Taking into account that the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ can be complex, we provide strips within which these zeros lie.

3.2 A Preliminary Result

We start this section by defining the sequences of elements $\{\mathfrak{p}_n\}_{n\geq 1}$ and $\{\mathfrak{q}_n\}_{n\geq 1}$, which will play an important role in the sequel. Let

$$\begin{aligned} \mathfrak{q}_{n+1} &= s\, n(n+1)\hbar_{n-1}^{\nu_1}\tau_n, \\ \mathfrak{p}_{n+1} &= \hbar_{n+1}^{\nu_0} + s\,(n+1)^2[\hbar_n^{\nu_1} + \hbar_{n-1}^{\nu_1}\tau_n^2], \end{aligned} \qquad n \geq 1, \tag{3.4}$$

with $\mathfrak{q}_1 = 0$ and $\mathfrak{p}_1 = \hbar_1^{\nu_0} + s\hbar_0^{\nu_1}$.

Now we consider the expression for $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ as a linear combination of the polynomials $\{\mathcal{P}_j(\nu_0; x)\}_{j=0}^n$. The next result follows analogously from a result shown on the unit circle (see [21]). For the sake of completeness, we also provide its proof.

Theorem 3.1. *Let $\{\nu_0, \nu_1\}$ be a CPPM2K on the real line. Then the sequence of MSOP $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$ is such that*

$$\mathcal{S}_0(\nu_0, \nu_1, s; x) = 1, \quad \mathcal{S}_1(\nu_0, \nu_1, s; x) = \mathcal{P}_1(\nu_0; x) \quad \text{and}$$

$$\mathcal{S}_{n+1}(\nu_0, \nu_1, s; x) = \mathcal{P}_{n+1}(\nu_0; x) \ + \ \sum_{j=1}^{n} \mathfrak{a}_{n+1,j} \mathcal{P}_j(\nu_0; x), \quad n \geq 1.$$

Here, $\mathbf{a}_n = [\mathfrak{a}_{n+1,1}, \mathfrak{a}_{n+1,2}, \ldots, \mathfrak{a}_{n+1,n}]^\top$ is the solution of the system of linear equations

$$\mathbf{T}_n\, \mathbf{a}_n = \mathfrak{q}_{n+1}\, \mathbf{e}_n,$$

where $\mathbf{e}_n$ is the n-th column of the $n\times n$ identity matrix and $\mathbf{T}_n$ is the $n\times n$ real symmetric tridiagonal matrix

$$\mathbf{T}_n = \begin{bmatrix} \mathfrak{p}_1 & -\mathfrak{q}_2 & & & \\ -\mathfrak{q}_2 & \mathfrak{p}_2 & -\mathfrak{q}_3 & & \\ \ddots & & \ddots & \ddots & \\ & & -\mathfrak{q}_{n-1} & \mathfrak{p}_{n-1} & -\mathfrak{q}_n \\ & & & -\mathfrak{q}_n & \mathfrak{p}_n \end{bmatrix}.$$

Proof. If we write

$$\mathcal{S}_{n+1}(\nu_0, \nu_1, s; x) = \sum_{j=0}^{n+1} \mathfrak{a}_{n+1,j} \mathcal{P}_j(\nu_0; x) \quad \text{with} \quad \mathfrak{a}_{n+1,n+1} = 1,$$

then by considering the orthogonality of $\mathcal{S}_{n+1}(\nu_0, \nu_1, s; x)$ we can deduce

$$\begin{aligned} 0 &= \langle \mathcal{P}_k(\nu_0; \cdot), \mathcal{S}_{n+1}(\nu_0, \nu_1, s; \cdot) \rangle_{\mathfrak{S}} \\ &= \langle \mathcal{P}_k(\nu_0; \cdot), \sum_{j=0}^{n+1} \mathfrak{a}_{n+1,j} \mathcal{P}_j(\nu_0; \cdot) \rangle_{\nu_0} + s\, \langle \mathcal{P}'_k(\nu_0; \cdot), \sum_{j=0}^{n+1} \mathfrak{a}_{n+1,j} \mathcal{P}'_j(\nu_0; \cdot) \rangle_{\nu_1} \\ &= \hbar_k^{\nu_0} \mathfrak{a}_{n+1,k} + s\, \langle \mathcal{P}'_k(\nu_0; \cdot), \sum_{j=0}^{n+1} \mathfrak{a}_{n+1,j} \mathcal{P}'_j(\nu_0; \cdot) \rangle_{\nu_1} \end{aligned} \tag{3.5}$$

for $n \geq 0$ and $k = 0, 1, \ldots, n$. From this, with the observation $\mathcal{P}'_0(\nu_0; x) = 0$, $\mathcal{P}'_1(\nu_0; x) = 1$ and $\mathcal{P}'_2(\nu_0; x) = 2[\mathcal{P}_1(\nu_1; x) - \tau_1]$ we find

$$\mathfrak{a}_{n+1,0} = 0 \quad \text{and} \quad [\hbar_1^{\nu_0} + s\, \hbar_0^{\nu_1}]\mathfrak{a}_{n+1,1} - 2s\hbar_0^{\nu_1} \tau_1 \mathfrak{a}_{n+1,2} = 0 \quad \text{for} \quad n \geq 0.$$

Substituting (3.1) into (3.5) gives

$$\begin{aligned} 0 = &-s\,(k-1)k\tau_{k-1} \hbar_{k-2}^{\nu_1} \mathfrak{a}_{n+1,k-1} \\ &+ \Big[\hbar_k^{\nu_0} + sk^2 \left(\hbar_{k-1}^{\nu_1} + \tau_{k-1}^2\, \hbar_{k-2}^{\nu_1}\right)\Big] \mathfrak{a}_{n+1,k} - \ s\, k(k+1)\tau_k \hbar_{k-1}^{\nu_1} \mathfrak{a}_{n+1,k+1}, \end{aligned}$$

for $k = 2, 3, \ldots, n$. Hence, since $\mathfrak{a}_{n+1,n+1} = 1$, we obtain the required system of linear equations. This completes the proof of the theorem. ■

Later in this chapter we show that the matrices $\mathbf{T}_n$ are positive definite. This fact guarantees the existence of a unique solution for the systems $\mathbf{T}_n\, \mathbf{a}_n = \mathfrak{q}_{n+1}\, \mathbf{e}_n$. Consequently, one can use this to verify the existence of the sequence of MSOP $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{S}}$.

3.3 The Simple Connection Formula

As we have mentioned in the introduction, one of the most important characteristics of the polynomials $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ is the simple connection formulas they satisfy. The following result states that two consecutive Sobolev polynomials are connected to an orthogonal polynomial corresponding to the measure ν_0.

Theorem 3.2. *Let $\{\nu_0, \nu_1\}$ be a CPPM2K on the real line. Then the sequence of MSOP $\{\mathcal{S}_n(\nu_0, \nu_1, s; x)\}_{n\geq 0}$ with respect to the inner product $\langle \cdot\,, \cdot\rangle_{\mathfrak{S}}$ satisfies*

$$\mathcal{S}_n(\nu_0, \nu_1, s; x) - \gamma_{n-1}\,\mathcal{S}_{n-1}(\nu_0, \nu_1, s; x) = \mathcal{P}_n(\nu_0; x), \quad n \geq 1, \tag{3.6}$$

where $\mathcal{S}_0(\nu_0, \nu_1, s; x) = 1$ and $\gamma_n = \gamma_n(\nu_0, \nu_1, s)$ are such that

$$\gamma_0 = 0 \qquad \textit{and} \qquad \gamma_n = \frac{\mathfrak{q}_{n+1}}{\mathfrak{p}_n - \mathfrak{q}_n\,\gamma_{n-1}}, \quad n \geq 1. \tag{3.7}$$

Here, the elements $\mathfrak{p}_n$ and $\mathfrak{q}_n$ are as in (3.4).

Proof. Let us consider the expansion

$$\mathcal{P}_n(\nu_0; x) = \mathcal{S}_n(\nu_0, \nu_1, s; x) + \sum_{j=0}^{n-1} \lambda_{n,j}\mathcal{S}_j(\nu_0, \nu_1, s; x),$$

where $\mathcal{S}_0(\nu_0, \nu_1, s; x) = 1$, $\mathcal{S}_1(\nu_0, \nu_1, s; x) = \mathcal{P}_1(\nu_0; x)$ and

$$\lambda_{n,j} = \frac{\langle \mathcal{P}_n(\nu_0;\cdot), \mathcal{S}_j(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}}}{\langle \mathcal{S}_j(\nu_0, \nu_1, s;\cdot), \mathcal{S}_j(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}}} = \frac{s\langle \mathcal{P}_n'(\nu_0;\cdot), \mathcal{S}_j'(\nu_0, \nu_1, s;\cdot)\rangle_{\nu_1}}{\langle \mathcal{S}_j(\nu_0, \nu_1, s;\cdot), \mathcal{S}_j(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}}}$$

for $n \geq 2$ and $j = 0, 1, \ldots, n-1$. Hence, substituting for $\mathcal{P}_n'(\nu_0; x)$ using (3.1) we get $\lambda_{n,j} = 0$ for $n \geq 2$ and $j = 0, 1, \ldots, n-2$. Moreover,

$$-\gamma_{n-1} = \lambda_{n,n-1} = \frac{s\langle \mathcal{P}_n'(\nu_0;\cdot), \mathcal{S}_{n-1}'(\nu_0, \nu_1, s;\cdot)\rangle_{\nu_1}}{\langle \mathcal{S}_{n-1}(\nu_0, \nu_1, s;\cdot), \mathcal{S}_{n-1}(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}}}, \quad n \geq 2.$$

Thus,

$$\gamma_0 = 0 \qquad \text{and} \qquad \gamma_n = \frac{\mathfrak{q}_{n+1}}{\langle \mathcal{S}_n(\nu_0, \nu_1, s;\cdot), \mathcal{S}_n(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}}}, \quad n \geq 1. \tag{3.8}$$

With the observation $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ is monic, we have

$$\begin{aligned}\langle \mathcal{S}_n(\nu_0, \nu_1, s;\cdot), \mathcal{S}_n(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}} &= \langle \mathcal{S}_n(\nu_0, \nu_1, s;\cdot), \mathcal{P}_n(\nu_0;\cdot)\rangle_{\mathfrak{S}} \\ &= \hbar_n^{\nu_0} + s\,\langle \mathcal{S}_n'(\nu_0, \nu_1, s;\cdot), \mathcal{P}_n'(\nu_0;\cdot)\rangle_{\nu_1}.\end{aligned}$$

Thus, from $\mathcal{S}_n'(\nu_0, \nu_1, s; x) = \mathcal{P}_n'(\nu_0; x) + \gamma_{n-1}\,\mathcal{S}_{n-1}'(\nu_0, \nu_1, s; x)$ and (3.1),

$$\begin{aligned}&\langle \mathcal{S}_n(\nu_0, \nu_1, s;\cdot), \mathcal{S}_n(\nu_0, \nu_1, s;\cdot)\rangle_{\mathfrak{S}} \\ &\qquad = \hbar_n^{\nu_0} + s\left[n^2\hbar_{n-1}^{\nu_1} + n^2\tau_{n-1}^2\hbar_{n-2}^{\nu_1}\right] - s\,(n-1)n\tau_{n-1}\hbar_{n-2}^{\nu_1}\gamma_{n-1} \qquad (3.9) \\ &\qquad = \mathfrak{p}_n - \mathfrak{q}_n\,\gamma_{n-1},\end{aligned}$$

which gives the statement of the theorem. ∎

Comparing the results in Theorems 3.1 and 3.2 we can write

$$\mathcal{S}_n(\nu_0,\nu_1,s;x) = \mathcal{P}_n(\nu_0;x) \ + \sum_{j=1}^{n-1} \mathfrak{a}_{n,j}\mathcal{P}_j(\nu_0;x), \quad n \geq 2, \tag{3.10}$$

where the elements $\mathfrak{a}_{n,j}$ appearing in Theorem 3.1 satisfy

$$\mathfrak{a}_{n,n-1} = \gamma_{n-1} \quad \text{and}$$
$$\mathfrak{a}_{n,j} = \gamma_j\, \mathfrak{a}_{n,j+1} = \gamma_j\, \gamma_{j+1}\, \ldots \gamma_{n-1}, \quad j = 1,2,\ldots,n-2.$$

The next theorem gives an upper bound for the ratio γ_n/τ_n.

Theorem 3.3. *The connection coefficients $\gamma_n = \gamma_n(\nu_0,\nu_1,s)$ that appear in* (3.6) *satisfy*

$$0 \ < \ \frac{\gamma_n(\nu_0,\nu_1,s)}{\tau_n(\nu_0,\nu_1)} \ < \ \frac{n+1}{n}, \quad n \geq 1. \tag{3.11}$$

Proof. Observe that (3.8) gives $\mathfrak{q}_{n+1}\gamma_n > 0$ for $n \geq 1$. Hence, from (3.4)

$$\tau_n\gamma_n > 0, \quad n \geq 1.$$

Now by combining (3.3) with the minimal norm properties of MOP, one finds

$$\langle \mathcal{S}_n(\nu_0,\nu_1,s;\cdot), \mathcal{S}_n(\nu_0,\nu_1,s;\cdot)\rangle_{\mathfrak{S}} \ >$$
$$\langle \mathcal{P}_n(\nu_0;\cdot), \mathcal{P}_n(\nu_0;\cdot)\rangle_{\nu_0} + sn^2\, \langle \mathcal{P}_{n-1}(\nu_1;\cdot), \mathcal{P}_{n-1}(\nu_1;\cdot)\rangle_{\nu_1}, \quad n \geq 1.$$

Then

$$\langle \mathcal{S}_n(\nu_0,\nu_1,s;\cdot), \mathcal{S}_n(\nu_0,\nu_1,s;\cdot)\rangle_{\mathfrak{S}} \ > \ \mathfrak{h}_n^{\nu_0} + sn^2\mathfrak{h}_{n-1}^{\nu_1}, \quad n \geq 1.$$

Thus, the desired inequality follows from (3.9). ■

Now, let us consider the sequences $\{\mathfrak{m}_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 1}$ which we define by

$$\mathfrak{m}_{n-1} = \frac{\mathfrak{q}_n\gamma_{n-1}}{\mathfrak{p}_n} \quad \text{and} \quad \sigma_n = \frac{\mathfrak{q}_{n+1}^2}{\mathfrak{p}_n\mathfrak{p}_{n+1}}, \quad n \geq 1. \tag{3.12}$$

Here, $\mathfrak{p}_n$ and $\mathfrak{q}_n$ are as in (3.4). It is easily verified from (3.7) that

$$\sigma_n = (1 - \mathfrak{m}_{n-1})\mathfrak{m}_n, \quad n \geq 1. \tag{3.13}$$

Thus, from the knowledge that $\mathfrak{m}_0 = 0$, $\mathfrak{m}_n > 0$ and $\sigma_n > 0$ for $n \geq 1$, there also holds $0 < \mathfrak{m}_n < 1$ for $n \geq 1$. Hence, the sequence $\{\sigma_n\}_{n\geq 1}$ is a positive chain sequence and $\{\mathfrak{m}_n\}_{n\geq 0}$ is its minimal parameter sequence.

By considering the tridiagonal matrices $\mathbf{T}_n$ in Theorem 3.1, we easily find that the respective monic characteristic polynomials $C_n(z)$ are such that

$$C_n(z) = (z - \mathfrak{p}_n)C_{n-1}(z) - \mathfrak{q}_n^2 C_{n-2}(z), \quad n \geq 2,$$

with $C_1(z) = z - \mathfrak{p}_1$ and $C_0(z) = 1$. Moreover, from $C_n(0) = (-1)^n det(\mathbf{T}_n)$,

$$det(\mathbf{T}_n) = \mathfrak{p}_n det(\mathbf{T}_{n-1}) - \mathfrak{q}_n^2 det(\mathbf{T}_{n-2}), \quad n \geq 2,$$

with $det(\mathbf{T}_0) = 1$ and $det(\mathbf{T}_1) = \mathfrak{p}_1$. From this,

$$\frac{det(\mathbf{T}_n)}{\mathfrak{p}_n det(\mathbf{T}_{n-1})}\left[1 - \frac{det(\mathbf{T}_{n+1})}{\mathfrak{p}_{n+1} det(\mathbf{T}_n)}\right] = \frac{\mathfrak{q}_{n+1}^2}{\mathfrak{p}_n \mathfrak{p}_{n+1}}, \quad n \geq 1.$$

Thus, comparing this with (3.13) there follows

$$0 \ < \ \frac{det(\mathbf{T}_n)}{\mathfrak{p}_n det(\mathbf{T}_{n-1})} = 1 - \mathfrak{m}_{n-1} \ < \ 1, \quad n \geq 2. \tag{3.14}$$

From (3.14) the principal minors $det(\mathbf{T}_k)$, $k = 1, 2, \ldots, n$ of the matrix $\mathbf{T}_n$ are all positive. Hence, by the Sylvester's criterion the matrix $\mathbf{T}_n$ is positive definite. We can therefore consider the Cholesky decomposition $\mathbf{G}_n^\top \mathbf{G}_n = \mathbf{T}_n$, where $\mathbf{G}_n$ is an upper triangular 2-bandwidth matrix with positive diagonal entries.

Theorem 3.4. *For $n \geq 2$, let $\mathbf{G}_n^\top \mathbf{G}_n$ be the Cholesky decomposition of $\mathbf{T}_n$. If*

$$\mathbf{G}_n = \begin{bmatrix} \mathfrak{g}_{1,1} & \mathfrak{g}_{1,2} & 0 & \ldots & 0 & 0 \\ 0 & \mathfrak{g}_{2,2} & \mathfrak{g}_{2,3} & \ldots & 0 & 0 \\ 0 & 0 & \mathfrak{g}_{3,3} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathfrak{g}_{n-1,n-1} & \mathfrak{g}_{n-1,n} \\ 0 & 0 & 0 & \ldots & 0 & \mathfrak{g}_{n,n} \end{bmatrix},$$

then $\mathfrak{g}_{1,1} = \sqrt{\mathfrak{p}_1}\sqrt{1 - \mathfrak{m}_0} = \sqrt{\mathfrak{p}_1}$ *and*

$$\begin{aligned} \mathfrak{g}_{k,k+1} &= -\frac{\mathfrak{q}_{k+1}}{\sqrt{\mathfrak{p}_k}\sqrt{1 - \mathfrak{m}_{k-1}}}, \\ \mathfrak{g}_{k+1,k+1} &= \sqrt{\mathfrak{p}_{k+1}}\sqrt{1 - \mathfrak{m}_k}, \end{aligned} \qquad k = 1, 2, \ldots n-1.$$

Moreover,

$$\gamma_n = \frac{\mathfrak{q}_{n+1}}{(1 - \mathfrak{m}_{n-1})\mathfrak{p}_n}, \quad n \geq 1.$$

Proof. By equating the corresponding nonzero entries in successive columns of $\mathbf{T}_n$ and $\mathbf{G}_n^\top \mathbf{G}_n$, one finds $\mathfrak{g}_{1,1}^2 = \mathfrak{p}_1$,

$$\begin{aligned} &\mathfrak{g}_{k,k}\, \mathfrak{g}_{k,k+1} = -\mathfrak{q}_{k+1}, \\ &\mathfrak{g}_{k,k+1}^2 + \mathfrak{g}_{k+1,k+1}^2 = \mathfrak{p}_{k+1}, \qquad k = 1, 2, \ldots, n-2, \\ &\mathfrak{g}_{k,k+1}\, \mathfrak{g}_{k,k} = -\mathfrak{q}_{k+1}, \end{aligned} \tag{3.15}$$

with $\mathfrak{g}_{n-1,n-1}\, \mathfrak{g}_{n-1,n} = -\mathfrak{q}_n$ and $\mathfrak{g}_{n-1,n}^2 + \mathfrak{g}_{n,n}^2 = \mathfrak{p}_n$.

Clearly, the formula in the theorem holds for $\mathfrak{g}_{1,1}$. To obtain the remaining elements we assume $\mathfrak{g}_{k,k} = \sqrt{\mathfrak{p}_k}\sqrt{1 - \mathfrak{m}_{k-1}}$ and get the required expressions for $\mathfrak{g}_{k,k+1}$ and $\mathfrak{g}_{k+1,k+1}$. From (3.15) we have

$$\mathfrak{g}_{k,k+1} = -\frac{\mathfrak{q}_{k+1}}{\mathfrak{g}_{k,k}} \qquad \text{and} \qquad \mathfrak{g}_{k+1,k+1}^2 = \mathfrak{p}_{k+1} - \mathfrak{g}_{k,k+1}^2 = \mathfrak{p}_{k+1} - \frac{\mathfrak{q}_{k+1}^2}{\mathfrak{g}_{k,k}^2}.$$

Thus, the expression for $\mathfrak{g}_{k,k+1}$ immediately follows. To obtain the latter part we observe that

$$\mathfrak{g}^2_{k+1,k+1} = \mathfrak{p}_{k+1} - \mathfrak{p}_{k+1}\frac{\sigma_k}{(1-\mathfrak{m}_{k-1})} = \mathfrak{p}_{k+1} - \mathfrak{p}_{k+1}\,\mathfrak{m}_k,$$

where σ_k are given by (3.12). Hence, this confirms the expression for $\mathfrak{g}_{k+1,k+1}$.

Now to obtain the expression for γ_n, we consider the two $n \times n$ systems of linear equations $\mathbf{G}_n^\top \mathbf{u}_n = \mathfrak{q}_{n+1}\,\mathbf{e}_n$ and $\mathbf{G}_n\mathbf{a}_n = \mathbf{u}_n$ that follow from $\mathbf{T}_n\,\mathbf{a}_n = \mathfrak{q}_{n+1}\,\mathbf{e}_n$. Here, we set $\mathbf{u}_n = [u_1, u_2, \ldots, u_n]^\top$. From the system $\mathbf{G}_n^\top\mathbf{u}_n = \mathfrak{q}_{n+1}\,\mathbf{e}_n$, we first obtain

$$u_1 = 0,\ u_2 = 0, \ldots, u_{n-1} = 0 \qquad \text{and} \qquad u_n = \frac{\mathfrak{q}_{n+1}}{\mathfrak{g}_{n,n}}.$$

Now computing the value of $\mathfrak{a}_{n+1,n}$ from the system $\mathbf{G}_n\mathbf{a}_n = \mathbf{u}_n$, we find

$$\gamma_n = \mathfrak{a}_{n+1,n} = \frac{u_n}{\mathfrak{g}_{n,n}} = \frac{\mathfrak{q}_{n+1}}{\mathfrak{g}^2_{n,n}}.$$

This completes the proof of the theorem. ∎

3.4 The Coefficients γ_n as Rational Functions

The connection coefficients $\gamma_n = \gamma_n(\nu_0, \nu_1, s)$ can be given as rational functions which are related to a sequence of MOP satisfying a simple TTRR. Firstly, let us assume the connection coefficients $\gamma_n = \gamma_n(\nu_0, \nu_1, s)$ that appear in (3.6), which depend on s, to be such that

$$\gamma_n = \frac{r_{n-1}(t)}{r_n(t)}, \quad n \geq 1, \tag{3.16}$$

where $r_0(t) = 1$ and $t = \lambda/s$. Here, we also assume $\lambda \neq 0$ to be an arbitrary real number.

Then from (3.7) one can write

$$\mathfrak{q}_{n+1}r_n(t) = \mathfrak{p}_n r_{n-1}(t) - \mathfrak{q}_n r_{n-2}(t), \quad n \geq 2, \tag{3.17}$$

with $\mathfrak{q}_2 r_1(t) = \mathfrak{p}_1 r_0(t)$. Thus, with the use of (3.4) we can state the following theorem.

Theorem 3.5. *Let $\mathcal{R}_0(t) = r_0(t)$ and $\mathcal{R}_n(t) = \lambda^n \Delta_n r_n(t)$, $n \geq 1$, where*

$$\Delta_0 = 1 \quad \textit{and} \quad \Delta_n = n(n+1)\tau_n \frac{\hbar^{\nu_1}_{n-1}}{\hbar^{\nu_0}_n}\Delta_{n-1}, \quad n \geq 1.$$

Then $\{\mathcal{R}_n\}_{n\geq 0}$ is a sequence of monic polynomials satisfying the TTRR

$$\mathcal{R}_{n+1}(t) = [t + (e_{n+1} + f_{n+1})\lambda]\mathcal{R}_n(t) - e_n f_{n+1}\lambda^2\mathcal{R}_{n-1}(t), \quad n \geq 1,$$

with $\mathcal{R}_0(t) = 1$ and $\mathcal{R}_1(t) = t + (e_1 + f_1)\lambda$, where $f_1 = 0$,

$$e_n = \frac{n^2\hbar^{\nu_1}_{n-1}}{\hbar^{\nu_0}_n} \qquad \textit{and} \qquad f_{n+1} = \frac{(n+1)^2\tau_n^2\hbar^{\nu_1}_{n-1}}{\hbar^{\nu_0}_{n+1}}, \quad n \geq 1.$$

Proof. Substitution of the formulas from (3.4) into (3.17) gives

$$\frac{t\mathfrak{q}_2}{\hbar_1^{\nu_0}} r_1(t) = \Big[t + \frac{\lambda \hbar_0^{\nu_1}}{\hbar_1^{\nu_0}}\Big] r_0(t),$$

$$\frac{t\mathfrak{q}_{n+1}}{\hbar_n^{\nu_0}} r_n(t) = \Big[t + \frac{\lambda n^2}{\hbar_n^{\nu_0}}\big(\hbar_{n-1}^{\nu_1} + \tau_{n-2}^2 \hbar_{n-2}^{\nu_1}\big)\Big] r_{n-1}(t) - \frac{\hbar_{n-1}^{\nu_0}}{\hbar_n^{\nu_0}} \frac{t\mathfrak{q}_n}{\hbar_{n-1}^{\nu_0}} r_{n-2}(t), \quad n \geq 2.$$

Since $t\mathfrak{q}_{n+1}$ for $n \geq 1$ are independent of t and

$$\lambda^n \Delta_n = \prod_{j=1}^{n} \frac{t\mathfrak{q}_{j+1}}{\hbar_j^{\nu_0}}, \quad n \geq 1,$$

we obtain the required recurrence relation for $\{\mathcal{R}_n\}_{n\geq 0}$. ■

With the observation $e_n f_{n+1} > 0$ for $n \geq 1$, by using Favard's Theorem 1.20 we can conclude that $\{\mathcal{R}_n\}_{n\geq 0}$ constitutes a sequence of MOP with respect to some positive measure ν supported on the real line.

The next result gives information on the location of the zeros of $\mathcal{R}_n$.

Theorem 3.6. *If $\lambda < 0$ (resp. $\lambda > 0$) then $\{\mathcal{R}_n\}_{n\geq 0}$ is a sequence of MOP with respect to a positive measure ν supported on the positive half (resp. negative half) of the real line.*

Proof. From the recurrence relation in Theorem 3.5 we have

$$[1 - \ell_{n-1}(t)]\ell_n(t) = \varrho_n(t) = \frac{e_n f_{n+1} \lambda^2}{[t + (e_n + f_n)\lambda][t + (e_{n+1} + f_{n+1})\lambda]}, \quad n \geq 1,$$

where $\ell_0(t) = 0$ and

$$1 - \ell_n(t) = \frac{\mathcal{R}_{n+1}(t)}{[t + (e_{n+1} + f_{n+1})\lambda]\mathcal{R}_n(t)}, \quad n \geq 1. \tag{3.18}$$

It is easy to show that $\{\varrho_n(0)\}_{n\geq 1}$ is a positive chain sequence where its minimal parameter sequence $\{\ell_n(0)\}_{n\geq 0}$ is such that

$$\frac{\mathcal{R}_{n+1}(0)}{(e_{n+1} + f_{n+1})\lambda\, \mathcal{R}_n(0)} = 1 - \ell_n(0) = \frac{e_{n+1}}{e_{n+1} + f_{n+1}}, \quad n \geq 0.$$

Thus, if t is such that $\lambda/t = s > 0$, then $\varrho_n(t) < \varrho_n(0)$, $n \geq 1$. In this case, by using Theorems 1.5 and 1.6 we obtain that the sequence $\{\varrho_n(t)\}_{n\geq 1}$ is also a positive chain sequence and its minimal parameter sequence $\{\ell_n(t)\}_{n\geq 0}$ is such that $\ell_n(t) < \ell_n(0)$ for $n \geq 1$. Therefore,

$$\frac{\mathcal{R}_{n+1}(t)}{[t + (e_{n+1} + f_{n+1})\lambda]\mathcal{R}_n(t)} = 1 - \ell_n(t) \ > \ 1 - \ell_n(0) = \frac{e_{n+1}}{e_{n+1} + f_{n+1}}, \quad n \geq 1. \tag{3.19}$$

From this, for t such that $\lambda/t = s > 0$, we find $\mathrm{sgn}(\lambda)\mathcal{R}_n(t)\mathcal{R}_{n+1}(t) > 0$ for $n \geq 1$. Thus, it is easily seen that if $\lambda < 0$ ($\lambda > 0$), then $\mathcal{R}_n$ has only positive (resp. negative) zeros. ■

The following theorem gives a monotonicity property associated with $\gamma_n(\nu_0, \nu_1, s)$.

Theorem 3.7. *Let $\gamma_n = \gamma_n(\nu_0, \nu_1, s)$ be the connection coefficients that appear in* (3.6)*. Then $|\gamma_n(\nu_0, \nu_1, s)|$ is a strictly increasing function of s for each $n \geq 1$.*

Proof. From (3.16) and Theorem 3.5, we have for $\lambda/t = s > 0$,

$$|\gamma_n| = \frac{|\Delta_n|}{|\Delta_{n-1}|} \frac{\lambda \mathcal{R}_{n-1}(t)}{\mathcal{R}_n(t)}, \quad n \geq 1.$$

Thus,

$$\frac{d}{dt}|\gamma_n| = -\frac{|\Delta_n|\lambda}{|\Delta_{n-1}|} \frac{\mathcal{R}'_n(t)\mathcal{R}_{n-1}(t) - \mathcal{R}_n(t)\mathcal{R}'_{n-1}(t)}{[\mathcal{R}_n(t)]^2}.$$

Observe that, from Theorem 1.21, $\mathcal{R}'_n(t)\mathcal{R}_{n-1}(t) - \mathcal{R}_n(t)\mathcal{R}'_{n-1}(t) > 0$ for $t \in \mathbb{R}$. Hence, we conclude the result of the theorem. ∎

The next result, when compared to (3.11), gives an improved upper bound for the ratio γ_n/τ_n.

Theorem 3.8. *Let $\gamma_n(\nu_0, \nu_1, s)$ be the connection coefficients in* (3.6) *and let $\tau_n(\nu_0, \nu_1)$ be the coefficients in the coherence formula* (3.1)*. Then, for $n \geq 1$,*

$$\frac{\gamma_n(\nu_0, \nu_1, s)}{\tau_n(\nu_0, \nu_1)} < \frac{n+1}{n} \frac{s(e_n + f_n)}{1 + s(e_n + f_n)} \quad \text{and} \quad \lim_{s\to\infty} \frac{\gamma_n(\nu_0, \nu_1, s)}{\tau_n(\nu_0, \nu_1)} = \frac{n+1}{n}.$$

Proof. Firstly, observe that

$$\frac{\Delta_n}{\Delta_{n-1}} = \frac{n+1}{n}\tau_n e_n, \quad n \geq 1.$$

Hence,

$$\gamma_n = \frac{\Delta_n}{\Delta_{n-1}} \frac{\lambda \mathcal{R}_{n-1}(t)}{\mathcal{R}_n(t)} = \frac{(n+1)\tau_n \lambda e_n}{n} \frac{\mathcal{R}_{n-1}(t)}{\mathcal{R}_n(t)}, \quad n \geq 1.$$

Now using (3.18) to express $\mathcal{R}_{n-1}(t)/\mathcal{R}_n(t)$, we get

$$\frac{\gamma_n}{\tau_n} = \frac{n+1}{n} \frac{\lambda e_n}{[t + (e_n + f_n)\lambda][1 - \ell_{n-1}(t)]}, \quad n \geq 1.$$

Therefore, from (3.19),

$$\frac{\gamma_n}{\tau_n} < \frac{n+1}{n} \frac{\lambda e_n}{[t + (e_n + f_n)\lambda][1 - \ell_{n-1}(0)]}, \quad n \geq 1,$$

for $s = \lambda/t > 0$. Moreover,

$$\lim_{s\to\infty} \frac{\gamma_n}{\tau_n} = \lim_{t\to 0} \frac{\gamma_n}{\tau_n} = \frac{n+1}{n} \frac{e_n}{(e_n + f_n)[1 - \ell_{n-1}(0)]}, \quad n \geq 1.$$

These, together with (3.19), give the required results of the Theorem. ∎

3.5 The Zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$

We now show that the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ can be obtained as the eigenvalues of a matrix which is a single line modification of the $n \times n$ monic Jacobi matrix associated with $\{\mathcal{P}_n(\nu_0; x)\}_{n\geq 0}$.

Let

$$\mathbf{s}_n(x) = [\mathcal{S}_0(\nu_0, \nu_1, s; x), \mathcal{S}_1(\nu_0, \nu_1, s; x), \ldots, \mathcal{S}_{n-1}(\nu_0, \nu_1, s; x)]^\top \quad \text{and}$$

$$\mathbf{p}_n^{\nu_0}(x) = [\mathcal{P}_0(\nu_0; x), \mathcal{P}_1(\nu_0; x), \ldots, \mathcal{P}_{n-1}(\nu_0; x)]^\top$$

for $n \geq 2$. Then, in a matrix form the connection formula (3.6) reads

$$\mathbf{\Gamma}_n \, \mathbf{s}_n(x) = \mathbf{p}_n^{\nu_0}(x), \quad n \geq 2, \tag{3.20}$$

where

$$\mathbf{\Gamma}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\gamma_0 & 1 & \cdots & 0 & 0 \\ 0 & -\gamma_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma_{n-2} & 1 \end{bmatrix}.$$

On the other hand, from (1.2) we can write

$$x \, \mathbf{p}_n^{\nu_0}(x) = \mathbf{J}_n^{\nu_0} \, \mathbf{p}_n^{\nu_0}(x) + \mathcal{P}_n(\nu_0; x) \, \mathbf{e}_n, \quad n \geq 2, \tag{3.21}$$

where $\mathbf{J}_n^{\nu_0}$ is the $n \times n$ monic Jacobi matrix associated with $\{\mathcal{P}_n(\nu_0; x)\}_{n\geq 0}$ and $\mathbf{e}_n$ is the n-th column of the $n \times n$ identity matrix.

Now let $x_{n,1}^{\nu_0} < x_{n,2}^{\nu_0} < \cdots < x_{n,n}^{\nu_0}$ denote the zeros of $\mathcal{P}_n(\nu_0; x)$, which are also known to be the eigenvalues of $\mathbf{J}_n^{\nu_0}$. Thus, for $n \geq 2$, one has

$$x_{n,j}^{\nu_0} \, \mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}) = \mathbf{J}_n^{\nu_0} \, \mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}), \quad j = 1, 2, \ldots, n.$$

Theorem 3.9. *For $n \geq 2$, let $\mathbf{A}_n$ be the $n \times n$ matrix*

$$\mathbf{A}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathfrak{a}_{n,1} & \cdots & \mathfrak{a}_{n,n-2} & \mathfrak{a}_{n,n-1} \end{bmatrix},$$

where the elements $\mathfrak{a}_{n,j}$ are as in Theorem 3.1 and satisfy

$$\mathfrak{a}_{n,j} = \gamma_j \, \gamma_{j+1} \, \ldots \gamma_{n-1}, \quad j = 1, 2, \ldots, n-1.$$

Then the zeros of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ are the eigenvalues of the matrix $\mathbf{J}_n^{\nu_0} - \mathbf{A}_n$.

Proof. Let

$$x\mathcal{S}_{n-1}(\nu_0, \nu_1, s; x) = \sum_{j=0}^{n-1} \theta_{n,j} \mathcal{S}_j(\nu_0, \nu_1, s; x) + \mathcal{S}_n(\nu_0, \nu_1, s; x), \quad n \geq 1.$$

Hence, for any $n \geq 2$ we can write

$$x\,\mathbf{s}_n(x) = \mathbf{U}_n\,\mathbf{s}_n(x) + \mathcal{S}_n(\nu_0,\nu_1,s;x)\,\mathbf{e}_n, \tag{3.22}$$

where

$$\mathbf{U}_n = \begin{bmatrix} \theta_{1,0} & 1 & \cdots & 0 & 0 \\ \theta_{2,0} & \theta_{2,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_{n-1,0} & \theta_{n-1,1} & \cdots & \theta_{n-1,n-2} & 1 \\ \theta_{n,0} & \theta_{n,1} & \cdots & \theta_{n,n-2} & \theta_{n,n-1} \end{bmatrix}$$

Observe that the zeros of $\mathcal{S}_n(\nu_0,\nu_1,s;x)$ are the eigenvalues of $\mathbf{U}_n$.

From (3.20) one can write $\mathbf{s}_n(x) = \mathbf{\Gamma}_n^{-1}\,\mathbf{p}_n^{\nu_0}(x)$ for $n \geq 2$. Thus, from (3.22) with the observation $\mathbf{\Gamma}_n\mathbf{e}_n = \mathbf{e}_n$, there holds

$$x\,\mathbf{p}_n^{\nu_0}(x) = \mathbf{\Gamma}_n\mathbf{U}_n\mathbf{\Gamma}_n^{-1}\,\mathbf{p}_n^{\nu_0}(x) + \mathcal{S}_n(\nu_0,\nu_1,s;x)\,\mathbf{e}_n. \tag{3.23}$$

Hence, combining (3.21) with (3.23) one finds

$$\mathbf{J}_n^{\nu_0}\,\mathbf{p}_n^{\nu_0}(x) + \mathcal{P}_n(\nu_0;x)\,\mathbf{e}_n = \mathbf{\Gamma}_n\mathbf{U}_n\mathbf{\Gamma}_n^{-1}\,\mathbf{p}_n^{\nu_0}(x) + \mathcal{S}_n(\nu_0,\nu_1,s;x)\,\mathbf{e}_n.$$

By evaluating these at the zeros of $\mathcal{P}_n(\nu_0;x)$ we get

$$\mathbf{J}_n^{\nu_0}\,\mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}) = \mathbf{\Gamma}_n\mathbf{U}_n\mathbf{\Gamma}_n^{-1}\,\mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}) + \mathcal{S}_n(\nu_0,\nu_1,s;x_{n,j}^{\nu_0})\,\mathbf{e}_n, \quad j = 1,2,\ldots,n.$$

On the other hand, since $\mathcal{S}_n(\nu_0,\nu_1,s;x) = \sum_{j=1}^{n-1}\mathfrak{a}_{n,j}\mathcal{P}_j(\nu_0;x) + \mathcal{P}_n(\nu_0;x)$, there follows

$$\mathcal{S}_n(\nu_0,\nu_1,s;x_{n,j}^{\nu_0})\,\mathbf{e}_n = \mathbf{A}_n\,\mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}), \quad j = 1,2,\ldots,n,$$

and hence,

$$\mathbf{J}_n^{\nu_0}\,\mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}) = \Big[\mathbf{\Gamma}_n\mathbf{U}_n\mathbf{\Gamma}_n^{-1} + \mathbf{A}_n\Big]\,\mathbf{p}_n^{\nu_0}(x_{n,j}^{\nu_0}), \quad j = 1,2,\ldots,n.$$

Thus, from the linear independence of the eigenvectors $\mathbf{p}_n^{\nu_0}(x_{n,1}^{\nu_0}), \mathbf{p}_n^{\nu_0}(x_{n,2}^{\nu_0}), \ldots, \mathbf{p}_n^{\nu_0}(x_{n,n}^{\nu_0})$ one finds

$$\mathbf{J}_n^{\nu_0} = \mathbf{\Gamma}_n\mathbf{U}_n\mathbf{\Gamma}_n^{-1} + \mathbf{A}_n.$$

Therefore, the matrices $\mathbf{J}_n^{\nu_0} - \mathbf{A}_n$ and $\mathbf{U}_n$ are similar and have the same eigenvalues. This completes the proof of the theorem. ■

Taking into account that the zeros of $\mathcal{S}_n(\nu_0,\nu_1,s;x)$ can be complex numbers, the following results provide strips within which these zeros lie.

Theorem 3.10. *If $\zeta_{n,j}$ is a zero of $\mathcal{S}_n(\nu_0,\nu_1,s;x)$ then the following can be stated*

(a) $|\operatorname{Im}(\zeta_{n,j})| < 1 + \max\limits_{1\leq k\leq n-1}|\mathfrak{a}_{n,k}|$;

(b) $|\operatorname{Im}(\zeta_{n,j})| \leq \dfrac{1}{2}\left(\displaystyle\sum_{k=1}^{n-1}\mathfrak{a}_{n,k}^2\,\frac{\hbar_k^{\nu_0}}{\hbar_{n-1}^{\nu_0}}\right)^{1/2}$.

Proof. The assertion (a) follows from Theorem 1.27 applied to (3.10). To obtain the latter we make use of Theorem 1.28 on (3.10) to find

$$\sum_{j=1}^{n} |\operatorname{Im}(\zeta_{n,j})| \leq \frac{1}{\sqrt{\hbar_{n-1}^{\nu_0}}} \Big(\sum_{k=1}^{n-1} \mathfrak{a}_{n,k}^2 \hbar_k^{\nu_0} \Big)^{1/2}.$$

Since complex zeros appear in conjugate pairs we have the latter result of the theorem. ∎

Finally, by using the bound (3.11) together with Theorem 3.10 we have the following.

Corollary 3.11. *If $\zeta_{n,j}$ is a zero of $\mathcal{S}_n(\nu_0, \nu_1, s; x)$ then the following can be stated*

(a) $$|\operatorname{Im}(\zeta_{n,j})| < 1 + \max_{1 \leq i \leq n-1} \Big\{ \frac{n}{i} \prod_{k=i}^{n-1} |\tau_k(\nu_0, \nu_1)| \Big\};$$

(b) $$|\operatorname{Im}(\zeta_{n,j})| < \frac{1}{2} \Big(\sum_{i=1}^{n} \frac{n^2 \hbar_i^{\nu_0}}{i^2 \hbar_{n-1}^{\nu_0}} \prod_{k=i}^{n-1} \tau_k^2(\nu_0, \nu_1) \Big)^{1/2}.$$

4 The CPPM2K-J and Associated Sobolev Orthogonal Polynomials

This chapter is devoted to the study of the Sobolev orthogonal polynomials that follow from a special example of a CPPM2K on the real line in which one of the measures is the Jacobi measure. In fact, this example provides a much more detailed study of the aspects of the polynomials and the connection coefficients that follow. The main results in this chapter were published in [18].

4.1 Introduction

In this chapter we study an example of a CPPM2K in which ν_1 is the Jacobi measure and ν_0 is a Geronimus perturbation of ν_1. Precisely, the example that we consider is such that

$$\text{CPPM2K-J:} \quad \{\nu_0, \nu_1\} = \{\tilde{\mu}^{(\alpha,\beta,q,\epsilon)},\, \mu^{(\alpha+1,\beta+1)}\},$$

where the measures $\mu^{(\alpha,\beta)}$ and $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$, which we consider to be probability measures, are defined as follows:

(i) $\mu^{(\alpha,\beta)}$ is the Jacobi probability measure on $[-1,1]$ given in (1.5). We write

$$d\mu^{(\alpha,\beta)}(x) = \eta^{(\alpha,\beta)}(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1, \tag{4.1}$$

with

$$\eta^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Hence, its corresponding sequence of MOP are the monic Jacobi classical orthogonal polynomials $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ (see Section 1.4).

(ii) $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ is the probability measure such that

$$d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) = (1-\epsilon)\frac{\tilde{\eta}^{(\alpha,\beta,q)}}{x-q}d\mu^{(\alpha,\beta)}(x) + \epsilon\,\boldsymbol{\delta}_q, \tag{4.2}$$

with $|q| \geq 1$ and $0 \leq \epsilon < 1$. Its support is given by

$$\mathcal{I}(q) = \begin{cases} [-1,1] & \text{if} \quad \epsilon = 0, \\ [-1,1]\cup\{q\} & \text{if} \quad 0 < \epsilon < 1. \end{cases}$$

We set the constant $\tilde{\eta}^{(\alpha,\beta,q)}$ as

$$\frac{1}{\tilde{\eta}^{(\alpha,\beta,q)}} = \int_{-1}^{1} \frac{1}{x-q} d\mu^{(\alpha,\beta)}(x),$$

so that $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ becomes a probability measure for every $\epsilon \in [0,1)$. Here, the values of α, β must be such that $1/\tilde{\eta}^{(\alpha,\beta,q)}$ exists. Thus, in general we assume $\alpha, \beta > -1$, with the exceptions: $\beta > 0$ when $q = -1$ and $\alpha > 0$ when $q = 1$.

In what follows, we denote by $\{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 0}$ the sequence of MOP with respect to the measure $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$.

Remark 4.1. Clearly the measure $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ constitutes the particular case of (2.4) in which $\nu = \mu^{(\alpha,\beta)}$ and $\tilde{\kappa}^{(q)} = -\tilde{\eta}^{(\alpha,\beta,q)}$. Therefore, we shall use the results contained in Chapter 2 to study the connection between the sequences of companion orthogonal polynomials $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ and $\{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 0}$.

The aim of this chapter is to consider the Sobolev orthogonal polynomials that follow from the pair CPPM2K-J. Precisely, we will look at the orthogonal polynomials with respect to the Sobolev-type inner product

$$\langle f, g\rangle_{\widetilde{\mathfrak{S}}^{(\alpha,\beta,q,\epsilon,s)}} = \int_{\mathcal{I}(q)} f(x)g(x)d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) + s\int_{-1}^{1} f'(x)g'(x)d\mu^{(\alpha+1,\beta+1)}(x), \quad s > 0.$$

In what follows, we will denote by $\{\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}\}_{n\geq 0}$ the sequence of MSOP with respect to the inner product $\langle \cdot\,,\cdot\rangle_{\widetilde{\mathfrak{S}}^{(\alpha,\beta,q,\epsilon,s)}}$. They satisfy the connection formula

$$\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x) - \tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}\,\mathcal{S}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}(x) = \widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x), \quad n \geq 1.$$

In the next section we provide some preliminary results to be used in the sequel that follow as direct applications of the results contained in Chapters 2 and 3. Some results regarding the connection coefficients $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ are stated. In particular, we consider the monic polynomials $\mathcal{R}_n^{(\alpha,\beta,q,\epsilon)}$ obtained in the special case $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\lambda/t)} = r_{n-1}^{(\alpha,\beta,q,\epsilon)}(t)/r_n^{(\alpha,\beta,q,\epsilon)}(t)$. In Section 4.4 we will see that when $\epsilon = 0$ and $|q| = 1$, the polynomials $\mathcal{R}_n^{(\alpha,\beta,q,\epsilon)}$ are related to a subfamily of the so-called Wilson polynomials. Section 4.3 is devoted to the asymptotic properties of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to the parameters s, q and n. Finally, in Section 4.5 we show that when $|q| = 1$ and s is sufficiently small, the zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ are real, simple and interlace with the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}$.

4.2 Preliminary Results

Consider the sequence of positive constants $\{\tilde{\hbar}_n^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 0}$ such that

$$\int_{\mathcal{I}(q)} \widetilde{\mathcal{P}}_m^{(\alpha,\beta,q,\epsilon)}(x)\,\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x)\,d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) = \tilde{\hbar}_n^{(\alpha,\beta,q,\epsilon)}\delta_{m,n}, \quad m, n = 0, 1, 2, \ldots.$$

Let $\{\tilde{c}_n^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 1}$ and $\{\tilde{d}_{n+1}^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 1}$ be the coefficients in the TTRR

$$\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(x) = (x - \tilde{c}_{n+1}^{(\alpha,\beta,q,\epsilon)})\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x) - \tilde{d}_{n+1}^{(\alpha,\beta,q,\epsilon)}\widetilde{\mathcal{P}}_{n-1}^{(\alpha,\beta,q,\epsilon)}(x), \quad n \geq 1,$$

with $\widetilde{\mathcal{P}}_1^{(\alpha,\beta,q,\epsilon)}(x) = x - \tilde{c}_1^{(\alpha,\beta,q,\epsilon)}$ and $\widetilde{\mathcal{P}}_0^{(\alpha,\beta,q,\epsilon)}(x) = 1$.

From (2.12) there holds the relation

$$\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(x) = \mathcal{P}_{n+1}^{(\alpha,\beta)}(x) - \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}\,\mathcal{P}_n^{(\alpha,\beta)}(x), \quad n \geq 0, \tag{4.3}$$

where

$$\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)} = -\frac{\tilde{\hbar}_{n+1}^{(\alpha,\beta,q,\epsilon)}}{(1-\epsilon)\tilde{\eta}^{(\alpha,\beta,q)}\,\hbar_n^{(\alpha,\beta)}}.$$

Differentiating both sides in (4.3) and then using (1.9), there follows

$$\frac{1}{n+1}\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)\prime}(x) = \mathcal{P}_n^{(\alpha+1,\beta+1)}(x) - \tilde{\tau}_n^{(\alpha,\beta,q,\epsilon)}\,\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1, \tag{4.4}$$

where

$$\tilde{\tau}_n^{(\alpha,\beta,q,\epsilon)} = \frac{n}{n+1}\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}.$$

Thus, (4.4) confirms that the pair of measures

$$\{\nu_0, \nu_1\} = \{\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}, \mu^{(\alpha+1,\beta+1)}\}$$

is a CPPM2K on the real line. As shown in [11], the CPPM2K-J verifies (3.2) with

$$\begin{aligned} A_2(x) &= \operatorname{sgn}(q)(x-q)[(\alpha+\beta+2)x - (\beta-\alpha)] \quad \text{and} \\ A_3(x) &= \operatorname{sgn}(q)(x-q)(x^2-1). \end{aligned}$$

Observing the sign of $\tilde{\eta}^{(\alpha,\beta,q)}$ we see that

$$\operatorname{sgn}(q)\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)} > 0, \quad n \geq 0.$$

Moreover, from (2.14) one can write

$$\left.\begin{aligned} &\frac{\tilde{d}_{n+2}^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} + \tilde{c}_{n+1}^{(\alpha,\beta,q,\epsilon)} + \tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)} = q = \frac{d_{n+1}^{(\alpha,\beta)}}{\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}} + c_{n+1}^{(\alpha,\beta)} + \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}, \\ &\frac{\tilde{d}_{n+2}^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} = \frac{d_{n+1}^{(\alpha,\beta)}}{\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}} \quad \text{and} \quad \tilde{c}_{n+1}^{(\alpha,\beta,q,\epsilon)} + \tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)} = c_{n+1}^{(\alpha,\beta)} + \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}, \end{aligned}\right\} \quad n \geq 1, \tag{4.5}$$

with the initial conditions

$$\tilde{c}_1^{(\alpha,\beta,q,\epsilon)} = c_1^{(\alpha,\beta)} + \tilde{\mathfrak{b}}_0^{(\alpha,\beta,q,\epsilon)} = q + (1-\epsilon)\tilde{\eta}^{(\alpha,\beta,q)} \quad \text{and} \quad \frac{\tilde{d}_2^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathfrak{b}}_0^{(\alpha,\beta,q,\epsilon)}} + \tilde{c}_1^{(\alpha,\beta,q,\epsilon)} = q.$$

Note that $\widetilde{\mathcal{P}}_n^{(\alpha,\beta,1,\epsilon)}$ and $\widetilde{\mathcal{P}}_n^{(\alpha,\beta,-1,\epsilon)}$ are special Koornwinder polynomials (see (2.36)). In particular, for $\alpha > 0$, $\beta > -1$ and $n \geq 0$,

$$\widetilde{\mathcal{P}}_n^{(\alpha,\beta,1,0)} = \mathcal{P}_n^{(\alpha-1,\beta)} \quad \text{and} \quad \tilde{\mathfrak{b}}_n^{(\alpha,\beta,1,0)} = \frac{2(n+1)(\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}. \tag{4.6}$$

Likewise, for $\alpha > -1$, $\beta > 0$ and $n \geq 0$,

$$\widetilde{\mathcal{P}}_n^{(\alpha,\beta,-1,0)} = \mathcal{P}_n^{(\alpha,\beta-1)} \quad \text{and} \quad \tilde{\mathfrak{b}}_n^{(\alpha,\beta,-1,0)} = -\frac{2(n+1)(\alpha+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}.$$

From Theorem 3.2 we obtain the following simple connection formula.

Theorem 4.2.

$$\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x) - \tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}\,\mathcal{S}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}(x) = \widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x), \quad n \geq 1, \tag{4.7}$$

where $\mathcal{S}_0^{(\alpha,\beta,q,\epsilon,s)}(x) = 1$ *and*

$$\tilde{\gamma}_0^{(\alpha,\beta,q,\epsilon,s)} = 0, \qquad \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} = \frac{\tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)} - \tilde{\mathfrak{q}}_n^{(\alpha,\beta,q,\epsilon,s)}\,\tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}}, \quad n \geq 1. \tag{4.8}$$

Here, the elements $\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)}$ *and* $\tilde{\mathfrak{q}}_n^{(\alpha,\beta,q,\epsilon,s)}$ *are such that*

$$\left.\begin{aligned} \tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)} &= sn^2\,\hbar_{n-1}^{(\alpha+1,\beta+1)}\,\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}, \\ \tilde{\mathfrak{p}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)} &= \tilde{\hbar}_{n+1}^{(\alpha,\beta,q,\epsilon)} + s\Big[(n+1)^2\,\hbar_n^{(\alpha+1,\beta+1)} + n^2\,\hbar_{n-1}^{(\alpha+1,\beta+1)}\,(\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)})^2\Big], \end{aligned}\right\} \quad n \geq 1, \tag{4.9}$$

with $\tilde{\mathfrak{q}}_1^{(\alpha,\beta,q,\epsilon,s)} = 0$ *and* $\tilde{\mathfrak{p}}_1^{(\alpha,\beta,q,\epsilon,s)} = \tilde{\hbar}_1^{(\alpha,\beta,q,\epsilon)} + s\,\hbar_0^{(\alpha+1,\beta+1)}$.

From Theorem 3.3 we get

$$0 \;<\; \frac{\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} \;<\; 1, \quad n \geq 1.$$

This also gives

$$\operatorname{sgn}(q)\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} > 0, \quad n \geq 1.$$

By using Theorems 3.5 and 3.6 we can write the connection coefficients $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\lambda/t)}$ as rational functions which are related to a sequence of MOP satisfying a simple TTRR.

Theorem 4.3. *Let* $\lambda \neq 0$ *be an arbitrary real number and write*

$$\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\lambda/t)} = \frac{\Delta_n^{(\alpha,\beta,q,\epsilon)}}{\Delta_{n-1}^{(\alpha,\beta,q,\epsilon)}}\frac{\lambda\mathcal{R}_{n-1}^{(\alpha,\beta,q,\epsilon)}(t)}{\mathcal{R}_n^{(\alpha,\beta,q,\epsilon)}(t)}, \quad n \geq 1,$$

where

$$\Delta_0^{(\alpha,\beta,q,\epsilon)} = 1 \quad \textit{and} \quad \Delta_n^{(\alpha,\beta,q,\epsilon)} = n^2\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}\frac{\hbar_{n-1}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_n^{(\alpha,\beta,q,\epsilon)}}\Delta_{n-1}^{(\alpha,\beta,q,\epsilon)}, \quad n \geq 1.$$

Then $\{\mathcal{R}_n^{(\alpha,\beta,q,\epsilon)}\}_{n\geq 0}$ *is a sequence of MOP satisfying the TTRR*

$$\mathcal{R}_{n+1}^{(\alpha,\beta,q,\epsilon)}(t) = [t + (e_{n+1} + f_{n+1})\lambda]\mathcal{R}_n^{(\alpha,\beta,q,\epsilon)}(t) - e_n f_{n+1}\lambda^2\mathcal{R}_{n-1}^{(\alpha,\beta,q,\epsilon)}(t), \quad n \geq 1,$$

with $\mathcal{R}_0^{(\alpha,\beta,q,\epsilon)}(t) = 1$ *and* $\mathcal{R}_1^{(\alpha,\beta,q,\epsilon)}(t) = t + (e_1 + f_1)\lambda$, *where* $f_1 = 0$,

$$\left.\begin{aligned} e_n = e_n^{(\alpha,\beta,q,\epsilon)} &= \frac{n^2\,\hbar_{n-1}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_n^{(\alpha,\beta,q,\epsilon)}}, \\ f_{n+1} = f_{n+1}^{(\alpha,\beta,q,\epsilon)} &= \frac{n^2[\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}]^2\,\hbar_{n-1}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_{n+1}^{(\alpha,\beta,q,\epsilon)}}, \end{aligned}\right. \quad n \geq 1.$$

4.3 Asymptotic Results

In this section we study the asymptotic behavior of $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to the parameters s, q and n.

4.3.1 Asymptotics for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to s

We now state the following results on the bounds and asymptotics of $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to s.

Theorem 4.4. *Let $n \geq 1$. Then the following statements hold:*

(i) $|\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}|$ *is a strictly increasing function of* s;

(ii) $\dfrac{\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} < \dfrac{s\big(\tilde{e}_n^{(\alpha,\beta,q,\epsilon)} + \tilde{f}_n^{(\alpha,\beta,q,\epsilon)}\big)}{1 + s\big(\tilde{e}_n^{(\alpha,\beta,q,\epsilon)} + \tilde{f}_n^{(\alpha,\beta,q,\epsilon)}\big)}$;

(iii) $\displaystyle\lim_{s\to\infty} \frac{\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} = 1$;

(iv) $\displaystyle\lim_{s\to\infty} \mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x) = \mathcal{P}_n^{(\alpha,\beta)}(x) - \prod_{j=0}^{n-1} \tilde{\mathfrak{b}}_j^{(\alpha,\beta,q,\epsilon)}$.

Proof. The assertions concerning the coefficients $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ follow from Theorems 3.7 and 3.8 combined with the coherence formula (4.4). Now, to obtain the asymptotic behavior of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$, by continuity we have from (4.7)

$$\lim_{s\to\infty} \mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x) - \tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)} \lim_{s\to\infty} \mathcal{S}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}(x) = \widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x), \quad n \geq 1.$$

On the other hand, observe that $[\mathcal{P}_n^{(\alpha,\beta)}(x)+a_n]' = n\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(x),\ n \geq 1$, for any constant a_n. Hence, by considering the inner product $s^{-1}\langle\cdot\,,\cdot\rangle_{\widetilde{\mathfrak{S}}^{(\alpha,\beta,q,\epsilon,s)}}$ and the corresponding orthogonal polynomials $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ as $s \to \infty$, there follows

$$\lim_{s\to\infty} \mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x) = \mathcal{P}_n^{(\alpha,\beta)}(x) + a_n, \quad n \geq 1,$$

where the constants a_n still need to be determined. By combining the two above formulas one deduces $\mathcal{P}_1^{(\alpha,\beta)}(x) + a_1 = \widetilde{\mathcal{P}}_1^{(\alpha,\beta,q,\epsilon)}(x)$ and

$$\left.\begin{aligned} \mathcal{P}_n^{(\alpha,\beta)}(x) - \tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}\, \mathcal{P}_{n-1}^{(\alpha,\beta)}(x) &= \widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x), \\ a_n - \tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}\, a_{n-1} &= 0, \end{aligned}\right\} \quad n \geq 2.$$

Thus, with the use of (4.3) we obtain the remaining assertion of the theorem. ∎

4.3.2 Asymptotics for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to q

From Corollary 2.12, if $0 \leq \epsilon < 1$ then

$$\tilde{\mathfrak{b}}_0^{(\alpha,\beta,q,\epsilon)} = \epsilon q - \epsilon c_1^{(\alpha,\beta)} + (1-\epsilon) d_2^{(\alpha,\beta)} q^{-1} + O\big(q^{-2}\big), \quad \text{as} \quad |q| \to \infty.$$

For $n \geq 1$, we get

$$\begin{aligned} \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,0)} &= d_{n+2}^{(\alpha,\beta)} q^{-1} + O\Big(q^{-2}\Big), \\ \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)} &= q - c_{n+1}^{(\alpha,\beta)} - d_{n+1}^{(\alpha,\beta)} q^{-1} + O\Big(q^{-2}\Big), \quad 0 < \epsilon < 1, \end{aligned} \qquad \text{as} \quad |q| \to \infty. \tag{4.10}$$

Now from Corollary 2.13, if $0 \leq \epsilon < 1$ then

$$q^{-2} \tilde{\hbar}_1^{(\alpha,\beta,q,\epsilon)} = (1-\epsilon)\Big[\epsilon - 2\epsilon c_1^{(\alpha,\beta)} q^{-1} + [(1-2\epsilon) d_2^{(\alpha,\beta)} + \epsilon (c_1^{(\alpha,\beta)})^2] q^{-2} + O\Big(q^{-3}\Big)\Big]$$

as $|q| \to \infty$. For $n \geq 1$, we have

$$\begin{aligned} \tilde{\hbar}_{n+1}^{(\alpha,\beta,q,0)} &= \hbar_{n+1}^{(\alpha,\beta)} + O\Big(q^{-1}\Big), \\ q^{-2} \tilde{\hbar}_{n+1}^{(\alpha,\beta,q,\epsilon)} &= (1-\epsilon) \hbar_n^{(\alpha,\beta)} \Big[1 - (c_1^{(\alpha,\beta)} + c_{n+1}^{(\alpha,\beta)}) q^{-1} \\ &\qquad + (c_1^{(\alpha,\beta)} c_{n+1}^{(\alpha,\beta)} - d_2^{(\alpha,\beta)} - d_{n+1}^{(\alpha,\beta)}) q^{-2} + O\Big(q^{-3}\Big)\Big], \quad 0 < \epsilon < 1, \end{aligned} \tag{4.11}$$

as $|q| \to \infty$.

Theorem 4.5. *Let $n \geq 1$ and $0 \leq \epsilon < 1$. Then*

$$\lim_{|q| \to \infty} \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} = 0.$$

Precisely,

$$\tilde{\gamma}_n^{(\alpha,\beta,q,0,s)} = \frac{s n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)} d_{n+2}^{(\alpha,\beta)}}{\hbar_n^{(\alpha,\beta)} + s n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)}} q^{-1} + O\Big(q^{-2}\Big), \quad n \geq 1,$$

as $|q| \to \infty$. If $0 < \epsilon < 1$, then

$$\begin{aligned} \tilde{\gamma}_1^{(\alpha,\beta,q,\epsilon,s)} &= \frac{s}{(1-\epsilon)\epsilon} q^{-1} + O\Big(q^{-2}\Big), \\ \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} &= \frac{s n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)}}{(1-\epsilon) \hbar_{n-1}^{(\alpha,\beta)} + s(n-1)^2 \hbar_{n-2}^{(\alpha+1,\beta+1)}} q^{-1} + O\Big(q^{-2}\Big), \quad n \geq 2, \end{aligned}$$

as $|q| \to \infty$.

Proof. We give the proof for $0 < \epsilon < 1$. Combining (4.9) with (4.10) and (4.11) one finds that for large $|q|$ there holds

$$q^{-2} \tilde{\mathfrak{p}}_1^{(\alpha,\beta,q,\epsilon,s)} = (1-\epsilon)\Bigg[\epsilon - 2\epsilon c_1^{(\alpha,\beta)} q^{-1} + \bigg((1-2\epsilon) d_2^{(\alpha,\beta)} + \epsilon (c_1^{(\alpha,\beta)})^2 + \frac{s}{1-\epsilon}\bigg) q^{-2} + O\Big(q^{-3}\Big)\Bigg]$$

and

$$\begin{aligned} q^{-2} \tilde{\mathfrak{p}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)} &= (1-\epsilon) \hbar_n^{(\alpha,\beta)} + s n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)} + O\Big(q^{-1}\Big), \\ q^{-2} \tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)} &= s n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)} q^{-1} \Big[1 - c_{n+1}^{(\alpha,\beta)} q^{-1} + O\Big(q^{-2}\Big)\Big], \end{aligned} \qquad n \geq 1.$$

From $\tilde{\gamma}_1^{(\alpha,\beta,q,\epsilon,s)} = \tilde{\mathfrak{q}}_2^{(\alpha,\beta,q,\epsilon,s)} / \tilde{\mathfrak{p}}_1^{(\alpha,\beta,q,\epsilon,s)}$ the asymptotic for $\tilde{\gamma}_1^{(\alpha,\beta,q,\epsilon,s)}$ follows.

To prove the asymptotic for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ for $n \geq 2$ we use the argument of induction. Suppose that $\tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)} = \text{const}\, q^{-1} + O\Big(q^{-2}\Big)$. Then

$$q^{-2}\Big[\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)} - \tilde{\mathfrak{q}}_n^{(\alpha,\beta,q,\epsilon,s)} \, \tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}\Big] = (1-\epsilon) \hbar_{n-1}^{(\alpha,\beta)} + s(n-1)^2 \hbar_{n-2}^{(\alpha+1,\beta+1)} + O\Big(q^{-1}\Big)$$

and

$$\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} = \frac{q^{-2}\tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}}{q^{-2}\left[\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)} - \tilde{\mathfrak{q}}_n^{(\alpha,\beta,q,\epsilon,s)}\,\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}\right]}$$

$$= \frac{sn^2 \mathcal{h}_{n-1}^{(\alpha+1,\beta+1)}}{(1-\epsilon)\mathcal{h}_{n-1}^{(\alpha,\beta)} + s(n-1)^2 \mathcal{h}_{n-2}^{(\alpha+1,\beta+1)}} q^{-1} + O\Big(q^{-2}\Big).$$

This completes the proof for $0 < \epsilon < 1$. The proof for $\epsilon = 0$ is analogous. ∎

Now we look at the asymptotic behavior of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to q.

Theorem 4.6. *Let $n \geq 1$. Then*

$$\lim_{|q|\to\infty} \mathcal{S}_n^{(\alpha,\beta,q,0,s)}(x) = \mathcal{P}_n^{(\alpha,\beta)}(x),$$

$$\lim_{|q|\to\infty} \frac{\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(x)}{\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}} = -\mathcal{P}_{n-1}^{(\alpha,\beta)}(x), \quad 0 < \epsilon < 1.$$

Proof. From (4.3) and (4.10),

$$\left.\begin{aligned} &\lim_{|q|\to\infty} \widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,0)}(x) = \mathcal{P}_n^{(\alpha,\beta)}(x), \\ &\lim_{|q|\to\infty} \frac{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(x)}{\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}} = -\mathcal{P}_{n-1}^{(\alpha,\beta)}(x), \quad 0 < \epsilon < 1, \end{aligned}\right. \quad n \geq 1. \tag{4.12}$$

We now give the proof for the case $0 < \epsilon < 1$. From (4.7) and (4.12),

$$\lim_{|q|\to\infty} \frac{\mathcal{S}_1^{(\alpha,\beta,q,\epsilon,s)}(x)}{\tilde{\mathfrak{b}}_0^{(\alpha,\beta,q,\epsilon)}} = \lim_{|q|\to\infty} \frac{\widetilde{\mathcal{P}}_1^{(\alpha,\beta,q,\epsilon)}(x)}{\tilde{\mathfrak{b}}_0^{(\alpha,\beta,q,\epsilon)}} = -\mathcal{P}_0^{(\alpha,\beta)}(x).$$

Assuming

$$\lim_{|q|\to\infty} \frac{\mathcal{S}_{k-1}^{(\alpha,\beta,q,\epsilon,s)}(x)}{\tilde{\mathfrak{b}}_{k-2}^{(\alpha,\beta,q,\epsilon)}} = -\mathcal{P}_{k-2}^{(\alpha,\beta)}(x)$$

holds for $k \geq 2$, the result follows by induction from

$$\frac{\mathcal{S}_k^{(\alpha,\beta,q,\epsilon,s)}(x)}{\tilde{\mathfrak{b}}_{k-1}^{(\alpha,\beta,q,\epsilon)}} - \tilde{\gamma}_{k-1}^{(\alpha,\beta,q,\epsilon,s)} \frac{\tilde{\mathfrak{b}}_{k-2}^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathfrak{b}}_{k-1}^{(\alpha,\beta,q,\epsilon)}} \frac{\mathcal{S}_{k-1}^{(\alpha,\beta,q,\epsilon,s)}(x)}{\tilde{\mathfrak{b}}_{k-2}^{(\alpha,\beta,q,\epsilon)}} = \frac{\widetilde{\mathcal{P}}_k^{(\alpha,\beta,q,\epsilon)}(x)}{\tilde{\mathfrak{b}}_{k-1}^{(\alpha,\beta,q,\epsilon)}}.$$

The proof for $\epsilon = 0$ is analogous. This concludes the required proof. ∎

4.3.3 Asymptotics for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to n

Now we look at the asymptotic behavior of $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to n. It is well known that the Jacobi measure $\mu^{(\alpha,\beta)}$ belongs to the Nevai class $M[0,1]$ (see [30], [31]) which means

$$\lim_{n\to\infty} c_n^{(\alpha,\beta)} = 0 \quad \text{and} \quad \lim_{n\to\infty} d_n^{(\alpha,\beta)} = \frac{1}{4}.$$

Thus, as we have seen in Subsection 2.3.3, one can prove that the measure $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ belongs to the same Nevai class $M[0,1]$ (see [8]). Let us remind that the support of a measure of the Nevai class $M[0,1]$ consists of the interval $[-1,1]$ and at most a countable set of real mass points located outside $[-1,1]$ which, when infinitely many, may accumulate only at the end points of the interval $[-1,1]$.

Let the complex function

$$\varphi(z) = z + \sqrt{z^2-1}, \quad \text{for} \quad z \in \mathbb{C} \setminus [-1,1],$$

be such that $\sqrt{z^2-1}$ is the branch of $(z^2-1)^{1/2}$ for which $|\varphi(z)| > 1$ when $z \in \mathbb{C}\setminus[-1,1]$. Clearly $\varphi(+\infty) = +\infty$ and $\varphi(-\infty) = -\infty$.

In order to study the asymptotics for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ with respect to n, let us first provide some properties of the coefficients $\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}$. From (2.26) we see that $|\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}|$ are bounded for any fixed $|q| \geq 1$. Since the measures $\mu^{(\alpha,\beta)}$ and $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ belong to the Nevai class $M[0,1]$, from (4.5) together with the boundedness of $|\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}|$ one has $\lim_{n\to\infty} \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)} = \tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)}$ exists and satisfies

$$\frac{1}{4\left[\operatorname{sgn}(q)\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)}\right]} + \operatorname{sgn}(q)\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)} = |q|. \tag{4.13}$$

Thus, $\operatorname{sgn}(q)\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)}$ is either $\frac{1}{2}[|q|+\sqrt{|q|^2-1}]$ or $\frac{1}{2}[|q|+\sqrt{|q|^2-1}]^{-1}$. On the other hand, from (4.10) by analyzing the behavior of $\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}$ for large $|q|$, we find

$$\begin{aligned} \tilde{\mathfrak{b}}^{(\alpha,\beta,q,0)} &= \frac{\operatorname{sgn}(q)}{2}\frac{1}{[|q|+\sqrt{|q|^2-1}]} = \frac{1}{2\varphi(q)}, \\ \tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)} &= \frac{\operatorname{sgn}(q)}{2}[|q|+\sqrt{|q|^2-1}] = \frac{\varphi(q)}{2}, \quad 0<\epsilon<1. \end{aligned} \tag{4.14}$$

The following lemma will be useful to study the asymptotic behavior of the ratio $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(z)/\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)$ with respect to n.

Lemma 4.7. *Let* $0 \leq \epsilon < 1$. *Then*

$$\lim_{n\to\infty} \frac{\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(z)}{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)} = \frac{\varphi(z)}{2},$$

uniformly in every compact subset of $\mathbb{C}\setminus\mathcal{I}(q)$. *Moreover, if* $0<\epsilon<1$, *then*

$$\lim_{n\to\infty} \frac{\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(q)}{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(q)} = \frac{1}{2\varphi(q)}.$$

Proof. The first part follows from the fact $\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}$ belongs to the Nevai class $M[0,1]$. Observe that from (2.15) we can write

$$\frac{\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(q)}{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(q)} = \frac{d_{n+1}^{(\alpha,\beta)}}{\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)}}, \quad n \geq 1,$$

and by letting $n\to\infty$, the latter part of the theorem follows. ■

With these results we can establish the limit of $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$ as $n \to \infty$.

Theorem 4.8.

$$\lim_{n\to\infty} \tilde{\gamma}_n^{(\alpha,\beta,q,0,\infty)} = \frac{1}{2\varphi(q)} \quad \textit{and} \quad \lim_{n\to\infty} \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\infty)} = \frac{\varphi(q)}{2} \quad \textit{for} \quad 0 < \epsilon < 1.$$

Moreover, in the case of $s < \infty$*, there holds*

$$\lim_{n\to\infty} \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} = \frac{1}{2\varphi(q)} \quad \textit{for} \quad 0 \le \epsilon < 1.$$

Proof. By Theorem 4.4 one gets $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\infty)} = \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}$ for $n \ge 1$ and hence the two initial limits follow from (4.14). To find the latter limit for $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$, let us consider the positive chain sequence $\{\tilde{\sigma}_n(s)\}_{n\ge1}$ and its minimal parameter sequence $\{\tilde{\mathfrak{m}}_n(s)\}_{n\ge0}$ given by

$$\tilde{\sigma}_n(s) = \frac{[\tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}]^2}{\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)}\tilde{\mathfrak{p}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}} \quad \text{and} \quad \tilde{\mathfrak{m}}_{n-1}(s) = \frac{\tilde{\mathfrak{q}}_n^{(\alpha,\beta,q,\epsilon,s)}\tilde{\gamma}_{n-1}^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{p}}_n^{(\alpha,\beta,q,\epsilon,s)}}, \qquad n \ge 1.$$

These results can be obtained from (4.8) and (4.9) similarly as in (3.12).

Clearly, $\tilde{\sigma}_n(s) > 0$ for any $s > 0$. By using (4.9) one can write $\tilde{\sigma}_n(s) = D_n(s) \times E_n(s)$ and $\tilde{\mathfrak{m}}_n(s) = D_n(s) \times \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$, where

$$D_n(s) = \frac{\tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{p}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}} = \frac{s\, n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathscr{h}}_{n+1}^{(\alpha,\beta,q,\epsilon)} + s[(n+1)^2 \mathscr{h}_n^{(\alpha+1,\beta+1)} + n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, (\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)})^2]},$$

$$E_n(s) = \frac{\tilde{\mathfrak{q}}_{n+1}^{(\alpha,\beta,q,\epsilon,s)}}{\tilde{\mathfrak{p}}_{n}^{(\alpha,\beta,q,\epsilon,s)}} = \frac{s\, n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathscr{h}}_{n}^{(\alpha,\beta,q,\epsilon)} + s[n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)} + (n-1)^2 \mathscr{h}_{n-2}^{(\alpha+1,\beta+1)}\, (\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)})^2]}.$$

From these, one can deduce

$$\frac{1}{D_n(s)} = K_n(s) + \frac{(n+1)^2 \mathscr{h}_n^{(\alpha+1,\beta+1)}}{n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} + \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)},$$

$$\frac{1}{E_n(s)} = \frac{\tilde{\mathscr{h}}_n^{(\alpha,\beta,q,\epsilon)}}{\tilde{\mathscr{h}}_{n+1}^{(\alpha,\beta,q,\epsilon)}} K_n(s) + \frac{1}{\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}} + \frac{(n-1)^2 \mathscr{h}_{n-2}^{(\alpha+1,\beta+1)}\, (\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,q,\epsilon)})^2}{n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, \tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}},$$

where

$$K_n(s) = \operatorname{sgn}(q) \frac{\tilde{\mathscr{h}}_{n+1}^{(\alpha,\beta,q,\epsilon)}}{s\, n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}\, |\tilde{\mathfrak{b}}_n^{(\alpha,\beta,q,\epsilon)}|}. \tag{4.15}$$

Using (4.3), $K_n(s)$ can be rewritten as

$$K_n(s) = \operatorname{sgn}(q) \frac{(1-\epsilon)|\tilde{\eta}^{(\alpha,\beta,q)}|\, \mathscr{h}_n^{(\alpha,\beta)}}{s\, n^2 \mathscr{h}_{n-1}^{(\alpha+1,\beta+1)}},$$

and hence we have $\lim_{n\to\infty} K_n(s) = 0$ if $s < \infty$. Thus, from (4.13)

$$\lim_{n\to\infty} \frac{1}{D_n(s)} = \frac{1}{4\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)}} + \tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)} = q,$$

$$\lim_{n\to\infty} \frac{1}{E_n(s)} = \frac{1}{\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)}} + 4\tilde{\mathfrak{b}}^{(\alpha,\beta,q,\epsilon)} = 4q.$$

Then

$$\lim_{n\to\infty} \tilde{\sigma}_n(s) = \lim_{n\to\infty} [D_n(s) \times E_n(s)] = \frac{1}{4q^2}.$$

In order to make use of Theorem 1.8 to find the limit for the sequence $\{\tilde{\mathfrak{m}}_n(s)\}_{n\geq 0}$, we now show that the positive chain sequence $\{\tilde{\sigma}_n(s)\}_{n\geq 1}$ has multiple parameter sequences. From (4.15) we can easily observe that if $0 < s_1 < s_2 < \infty$ then

$$0 = \mathrm{sgn}(q) K_n(\infty) < \mathrm{sgn}(q) K_n(s_2) < \mathrm{sgn}(q) K_n(s_1) < \infty.$$

Hence, from $\tilde{\sigma}_n(s) = \mathrm{sgn}(q) D_n(s) \times \mathrm{sgn}(q) E_n(s)$ one finds

$$0 < \tilde{\sigma}_n(s_1) < \tilde{\sigma}_n(s_2) < \tilde{\sigma}_n(\infty) \quad \text{for} \quad 0 < s_1 < s_2 < \infty.$$

With the observation $\tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,\infty)}$ exists, one can conclude that the sequence $\{\tilde{\sigma}_n(\infty)\}_{n\geq 1}$ is also a positive chain sequence. Thus, from Theorem 1.5 we find that for any $0 < s < \infty$, the sequence $\{\tilde{\sigma}_n(s)\}_{n\geq 0}$ is a multiple parameter positive chain sequence. Therefore, from Theorem 1.8 we find

$$\lim_{n\to\infty} \tilde{\mathfrak{m}}_n(s) = \frac{1}{2|q|[|q| + \sqrt{|q|^2 - 1}]}.$$

On the other hand, by writing $\tilde{\mathfrak{m}}_n(s) = D_n(s) \times \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)}$, we get

$$\lim_{n\to\infty} \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} = \frac{q}{2|q|} \frac{1}{[|q| + \sqrt{|q|^2 - 1}]}.$$

This completes the proof of the theorem. ■

Now, we are ready to prove the following asymptotic relation involving the polynomials $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ and $\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}$.

Theorem 4.9. *Let $0 \leq \epsilon < 1$ and $0 < s < \infty$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(z)}{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)} = \frac{\varphi(q)\,\varphi(z)}{\varphi(q)\,\varphi(z) - 1},$$

uniformly in every compact subset of $\mathbb{C} \setminus \mathcal{I}(q)$.

Proof. Let us denote by $\hat{\mathcal{I}}(q)$ the smallest closed interval containing $\mathcal{I}(q)$. Note from (4.7) that we can write

$$F_{n+1}(z) - G_n(z) F_n(z) = 1, \quad n \geq 0, \tag{4.16}$$

where

$$F_n(z) = \frac{\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}(z)}{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)} \quad \text{and} \quad G_n(z) = \tilde{\gamma}_n^{(\alpha,\beta,q,\epsilon,s)} \frac{\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)}{\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}(z)},$$

are analytic functions in $\mathbb{C} \setminus \hat{\mathcal{I}}(q)$. Hence, with the use of Lemma 4.7 and Theorem 4.8, for $s < \infty$ we find

$$\lim_{n\to\infty} G_n(z) = G(z) = \frac{1}{\varphi(q)\,\varphi(z)},$$

in every compact subset of $\mathbb{C} \setminus \hat{\mathcal{I}}(q)$. Note also that $|G(z)| < 1$ for $z \in \mathbb{C} \setminus \hat{\mathcal{I}}(q)$. Thus, in every compact subset of $\mathbb{C} \setminus \hat{\mathcal{I}}(q)$, there exist $B \in (0, 1)$ and a positive integer N_0 such that

$$|G_n(z)| \leq B \;\text{ for all }\; n \geq N_0.$$

Consequently, from (4.16)

$$|F_n(z)| \le B\,|F_{n-1}(z)| + 1 \quad \text{for all} \quad n \ge N_0.$$

Now, let us consider the sequence $\{V_n(z)\}_{n\ge 0}$ given by

$$V_n(z) = \begin{cases} |F_n(z)|, & 0 \le n \le N_0 - 1, \\ BV_{n-1}(z) + 1, & n \ge N_0. \end{cases}$$

For all $n \ge N_0 - 1$ and $m \ge 1$, we have

$$V_{n+m}(z) = B^m\, V_n(z) + \frac{1 - B^m}{1 - B}.$$

Thus, by taking the limit as $m \to \infty$, we obtain

$$\lim_{n\to\infty} V_n(z) = \frac{1}{1 - B}.$$

This means that $V_n(z)$ is uniformly bounded for all n large enough. Furthermore, $0 \le |F_n(z)| \le V_n(z)$ for every positive integer n, and hence $F_n(z)$ is also uniformly bounded for all n large enough in every compact subset of $\mathbb{C} \setminus \hat{\mathcal{I}}(q)$. In other words, the sequence $\{F_n(z)\}_{n\ge 0}$ is a normal family of analytic functions and, thus, there exists a function $F(z)$ analytic in $\mathbb{C} \setminus \hat{\mathcal{I}}(q)$ such that

$$\lim_{n\to\infty} F_n(z) = F(z).$$

Taking the limit in (4.16) we have

$$F(z) - G(z)F(z) = 1,$$

from which

$$F(z) = \frac{\varphi(q)\,\varphi(z)}{\varphi(q)\,\varphi(z) - 1}.$$

To extend the results for $\mathbb{C} \setminus \mathcal{I}(q)$ we observe that $F(z)$ is analytic in $\mathbb{C} \setminus \mathcal{I}(q)$ and that, if $z_n^{(\alpha,\beta,q,\epsilon)}$ is the zero of $\widetilde{\mathcal{P}}_n^{(\alpha,\beta,q,\epsilon)}(z)$ which lies outside $[-1, 1]$, then it is known that it converges to q. This completes the proof of the theorem. ■

4.4 The Coefficients $\tilde{\gamma}_n^{(\alpha,\beta,1,0,s)}$ and the Wilson Polynomials

Wilson polynomials were first introduced in [39] by J.A. Wilson (see also [1, p. 157] and [16]). They are usually denoted by $W_n(x; a, b, c, d)$ and are given by

$$\frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\left. \begin{matrix} -n,\ n+a+b+c+d-1,\ a+ix,\ a-ix \\ a+b,\ a+c,\ a+d \end{matrix} \right| 1 \right), \quad n \ge 0.$$

We denote by $\widetilde{W}_n(x; a, b, c, d)$ the monic Wilson polynomials. They satisfy the TTRR

$$\widetilde{W}_{n+1}(x; a, b, c, d) = (x - A_n - C_n + a^2)\widetilde{W}_n(x; a, b, c, d) - A_{n-1}C_n\widetilde{W}_{n-1}(x; a, b, c, d),$$

for $n \geq 1$, with $\widetilde{W}_1(x;a,b,c,d) = x - A_0 + a^2$ and $\widetilde{W}_0(x;a,b,c,d) = 1$, where

$$A_n = \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)},$$

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}.$$

We now see how the coefficients $\tilde{\gamma}_n^{(\alpha,\beta,1,0,s)}$ are related to a subclass of the Wilson polynomials. Here, we must assume $\alpha > 0$ and $\beta > -1$.

From Theorem 4.3, for the sequence of monic polynomials $\{\mathcal{R}_n^{(\alpha,\beta,1,0)}\}_{n\geq 0}$ given by

$$\tilde{\gamma}_n^{(\alpha,\beta,1,0,\lambda/t)} = \frac{\Delta_n^{(\alpha,\beta,1,0)}}{\Delta_{n-1}^{(\alpha,\beta,1,0)}} \frac{\lambda \mathcal{R}_{n-1}^{(\alpha,\beta,1,0)}(t)}{\mathcal{R}_n^{(\alpha,\beta,1,0)}(t)}, \quad n \geq 1,$$

where

$$\Delta_0^{(\alpha,\beta,1,0)} = 1 \quad \text{and} \quad \Delta_n^{(\alpha,\beta,1,0)} = n^2 \tilde{\mathfrak{b}}_n^{(\alpha,\beta,1,0)} \frac{\hbar_{n-1}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_n^{(\alpha,\beta,1,0)}} \Delta_{n-1}^{(\alpha,\beta,1,0)}, \quad n \geq 1,$$

there holds the TTRR

$$\mathcal{R}_n^{(\alpha,\beta,1,0)}(t) = [t + (e_n + f_n)\lambda]\mathcal{R}_{n-1}^{(\alpha,\beta,1,0)}(t) - e_{n-1} f_n \lambda^2 \mathcal{R}_{n-2}^{(\alpha,\beta,1,0)}(t), \quad n \geq 2,$$

with $\mathcal{R}_0^{(\alpha,\beta,1,0)}(t) = 1$ and $\mathcal{R}_1^{(\alpha,\beta,1,0)}(t) = t + (e_1 + f_1)\lambda$. Here,

$$e_n = e_n^{(\alpha,\beta,1,0)} = \frac{n^2 \hbar_{n-1}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_n^{(\alpha,\beta,1,0)}} \quad \text{and} \quad f_n = f_n^{(\alpha,\beta,1,0)} = \frac{(n-1)^2[\tilde{\mathfrak{b}}_{n-1}^{(\alpha,\beta,1,0)}]^2 \hbar_{n-2}^{(\alpha+1,\beta+1)}}{\tilde{\hbar}_n^{(\alpha,\beta,1,0)}},$$

for $n \geq 2$, with $f_1 = 0$ and $e_1 = e_1^{(\alpha,\beta,1,0)} = \hbar_0^{(\alpha+1,\beta+1)}/\tilde{\hbar}_1^{(\alpha,\beta,1,0)}$.

Hence, from (1.6) and (4.6),

$$\left.\begin{aligned} \frac{4(\alpha)_2(\beta+1)}{(\alpha+\beta+1)_3} e_{n+1}^{(\alpha,\beta,1,0)} &= \frac{(n+1)(\alpha+n+1)(\alpha+\beta+n+1)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)}, \\ \frac{4(\alpha)_2(\beta+1)}{(\alpha+\beta+1)_3} f_{n+1}^{(\alpha,\beta,1,0)} &= \frac{n(n+1)(\beta+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}, \end{aligned}\right\} \quad n \geq 0.$$

To obtain the connection between the Wilson polynomials $\{\widetilde{W}_n(x;a,b,c,d)\}_{n\geq 0}$ and the polynomials $\{\mathcal{R}_n^{(\alpha,\beta,1,0)}(t)\}_{n\geq 0}$, we take

$$c + d = 2; \quad b + c = \beta + 2; \quad b + d = \alpha + \beta + 2; \quad a + b + c + d = \alpha + \beta + 3.$$

From which we can state the following.

Theorem 4.10. *Let $\alpha > 0$ and $\beta > -1$. If*

$$a = \frac{\alpha}{2}, \quad b = 1 + \frac{\alpha}{2} + \beta, \quad c = 1 - \frac{\alpha}{2}, \quad d = 1 + \frac{\alpha}{2}$$

and

$$\lambda = -\frac{4(\alpha)_2(\beta+1)}{(\alpha+\beta+1)_3},$$

then

$$\widetilde{W}_n(t - a^2; a,b,c,d) = \mathcal{R}_n^{(\alpha,\beta,1,0)}(t), \quad n \geq 0.$$

4.5 The Real Zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$

The zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ in general can be complex numbers and determined from results given in Section 3.5. However, when $|q|=1$ and s is sufficiently small, we obtain that the zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ are real, simple and, even more, interlace with the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}$. In order to study the zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ for $|q|=1$, we consider an approach used in [29].

Let

$$\left.\begin{aligned} \mathcal{U}_n^{(\alpha,\beta,q,\epsilon)}(f) &= \int_{\mathcal{I}(q)} (x-q)^n f(x)\, d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x), \\ \mathcal{V}_n^{(\alpha,\beta,q)}(f) &= \int_{-1}^{1} (x-q)^n f'(x)\, d\mu^{(\alpha+1,\beta+1)}(x), \end{aligned}\right\} \quad n\ge 0,$$

which are, for our purposes, defined for $f\in C^1[-1,1]$ such that $\mathcal{U}_0^{(\alpha,\beta,q,\epsilon)}(f)$ exists. Then the following can be stated.

Lemma 4.11. *Let $n\ge 0$. Then*

$$\begin{aligned} \frac{(1-\epsilon)\tilde{\eta}^{(\alpha,\beta,q)}\eta^{(\alpha,\beta)}}{\eta^{(\alpha+1,\beta+1)}}\, \mathcal{V}_n^{(\alpha,\beta,q)}(f) &= (\alpha+\beta+n+2)\,\mathcal{U}_{n+2}^{(\alpha,\beta,q,\epsilon)}(f) \\ &\quad + [(\alpha+1)(q+1)+(\beta+1)(q-1)+2nq]\,\mathcal{U}_{n+1}^{(\alpha,\beta,q,\epsilon)}(f) \\ &\quad + n(q^2-1)\,\mathcal{U}_n^{(\alpha,\beta,q,\epsilon)}(f), \end{aligned}$$

where we assume $n(q^2-1)\mathcal{U}_n^{(\alpha,\beta,q,\epsilon)}(f)=0$ when $n=0$. Here, $\eta^{(\alpha,\beta)}$ is as in (4.1).

Proof. We start with the following result from [29, Lemma 3.1], which is valid for any f for which the respective integrals hold:

$$\begin{aligned} &\frac{1}{\eta^{(\alpha+1,\beta+1)}}\, \mathcal{V}_n^{(\alpha,\beta,q)}(f) \\ &\quad = (\alpha+\beta+n+2)\int_{-1}^{1}(x-q)^{n+1}f(x)(1-x)^\alpha(1+x)^\beta dx \\ &\qquad + [(\alpha+1)(q+1)+(\beta+1)(q-1)+2nq]\int_{-1}^{1}(x-q)^{n}f(x)(1-x)^\alpha(1+x)^\beta dx \\ &\qquad + n(q^2-1)\int_{-1}^{1}(x-q)^{n-1}f(x)(1-x)^\alpha(1+x)^\beta dx, \end{aligned}$$

for $n\ge 0$. Here, the last term on the right hand side does not exist if $n=0$ and hence, we assume it to be zero for $n=0$. From (4.1) and (4.2), observe that

$$\mathcal{U}_n^{(\alpha,\beta,q,\epsilon)}(f) = (1-\epsilon)\tilde{\eta}^{(\alpha,\beta,q)}\eta^{(\alpha,\beta)}\int_{-1}^{1}(x-q)^{n-1}f(x)(1-x)^\alpha(1+x)^\beta dx, \quad n\ge 1.$$

Thus, the statement of the lemma follows. ∎

Notice that

$$\begin{aligned} 0 &= \langle (x-q)^j, \mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}\rangle_{\mathfrak{S}^{(\alpha,\beta,q,\epsilon,s)}} \\ &= \int_{\mathcal{I}(q)}(x-q)^j\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}d\tilde{\mu}^{(\alpha,\beta,q,\epsilon)}(x) \;+\; s\int_{-1}^{1} j(x-q)^{j-1}\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)\,\prime}(x)d\mu^{(\alpha+1,\beta+1)}(x), \end{aligned}$$

for $n \ge 1$ and $j = 0, 1, \ldots, n-1$. Thus, for $n \ge 1$,

$$\mathcal{U}_0^{(\alpha,\beta,q,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}) = 0,$$

$$\mathcal{U}_j^{(\alpha,\beta,q,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}) = -s\,j\,\mathcal{V}_{j-1}^{(\alpha,\beta,q)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}), \quad j = 1, 2, \ldots, n-1.$$

Thus, from Lemma 4.11

$$\begin{aligned}
&(\alpha+\beta+j+2)\mathcal{U}_{j+2}^{(\alpha,\beta,q,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}) \\
&= -\Big[\frac{(1-\epsilon)\tilde{\eta}^{(\alpha,\beta,q)}\eta^{(\alpha,\beta)}}{s(j+1)\eta^{(\alpha+1,\beta+1)}} + \{(\alpha+1)(q+1) + (\beta+1)(q-1) + 2jq\}\Big]\mathcal{U}_{j+1}^{(\alpha,\beta,q,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}) \\
&\qquad - j(q^2-1)\mathcal{U}_j^{(\alpha,\beta,q,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}),
\end{aligned}$$

for $j = 0, 1, \ldots, n-2$.

For example, if $q = 1$, then

$$\begin{aligned}
\mathcal{U}_{j+2}^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)}) &= -\Big[\frac{(1-\epsilon)\tilde{\eta}^{(\alpha,\beta,1)}\eta^{(\alpha,\beta)}}{s(j+1)\eta^{(\alpha+1,\beta+1)}} + \{2(\alpha+1) + 2j\}\Big]\frac{\mathcal{U}_{j+1}^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)})}{(\alpha+\beta+j+2)} \\
&= 2\Big[\frac{(1-\epsilon)4\alpha(\alpha+1)(\beta+1)}{s(j+1)(\alpha+\beta+1)_3} - (\alpha+j+1)\Big]\frac{\mathcal{U}_{j+1}^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)})}{(\alpha+\beta+j+2)}
\end{aligned}$$

for $j = 0, 1, \ldots, n-2$. Thus,

$$\mathcal{U}_{j+1}^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n) = \frac{2(\alpha+j)}{s(\alpha+\beta+j+1)}\Big[\frac{4(1-\epsilon)\alpha(\alpha+1)(\beta+1)}{j(\alpha+j)(\alpha+\beta+1)_3} - s\Big]\mathcal{U}_j^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n), \tag{4.17}$$

for $j = 1, 2, \ldots, n-1$, where $\mathcal{S}_n = \mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)}$.

Likewise, if $q = -1$, then

$$\mathcal{U}_{j+1}^{(\alpha,\beta,-1,\epsilon)}(\mathcal{S}_n) = \frac{-2(\beta+j)}{s(\alpha+\beta+j+1)}\Big[\frac{4(1-\epsilon)(\alpha+1)\beta(\beta+1)}{j(\beta+j)(\alpha+\beta+1)_3} - s\Big]\mathcal{U}_j^{(\alpha,\beta,-1,\epsilon)}(\mathcal{S}_n),$$

for $j = 1, 2, \ldots, n-1$, where $\mathcal{S}_n = \mathcal{S}_n^{(\alpha,\beta,-1,\epsilon,s)}$.

Clearly,

$$\frac{4(1-\epsilon)\alpha(\alpha+1)(\beta+1)}{j(\alpha+j)(\alpha+\beta+1)_3} > \frac{4(1-\epsilon)\alpha(\alpha+1)(\beta+1)}{(n-1)(\alpha+n-1)(\alpha+\beta+1)_3}, \tag{4.18}$$

for $j = 1, 2, \ldots, n-1$, and

$$\frac{4(1-\epsilon)(\alpha+1)\beta(\beta+1)}{j(\beta+j)(\alpha+\beta+1)_3} > \frac{4(1-\epsilon)(\alpha+1)\beta(\beta+1)}{(n-1)(\beta+n-1)(\alpha+\beta+1)_3}, \quad j = 1, 2, \ldots, n-1.$$

Lemma 4.12. *Let $n \ge 2$ and $\mathcal{S}_n = \mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)}$. If $\alpha > 0$, $\beta > -1$ and*

$$s \le \frac{4(1-\epsilon)\alpha(\alpha+1)(\beta+1)}{(n-1)(\alpha+n-1)(\alpha+\beta+1)_3},$$

then

$$\operatorname{sgn}[\mathcal{U}_j^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)] = \operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)] \quad \text{for} \quad j = 2, 3, \ldots, n.$$

Moreover, if π is a polynomial of exact degree n such that $\pi'(1) > 0$ with all its zeros inside $(-1, 1)$, then

$$\operatorname{sgn}[\mathcal{U}_0^{(\alpha,\beta,1,\epsilon)}(\pi\,\mathcal{S}_n)] = \operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)].$$

Proof. The first part of the lemma follows from (4.17) and (4.18). To obtain the latter part, let $1-y_j$, $j=1,2,\ldots,n$, denote the zeros of π, where y_j are positive real numbers. Then

$$\begin{aligned}\pi(x) &= \text{const}[(x-1)+y_1][(x-1)+y_2]\cdots[(x-1)+y_n]\\ &= c_0+c_1(x-1)+c_2(x-1)^2+\cdots+c_n(x-1)^n,\end{aligned}$$

where const >0 and $c_j>0$ for $j=0,1,\ldots,n$. Hence,

$$\begin{aligned}\langle \pi, \mathcal{S}_n\rangle_{\tilde{\mu}^{(\alpha,\beta,1,\epsilon)}} = \mathcal{U}_0^{(\alpha,\beta,1,\epsilon)}(\pi\,\mathcal{S}_n) &= \sum_{j=0}^{n} c_j\mathcal{U}_0^{(\alpha,\beta,1,\epsilon)}((x-1)^j\,\mathcal{S}_n)\\ &= \sum_{j=0}^{n} c_j\mathcal{U}_j^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)\\ &= \sum_{j=1}^{n} c_j\mathcal{U}_j^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n).\end{aligned}$$

Thus, we can conclude the latter statement of the lemma. ∎

Likewise, we have the following.

Lemma 4.13. *Let $n\geq 2$ and $\mathcal{S}_n=\mathcal{S}_n^{(\alpha,\beta,-1,\epsilon,s)}$. If $\alpha>-1$, $\beta>0$ and*

$$s\leq \frac{4(1-\epsilon)(\alpha+1)\beta(\beta+1)}{(n-1)(\beta+n-1)(\alpha+\beta+1)_3},$$

then

$$\operatorname{sgn}[\mathcal{U}_j^{(\alpha,\beta,-1,\epsilon)}(\mathcal{S}_n)] = (-1)^{j-1}\operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,-1,\epsilon)}(\mathcal{S}_n)] \quad \textit{for} \quad j=2,3,\ldots,n.$$

Moreover, if π is a polynomial of exact degree n such that $\pi'(1)>0$ with all its zeros inside $(-1,1)$, then

$$\operatorname{sgn}[\mathcal{U}_0^{(\alpha,\beta,-1,\epsilon)}(\pi\,\mathcal{S}_n)] = \operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,-1,\epsilon)}(\mathcal{S}_n)].$$

The main results in this section are the next two theorems which state that when $|q|=1$ and s is sufficiently small, the zeros of $\mathcal{S}_n^{(\alpha,\beta,q,\epsilon,s)}$ are real, simple and interlace with the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,q,\epsilon)}$.

Theorem 4.14. *Let α, β and s be as in Lemma 4.12. Then the zeros of $\mathcal{S}_n(x)=\mathcal{S}_n^{(\alpha,\beta,1,\epsilon,s)}(x)$ and the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)}(x)$ interlace.*

Proof. Let $x_{n+1,j}=x_{n+1,j}^{(\alpha,\beta,1,\epsilon)}$ be a zero of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)}(x)$. Then from Lemma 4.12,

$$\operatorname{sgn}[\langle \pi, \mathcal{S}_n\rangle_{\tilde{\mu}^{(\alpha,\beta,1,\epsilon)}}] = \operatorname{sgn}[\mathcal{U}_0^{(\alpha,\beta,1,\epsilon)}(\pi\,\mathcal{S}_n)] = \operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)],$$

where $\pi(x)=\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)}(x)/(x-x_{n+1,j})$. On the other hand, using the Gaussian quadrature rule based on the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)}(x)$,

$$\langle \pi, \mathcal{S}_n\rangle_{\tilde{\mu}^{(\alpha,\beta,1,\epsilon)}} = \lambda_{n+1,j}\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)\prime}(x_{n+1,j})\,\mathcal{S}_n(x_{n+1,j}),$$

where $\lambda_{n+1,j}$ is the j-th associated quadrature weight. Thus,

$$\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)\prime}(x_{n+1,j})\,\mathcal{S}_n(x_{n+1,j}) = \operatorname{sgn}[\mathcal{U}_1^{(\alpha,\beta,1,\epsilon)}(\mathcal{S}_n)], \quad j=1,2,\ldots,n+1.$$

Let $x_{n+1,j}$ and $x_{n+1,j+1}$ be consecutive zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)}$. This means

$$\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)\prime}(x_{n+1,j})\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,1,\epsilon)\prime}(x_{n+1,j+1}) < 0.$$

Hence $\mathcal{S}_n(x_{n+1,j})\mathcal{S}_n(x_{n+1,j+1}) < 0$. Thus, we conclude that there exists exactly one zero of $\mathcal{S}_n(x)$ between $x_{n+1,j}$ and $x_{n+1,j+1}$. ∎

Finally, in the same way we can prove the following result.

Theorem 4.15. *Let α, β and s be as in Lemma 4.13. Then the zeros of $\mathcal{S}_n(x) = \mathcal{S}_n^{(\alpha,\beta,-1,\epsilon,s)}(x)$ and the zeros of $\widetilde{\mathcal{P}}_{n+1}^{(\alpha,\beta,-1,\epsilon)}(x)$ interlace.*

5 Conclusions

The results presented in this work contribute to the study of the monic orthogonal polynomials $\mathcal{S}_n(x) = \mathcal{S}_n(\nu_0, \nu_1, s; x)$ with respect to the positive definite Sobolev-type inner product

$$\begin{aligned}\langle f, g\rangle_{\mathfrak{S}} &= \langle f, g\rangle_{\nu_0} + s\,\langle f', g'\rangle_{\nu_1}\\ &= \int f(x)g(x)d\nu_0(x) + s\int f'(x)g'(x)d\nu_1(x),\end{aligned}$$

with $s > 0$, where the pair of positive measures $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line.

We have provided an extensive study of the connection formulas between the Sobolev orthogonal polynomials and the orthogonal polynomial with respect to the measure ν_0. We have also shown that the zeros of $\mathcal{S}_n(x)$ are the eigenvalues of a matrix which is a single line modification of the $n \times n$ monic Jacobi matrix associated with the measure ν_0. Some results emerged in the special case the connection coefficients are assumed to be rational functions.

Finally, we have worked with the Sobolev orthogonal polynomials that follow from the pair CPPM2K-J. We have studied the asymptotic properties of the Jacobi-Sobolev orthogonal polynomials and the associated connection coefficients with respect to their parameters. Some of these asymptotic results were achieved with the help of a novel representation to the parameter sequences of a traditional positive chain sequence in the theory of orthogonal polynomials. We have also proved that under certain conditions the connection coefficients are related to a subfamily of the Wilson polynomials.

5.1 Future Work

In this section we discuss some future directions of research that we plan to undertake.

5.1.1 Associated Fourier Approximation

Orthogonal polynomials with respect to Sobolev-type inner products have been vastly studied over the past decades. In addition to its interesting theoretical aspect, this topic provides important tools in the framework of approximation theory. In fact, this was the main motivation in [13] for introducing the concept of coherence between pairs of positive measures on the real line. To be more precise, the authors of [13] considered Fourier expansions of functions with respect to the Sobolev-type inner product $\langle\cdot\,,\cdot\rangle_{\mathfrak{S}}$ when $\{\nu_0, \nu_1\}$ is a coherent pair of positive measures on the real line.

Some interesting numerical tests comparing the Legendre-Sobolev Fourier series expansions and the ordinary Legendre-Fourier series expansions had already been considered in [12]. Moreover, in the context of coherent pairs of measures it is possible to obtain the associated Sobolev-Fourier coefficients with low computational cost (see [9]).

One of our object of studies for the near future is to carry out an analysis of the Fourier expansions of functions in terms of the sequences of polynomials orthogonal with respect to the Sobolev inner product $\langle \cdot, \cdot \rangle_{\mathfrak{S}}$ when $\{\nu_0, \nu_1\}$ is a CPPM2K on the real line.

To be more precise, assume f to be a function such that $\langle f, x^j \rangle_{\mathfrak{S}}$ exist for $0 \le j \le N$. Let $Q_N^{\mathfrak{S}}$ denote the best approximation of f by a polynomial of degree N with respect to the norm $\|f\|_{\mathfrak{S}} = (\langle f, f \rangle_{\mathfrak{S}})^{1/2}$. In other words,

$$\|f - Q_N^{\mathfrak{S}}\|_{\mathfrak{S}} = \min_{\pi \in \mathbb{P}_N} \|f - \pi\|_{\mathfrak{S}},$$

where $\mathbb{P}_N$ denotes the set of all polynomials of degree N with real coefficients.

Let $\{\hbar_n^{\mathfrak{S}}\}_{n \ge 0}$ be the positive constants such that

$$\langle \mathcal{S}_m, \mathcal{S}_n \rangle_{\mathfrak{S}} = \hbar_n^{\mathfrak{S}} \delta_{m,n}, \quad m, n = 0, 1, 2, \ldots.$$

By least squares method one finds

$$Q_N^{\mathfrak{S}}(x) = \sum_{j=0}^{N} b_j^{\mathfrak{S}} \mathcal{S}_j(x),$$

where

$$b_j^{\mathfrak{S}} = \frac{\langle f, \mathcal{S}_j \rangle_{\mathfrak{S}}}{\hbar_j^{\mathfrak{S}}}, \quad j = 0, 1, 2, \ldots, N.$$

We observe that $\hbar_j^{\mathfrak{S}}$ and $\langle f, \mathcal{S}_j \rangle_{\mathfrak{S}}$ can be recursively obtained using (see [11]):

$$\begin{aligned} \hbar_1^{\mathfrak{S}} &= \hbar_1^{\nu_0} + s\hbar_0^{\nu_1}, \\ \hbar_j^{\mathfrak{S}} &= -\gamma_{j-1}^2 \hbar_{j-1}^{\mathfrak{S}} + \hbar_j^{\nu_0} + sj^2[\hbar_{j-1}^{\nu_1} + \tau_{j-1}^2 \hbar_{j-2}^{\nu_1}], \quad j \ge 2, \end{aligned}$$

and

$$\begin{aligned} &\langle f, \mathcal{S}_0 \rangle_{\mathfrak{S}} = \langle f, 1 \rangle_{\nu_0}, \quad \langle f, \mathcal{S}_1 \rangle_{\mathfrak{S}} = \langle f, \mathcal{P}_1(\nu_0; \cdot) \rangle_{\nu_0} + s\langle f', 1 \rangle_{\nu_1}, \\ &\langle f, \mathcal{S}_j \rangle_{\mathfrak{S}} = \gamma_{j-1} \langle f, \mathcal{S}_{j-1} \rangle_{\mathfrak{S}} + \langle f, \mathcal{P}_j(\nu_0; \cdot) \rangle_{\nu_0} \\ &\qquad + sj\langle f', \mathcal{P}_{j-1}(\nu_1; \cdot) \rangle_{\nu_1} - sj\tau_{j-1} \langle f', \mathcal{P}_{j-2}(\nu_1; \cdot) \rangle_{\nu_1}, \quad j \ge 2. \end{aligned}$$

Thus, the easy determination of the Fourier coefficients $b_n^{\mathfrak{S}}$ makes this a convenient technique of approximation. This constitutes a nice future problem that should be further explored.

5.1.2 The CPPM2K-L and Associated Laguerre-Sobolev Orthogonal Polynomials

In [11] has been stated the special example of a CPPM2K on the real line $\{\nu_0, \nu_1\}$ in which ν_1 is given by the generalized Laguerre measure (see Table 1.1). Precisely, this example is such that

$$\text{CPPM2K-L:} \quad \{\nu_0, \nu_1\},$$

where the measures ν_0 and ν_1 are defined as follows:

$$\nu_0(x) = \frac{x^\alpha e^{-x}}{x-q} dx + \epsilon\, \boldsymbol{\delta}(q),$$

$$\nu_1(x) = x^{\alpha+1} e^{-x} dx,$$

with $\alpha > -1$ and $q \in (-\infty, 0]$. As shown in [11], the pair CPPM2K-L verifies (3.2) with

$$\begin{aligned} A_2(x) &= (x-q)(-x+\alpha+1) \quad \text{and} \\ A_3(x) &= x(x-q). \end{aligned}$$

It would be interesting to use the results contained in Chapters 2 and 3 of this thesis to study the Sobolev orthogonal polynomials and the associated connection coefficients that follow from the pair CPPM2K-L.

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