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Pure Spinor Superstring Partition Function

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Resumo

Nesta tese, mostramos o cálculo da função de partição dos espinores puros. O cálculo será executado de dois modos diferentes usando o método de fantasma-para-fantasma (até o décimo segundo nìvel massivo) e usando o método do ponto fixo (até o quinto nível massivo). Após incluir a contribuição das variáveis do setor da matéria $(x^m, \theta^\alpha, p_\alpha)$, nós derivamos o espectro massivo da supercorda aberta.

Embora os espinores puros sejam variáveis bosônicas, a função de partição dos espinores puros contém estados fermiônicos os quais começam aparecer a partir do segundo nível massivo. Estos estados fermiônicos vêm de funções que não são bem definidas globalmente no espaço dos espinores puros, e estão relacionados ao fantasma b no formalismo de spinores puros para a supercorda.

Palavras Chaves: Supersimetria; Supercordas; CFT; BRST; Espinores Puros; Quantização Covariante

Áreas do conhecimento: Física de Partículas e Teoría de Campos

Abstract

In this thesis, we have calculated the partition function of pure spinors. The computation is performed by using two different methods, namely ghosts-for-ghosts (up to the twelfth mass-level) and fixed point (up to the fifth mass-level) techniques. After adding the contribution from the $(x^m, \theta^\alpha, p_\alpha)$ matter variables, we reproduce the massive open superstring spectrum.

Even though pure spinor variables are bosonic, the pure spinor partition function contains fermionic states which first appear at the second mass-level. These fermionic states come from functions which are not globally defined in pure spinor space, and are related to the b ghost in the pure spinor formalism for the superstring.

Index

1 Introduction **1**

2 Partition functions for bosonic and fermionic variables **7**

2.1 Free bosonic $\beta\gamma$ and fermionic bc system 7

2.2 Bosonic string partition function 10

2.3 Superstring partition function 11

3 A simple constrained model **13**

3.1 The model 13

3.2 The partition function 14

3.3 BRST description of the model 16

3.4 BRST cohomology and symmetries of partition function 17

4 Pure spinor formalism for the superstring **20**

4.1 The action 21

4.2 Physical states 23

4.3 Massless pure spinor superstring state: fields and antifields 24

5 Naive partition function of pure spinors and missing states **28**

5.1 Definition of the partition function 28

5.2 Cohomology via partition function 29

5.3 Counting of gauge invariant polynomials and the missing states . . . 30

5.3.1 Level 0 gauge invariants 31

5.3.2 Field-antifield symmetry 31

5.3.3 Level 1 gauge invariants 32

5.3.4 Level 2 gauge invariants and a missing state 33

6 Partition function of pure spinors and its symmetries **36**

6.1 Ghost-for-ghost description of pure spinors 36

6.2 Partition function of the mini-BRST cohomology 39

6.3 Symmetries of the partition function 40

6.4 Padé approximants . . . 41

7 Fixed point formulas for the full partition function **46**

7.1 Review of fixed point formula for level zero partition function . . . 46

7.2 Fixed point formulas for the full partition function . . . 48

7.3 Partition functions for non-zero modes with spin character . . . 49

8 Pure spinor cohomology: Čech and Dolbeault descriptions **52**

8.1 Pure spinor sector as a curved $\beta\gamma$ system . . . 52

8.2 Čech description . . . 53

8.3 Čech/Dolbeault descriptions and the operator b_3 . . . 55

9 Derivation of the lightcone spectrum **58**

9.1 Lightcone spectrum of bosonic string from covariant partition function 59

9.2 Lightcone spectrum from pure spinor partition function . . . 60

9.2.1 Twisting of t-charge . . . 61

9.2.2 Massless states . . . 62

9.2.3 First massive states . . . 62

9.2.4 Higher massive states . . . 63

9.3 Q_0-cohomology and absence of on-shell condition . . . 63

9.3.1 Ghost-for-ghost method with an $SO(8)$ parameterization of pure spinor . . . 65

9.3.2 Lightcone "off-shell" spectrum from Q_0-cohomology . . . 66

10 Summary and future applications **68**

A $SO(10)$ conventions and formulas **72**

A.1 Dynkin labels . . . 72

A.2 Some dimension formulas . . . 73

B Table of partition functions **74**

B.1 Partition functions without spin: number of states . . . 74

B.2 Spin partition functions . . . 74

C Computation of higher level character formulas **78**

References **81**

Chapter 1

Introduction

String theory is one of the most promising candidate theory we have for a unified description of the fundamental particles and forces in nature including gravity. Its basic building blocks are one dimensional extended objects (strings) which for their mathematical consistency can be defined in 26 (bosonic string) or in 10 spacetime dimensions (superstrings). In contrast with particle theories, string theory is highly constrained in the choice of interactions, supersymmetries and gauge groups. In fact, all the usual particles emerge as excitations of the string and the interactions are simply given by the geometric splitting and joining of these strings.

There are five consistent superstring theories, all of them are related by a net of dualities [1]: type I, type IIA, type IIB, heterotic $E_8 \times E_8$ and heterotic $SO(32)$. These five theories are described using two standard formalisms: the Ramond-Neveu-Schwarz (RNS) formalism and Green-Schwarz (GS) formalism. The RNS formalism has manifest worldsheet supersymmetry, but the spacetime supersymmetry is not. The spacetime supersymmetry can be verified after perform the so-called GSO projection (Gliozzi-Scherk-Olive) [2]. The fact that this formalism does not have a manifest spacetime supersymmetry makes it hard to perform scattering amplitude computations as the vertex operators which describe fermions in spacetime are complicated, in addition to the technical problem of summing over spin structures (different periodic or antiperiodic boundary conditions for the worldsheet fermions around various cycles) [3].

To solve the issues we have just mentioned, it would be nice to have a formulation with manifest spacetime supersymmetry. In fact there is such formulation, it is the GS formalism, although covariant quantization of this theory seems hard because the existence of a technical problem dealing with mixture of first and second class constraints, the GS superstring can be quantized in the light cone gauge, but due to the lost of manifest covariance, problems in computing scattering amplitudes arise. And in fact using this formalism only four-point tree and one loop scattering amplitudes were computed explicitly [4].

About seven years ago, a new formalism for the superstring was proposed which is manifestly super-Poincaré covariant and which can be easily quantized [5]. This pure spinor formalism for the superstring has passed various consistency checks and has been used to compute multiloop amplitudes and to describe Ramond-Ramond backgrounds in a super-Poincaré covariant manner.

One of the key ingredients of the formalism is the use of a bosonic variable λ^α transforming as an $SO(10)$ spinors and satisfying a pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ for $m = 1$ to 10. Thanks to this constraint, the BRST operator $Q = \int dz \lambda^\alpha d_\alpha$ is nilpotent. In a sense, λ^α can be thought as the ghost for the Green-Schwarz-Siegel worldsheet constraint d_α. Although the use of such a constrained ghost system is unconventional, it can be used to construct vertex operators and to define string amplitudes as worldsheet correlation functions [5, 6, 7]. Dependence of the amplitudes on the non-zero modes of λ^α and its conjugate ω_α is fixed by the operator product expansions OPES, and the functional integral over the zero-modes can be inferred by requiring BRST and super-Poincaré invariance.

Although the basic ingredients for computing on-shell amplitudes are already there, it would be useful to understand the functional integral over λ^α without relying on the BRST invariance, or equivalently, to understand the nature of the Hilbert space in the operator formalism. This would be necessary, for example, if one wishes to apply the formalism to construct a superstring field theory. In this thesis, this Hilbert space question will be answered by explicitly computing the partition function of pure spinor variables.

There are two basic strategies to study the structure of the Hilbert space for the pure spinors. The first is to deal directly with the constrained variables (this approach is known as the curved $\beta\gamma$ description), and define the Hilbert space as the space of operators that are consistent with the pure spinor constraint [5]. To be consistent with the constraint, the conjugate ω_α has to appear in combinations invariant under the "gauge transformations" $\delta_\Lambda \omega_\alpha = \Lambda_m (\gamma^m \lambda)_\alpha$ generated by the constraint $\lambda \gamma^m \lambda$. The other is to try to remove the constraint by introducing BRST ghosts. The constraint is then expressed effectively as the cohomology condition of a BRST operator D [8] *.

In order to properly understand the Hilbert space of the pure spinors variables, in a recent work [10], we have considered models with a single irreducible quadratic constraint. It has been argued that the curved $\beta\gamma$ and BRST formalisms provide

*D should not to be confused with the "physical" BRST operator $Q = \int \lambda^\alpha d_\alpha$ of the pure spinor formalism. (Because a possible use of D is to combine it with Q to construct a single nilpotent operator $\hat{Q} = D + Q + \cdots$, we called D a "mini-BRST" operator in [9].)

equivalent classical descriptions of these models, although, quantum mechanically, the Hilbert spaces of the two descriptions differ slightly due to the different normal ordering prescriptions used. Nevertheless, since our partition function is defined so that it is insensitive to quantum corrections, the two descriptions lead to the same partition function even quantum mechanically. We shall use the partition function as a guide to study the structure of the Hilbert space.

One of the virtues for studying these simpler models is that the BRST description is very effective, allowing a close study of its cohomology. In particular, the full partition function of the BRST cohomology can be easily computed and it manifestly possesses two important symmetries that we shall call "field-antifield" and "$*$-conjugation" symmetries. The former implies that, after coupling to "matter" variables (p, θ), the cohomology of the "physical" BRST operator $Q = \int \lambda p$ comes in field-antifield pairs.

Coming back to the case of pure spinor variables; for states depending only on the zero modes of λ^α, the Hilbert space of states in the pure spinor formalism is easily understood and is given by arbitrary polynomials in λ^α. Since λ^α is constrained to satisfy $\lambda\gamma^m\lambda = 0$, these polynomials are parameterized by constants $f_{((\alpha_1 \ldots \alpha_n))}$ for $n = 0$ to ∞ which are symmetric in their spinor indices and which satisfy $\gamma_m^{\alpha_1\alpha_2} f_{((\alpha_1 \ldots \alpha_n))} = 0$.

As shown in [11, 12], this Hilbert space for the zero modes is described by the partition function

$$Z_0(t) = (1-t)^{-16}(1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8)$$

where λ^α carries $+1$ t-charge. Expanding $Z_0(t)$ in powers of t, one reproduces the independent number of $f_{((\alpha_1 \ldots \alpha_n))}$'s at order t^n. After multiplying by $(1-t)^{16}$ which comes from the partition function for the 16 θ^α zero modes, $(1-t)^{16} Z_0(t)$ describes the x-independent degrees of freedom for the massless sector of the open superstring. For example, 1 describes the Maxwell ghost, $-10t^2$ describes the photon, $+16t^3$ describes the photino, and the remaining terms describe the antifields for these states. Note that $Z_0(t)$ satisfies the identity $Z_0(1/t) = -t^8 Z_0(t)$ which implies a symmetry between the fields and antifields.

We shall perform a similar analysis for the non-zero modes of λ^α, as well as the modes of its conjugate momentum w_α. The partition function for the lowest non-zero mode was already computed in [12], and we shall extend this computation for higher massive modes. The computation will be performed in two ways, firstly using the ghosts-for-ghosts method (up to the twelfth mass-level) [13] and secondly using the fixed point method (up to the fifth mass-level) [9, 11].

Due to the fact that the pure spinor constraint is infinitely reducible, an infinity chain of ghosts is required. Using the multiplicities N_k of these ghosts, we have written a formal expression for the partition function of pure spinors [9, 13]

$$Z(q,t) = \prod_{k=1}^{\infty} [(1-t^k)^{-N_k} \prod_{h=1}^{\infty} (1-q^h t^k)^{-N_k} (1-q^h t^{-k})^{-N_k}] \, .$$

Although it may seem difficult to extract useful information from this formal expression, by appealing to some regularization procedure in order to guarantee the convergence of the infinite product over k, character formulas $Z_h(t)$ were calculated up to the twelfth mass-level ($h = 12$). A suitable regularization procedure which respects the two important symmetries of the partition function has been used in [13]. This prescription for computing higher mass-level character formulas is based on Padé approximants. It was shown by explicit computation that the firsts five character formulas obtained by means of Padé approximants are in agreement with the ones found by means of fixed point technique [9].

Nevertheless using the ghosts-for-ghosts method (Padé approximants) the partition function has been computed without the spin dependence on the states. Spin dependence is crucial if we want to prove that the full partition function (including the contribution of the worldsheet matter sector) correctly reproduces the light cone open superstring spectrum [9]. Therefore, it would be interesting to know the character formula with the spin dependence in the ghosts-for-ghosts scheme and implement another prescription like Padé approximants which takes into account the spin dependence on the states.

After including the contribution from the matter variables $(x^m, \theta^\alpha, p_\alpha)$, we show that the partition function, up to the fifth mass-level, correctly describes the massive levels of the open superstring spectrum [9].

In computing the partition function for the non-zero modes of λ^α and w_α, we will discover a surprise. Because the constraint $\lambda\gamma^m\lambda = 0$ generates the gauge transformation $\delta_\Lambda w_\alpha = \Lambda^m(\gamma_m\lambda)_\alpha$ for the conjugate momentum, one naively expects that the Hilbert space is described by polynomials of λ^α and w_α (and their worldsheet derivatives) which are invariant under this gauge transformation. However, in addition to these ordinary gauge invariant states, we will discover that field-antifield symmetry implies that there are additional states starting at the second mass level which contribute to the partition function with a minus sign. These additional states should therefore be interpreted as fermions, which is surprising since λ^α and w_α are bosonic variables.

We will argue that these extra fermionic states are related to the b ghost in the pure spinor formalism, and come from functions which are not globally defined on the space of the pure spinors. Remember that the b ghost satisfying $\{Q, b\} = T$ is a composite operator constructed from both the matter variables $(x^m, \theta^\alpha, p_\alpha)$ and ghost variables $(\lambda^\alpha, w_\alpha)$. This composite operator cannot be globally defined on all patches, and in the non-minimal pure spinor formalism, is described by the sum of a zero-form, one-form, two-form and three-form [14]. The three-form in the b ghost is independent of the matter variables $(x^m, \theta^\alpha, p_\alpha)$, and will be identified with a fermionic scalar in the pure spinor partition function at the second mass level. At higher mass levels, the extra fermionic states in the pure spinor partition function can be similarly identified with products of this fermionic three-form with polynomials of λ^α and w_α (and their worldsheet derivatives).

In hindsight, the appearance of the b ghost in the $(\lambda^\alpha, w_\alpha)$ partition function is not surprising since any covariant description of massive states is expected to include auxiliary spacetime fields whose vertex operator involves the b ghost. Nevertheless, the manner in which the b ghost appears in a partition function for bosonic worldsheet variables is quite remarkable and suggests that many important features of the b ghost can be learned by studying the pure spinor partition function.

The plan of this thesis is as follows: In chapter 2, we study the partition function of free bosonic and fermionic conformal field theories. These theories are of particular importance since they are very used in bosonic string as well as in superstring theories.

In chapter 3, we study the partition function of a simple model defined by a single quadratic constraint. The model is studied using two different approach, namely the intrinsic curved $\beta\gamma$ description and the BRST description.

In chapter 4, we review the basics of the pure spinor formalism for the superstring and study in detail the lowest open superstring excitation which describes on-shell super-Yang-Mills theory.

In chapter 5, the partition function of gauge invariant polynomials constructed out of $(\lambda^\alpha, w_\alpha)$ and their derivatives are computed by explicitly constructing them at lower Virasoro levels. We point out that the space of gauge invariants is insufficient if one requires field-antifield symmetry; in particular, a fermionic state is found to be missing at level 2, which will be identified as a term in the composite b ghost.

Chapter 6 is devoted to the computation of the partition function including the missing states found in the previous chapter. We use two methods for the computation, each with its advantages and disadvantages. The first method utilizes Chesterman's BRST description of the pure spinor system [8] involving ghosts-for-ghosts. A nice feature of this method is that two important symmetries—field-antifield and "$*$-conjugation" symmetries—are (formally) manifest. However, since this description requires an infinite tower of ghosts-for-ghosts, the expression for the partition function is not rigorously defined. Nevertheless, we show that there is an unambiguous way to compute the partition function level by level respecting the two symmetries.

In chapter 7, we use the second method for computing the partition function, namely a fixed point formula, which generalizes the zero mode result of [11]. The formula includes the spin dependence of the states, and the computation is fairly straightforward. However, it misses some finite number of states that must be recovered by imposing the two symmetries given in chapter 6.

In chapter 8, we explain the structure of the Hilbert space of the pure spinor system using Čech and Dolbeault descriptions. We give an explanation of the states which does not correspond to the usual gauge invariant polynomials, those missing fermionic states will be identified as elements of the third Čech cohomology.

In chapter 9, we relate the partition function and the superstring spectrum. After including the contribution from the matter variables, we show that a simple twisting of the charges gives rise to the partition function of lightcone fields and their antifields.

A summary and further interesting applications are given in chapter 10. Several appendices are included for convenience. Some group theoretical formulas are collected in appendix A, and a list of partition functions can be found in appendix B. Finally in appendix C, we present some details involved in the computations of higher level character formulas.

Chapter 2

Partition functions for bosonic and fermionic variables

In this chapter, we are going to study the partition function of free bosonic and fermionic conformal field theories (CFT). These theories are of particular importance since they are very used in bosonic string as well as in superstring theories. For the purposes of this thesis, let us study the partition function of these theories as simplest examples before going into the problems dealing with constrained CFT models as the pure spinors case.

2.1 Free bosonic $\beta\gamma$ and fermionic bc system

Let us start with the simplest case, namely the free $\beta\gamma$ CFT defined by commuting variables β and γ which have conformal weights $(1,0)$ and $(0,0)$ respectively, with action

$$S = \int d^2 z \beta \bar{\partial} \gamma \, . \tag{2.1}$$

These fields are holomorphic by the equations of motion,

$$\bar{\partial}\beta = \bar{\partial}\gamma = 0 \, . \tag{2.2}$$

The equations of motion and the operator products OPES are derived in the standard way.

Because the statistic is bosonic, some signs in operator products are different,

$$\beta(x)\gamma(y) \sim \frac{1}{x-y} \, , \qquad \gamma(x)\beta(y) \sim -\frac{1}{x-y} \, . \tag{2.3}$$

The energy momentum tensor is

$$T = \beta \partial \gamma \, , \tag{2.4}$$

using the OPES (2.3), it is easy to show that the central charge is given by $c = 2$.

The action (2.1) has the following global $U(1)$ symmetry: $\delta\gamma = +\epsilon\gamma$, $\delta\beta = -\epsilon\beta$. The corresponding Noether current is

$$J = \beta\gamma\,. \tag{2.5}$$

Therefore the $U(1)$ charge for the fields β and γ are given by -1 and $+1$ respectively.

Now let us define and compute the partition function for this system. Our partition function is defined as the trace

$$Z(q,t) = \mathrm{Tr}_{\mathcal{H}}[(-1)^F q^{L_0} t^{J_0}], \tag{2.6}$$

over the Hilbert space $\mathcal{H}$ (which will be defined soon), where $(-1)^F = +1$ for bosonic and $(-1)^F = -1$ for fermionic states. L_0 and J_0 are the zero mode of the energy momentum tensor (2.4) and the $U(1)$ current (2.5).

As we know by the equation of motion (2.2), all fields are holomorphic. One can then expand a field f in modes, $f(z) = \sum_{k=-\infty}^{\infty} f_k/z^{k+h}$, where h is the conformal weight of f. The vacuum state $|0\rangle$ is defined so that $f_{k+h}|0\rangle = 0$, for $k \geq 1$. More explicitly, in terms of the modes of the basic fields β and γ, the vacuum is defined such that is annihilated by the modes: $\beta_{k-1}|0\rangle = \gamma_k|0\rangle = 0$, for $k \geq 1$.

The states are built by acting on the vacuum with the remaining field modes. Instead of working with the modes, for convenience we are going to use local operators, in accord with the state-operator isomorphism. For instance, the state $\gamma_k|0\rangle$ corresponds to the operator $\partial^k\gamma$. A general state is then a polynomial in the fields and their derivatives. The set of these states is that we have denoted by $\mathcal{H}$.

As a pedagogical illustration, let us compute the level $h = 0$, $h = 1$ and $h = 2$ character formulas and then write down the full partition function.

Weight 0 At the lowest level $h = 0$, the states are given by

$$\mathcal{H}_0 = \{\gamma^n\}\,, \qquad n \geq 0\,, \tag{2.7}$$

therefore the partition function (character) at this level reads

$$Z_0(t) = \mathrm{Tr}_{\mathcal{H}_0}[(-1)^F t^{J_0}] = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}\,. \tag{2.8}$$

Weight 1 At level $h = 1$, the states are given by

$$\mathcal{H}_1 = \{\beta\gamma^n,\ \gamma^n\partial\gamma\}\,, \qquad n \geq 0\,, \tag{2.9}$$

therefore the partition function at this level is given by

$$Z_1(t) = \frac{t^{-1}}{1-t} + \frac{t}{1-t} = \Big(t^{-1} + t\Big) Z_0(t)\,. \tag{2.10}$$

Weight 2   At level $h = 2$, the states are given by

$$\mathcal{H}_2 = \{\gamma^n \beta \partial \gamma,\ \gamma^n \partial \beta,\ \gamma^n \partial^2 \gamma,\ \beta^2 \gamma^n, \gamma^n (\partial \gamma)^2\}\ , \qquad n \geq 0\ , \tag{2.11}$$

therefore the partition function at this level is given by

$$Z_2(t) = \Big(1 + t^{-1} + t + t^{-2} + t^2\Big) Z_0(t)\,. \tag{2.12}$$

Full expression for the partition function   As we can see, the partition function $Z(q,t)$ can be expanded like

$$Z(q,t) = \sum_{h=0}^{\infty} Z_h(t) q^h\ , \tag{2.13}$$

where we have computed explicitly up to level $h = 2$ the character formulas $Z_h(t)$. From the results (2.8), (2.10) and (2.12), we can easily obtain the expression which reproduces these character formulas

$$Z(q,t) = \frac{1}{1-t} \prod_{h=1}^{\infty} \frac{1}{(1-t^{-1}q^h)(1-tq^h)}\,. \tag{2.14}$$

In general if we have a field γ with N components, i.e. $\gamma = (\gamma^1, \gamma^2, \cdots, \gamma^N)$, the full partition function reads

$$Z(q,t) = \frac{1}{(1-t)^N} \prod_{h=1}^{\infty} \frac{1}{(1-t^{-1}q^h)^N (1-tq^h)^N}\,. \tag{2.15}$$

For the case of a free bc CFT defined by anticommuting variables b and c which have conformal weights $(1,0)$ and $(0,0)$ and $U(1)$ t-charge -1 and $+1$ respectively, the partition function is given by

$$Z(q,t) = (1-t) \prod_{h=1}^{\infty} (1-t^{-1}q^h)(1-tq^h)\,. \tag{2.16}$$

In general if we have a fermionic field c with N components, i.e. $c = (c^1, c^2, \cdots, c^N)$, the full partition function reads

$$Z(q,t) = (1-t)^N \prod_{h=1}^{\infty} (1-t^{-1}q^h)^N (1-tq^h)^N\,. \tag{2.17}$$

Let us use the ideas given in this section and study the partition function of more interesting models, namely the bosonic string and superstring. We are going to compute the partition function of these models using the lightcone coordinates since the lightcone is the quickest route to obtain the physical spectrum. We left the covariant computations for chapters 6, 7 and section 9.1.

2.2 Bosonic string partition function

As it is well known (for instance see reference [15]), the bosonic string action in the lightcone gauge is given by

$$S = \int d^2z \, \frac{1}{2} \partial X^i \bar{\partial} X^i \,, \tag{2.18}$$

where only the transverse coordinates X^i, $i = 1, 2, \cdots, 24$ are presented. We can expand the fields in modes, for instance

$$\partial X^i(z) = -\frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^i}{z^{n+1}} \,, \qquad \bar{\partial} X^i(\bar{z}) = -\frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^i}{\bar{z}^{n+1}} \,. \tag{2.19}$$

In this gauge all string excitations are generated by the transverse oscillators α_n^i. Thus, for example, the first exited open string state is given by $\alpha_{-1}^i|0\rangle$. A list of the firsts 3 open string states is given in the following table

Open bosonic string spectrum		
Mass	States	# of degrees
$M^2 = -1$	$\vert 0\rangle$	**1**
$M^2 = 0$	$\alpha_{-1}^i \vert 0\rangle$	**24**
$M^2 = 1$	$\alpha_{-2}^i \vert 0\rangle$, $\alpha_{-1}^i \alpha_{-1}^j \vert 0\rangle$	**24+300**

(2.20)

The mass square of the open string states is given by the eigenvalue of the operator

$$M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - 1 = N - 1 \,, \tag{2.21}$$

where N is the *level*

$$N = \sum_{i=1}^{24} \sum_{n=1}^{\infty} n N_{in} \,. \tag{2.22}$$

The mass of each state is thus determined in terms of the level of excitation.

The light cone open bosonic string partition function

$$Z_{lc}(q) = \mathrm{Tr}_{\mathcal{H}_{lc}}[(-1)^F q^{N-1}], \tag{2.23}$$

can be computed as the trace over the occupation number N_{in} and it breaks up into a sum

$$Z_{lc}(q) = q^{-1} \prod_{i=1}^{24} \prod_{n=1}^{\infty} \sum_{N_{in}=0}^{\infty} q^{nN_{in}} \,, \tag{2.24}$$

the various sums are geometric,

$$\sum_{N=0}^{\infty} q^{nN} = (1-q^n)^{-1}\,, \tag{2.25}$$

and so we obtain

$$Z_{lc}(q) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-\mathbf{24}}\,. \tag{2.26}$$

Expanding this lasts expression (2.26) in powers of q we get

$$Z_{lc}(q) = \mathbf{1}q^{-1} + \mathbf{24} + \mathbf{324}q + \mathbf{3200}q^2 + \cdots\,, \tag{2.27}$$

and therefore the partition function is describing the number of open string physical degrees at each mass level as it is shown in table (2.20).

2.3 Superstring partition function

For the superstring case, we are going to use the lightcone version of Green-Schwarz (GS) formalism [16] since this formalism provides us the lightcone spectrum without using any GSO projection *. The action of the GS superstring in the lightcone gauge is given by

$$S = \int d^2z\,\Big(\frac{1}{2}\partial X^i\bar{\partial}X^i + \theta^a\bar{\partial}\theta^a\Big)\,, \tag{2.28}$$

where only the transverse coordinates X^i, $i = 1,2,\cdots,8$ and θ^a, $a = 1,2,\cdots,8$ are presented. Note that the X^i variable carries vectorial index while the θ^a variable carries quiral spinorial index a of the group $SO(8)$. We can expand the fields in modes, for instance

$$\partial X^i(z) = -\frac{i}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\frac{\alpha_n^i}{z^{n+1}}\,,\qquad \theta^a(z) = \sum_{n=-\infty}^{\infty}\frac{\theta_n^a}{z^{n+\frac{1}{2}}}\,. \tag{2.29}$$

In this gauge all superstring excitations are generated by the transverse oscillators α_n^i and θ_n^a. A list of the firsts 3 open superstring states is given in the following table

Open superstring spectrum			
Mass	States	bosonic	fermionic
$M^2=0$	$\lvert 0\rangle^{\dot{a},j}$	**8**	**8**
$M^2=1$	$\alpha_{-1}^i\lvert 0\rangle^{\dot{a},j}$, $\theta_{-1}^a\lvert 0\rangle^{\dot{a},j}$	**128**	**128**
$M^2=2$	$\alpha_{-2}^i\lvert 0\rangle^{\dot{a},j}$, $\alpha_{-1}^i\alpha_{-1}^j\lvert 0\rangle^{\dot{a},j}$, $\theta_{-2}^a\lvert 0\rangle^{\dot{a},j}$ $\theta_{-1}^a\theta_{-1}^b\lvert 0\rangle^{\dot{a},j}$, $\theta_{-1}^a\alpha_{-1}^i\lvert 0\rangle^{\dot{a},j}$	**1152**	**1152**

(2.30)

*In the RNS formalism the GSO projection is required in order to achieve spacetime supersymmetry.

The mass square of the open superstring states is given by the eigenvalue of the operator

$$M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n\theta^a_{-n}\theta^a_n = N\,, \tag{2.31}$$

where N is the *level*. The mass of each state is thus determined in terms of the level of excitation.

Due to supersymmetry the light cone open superstring partition function

$$Z_{lc}(q) = \mathrm{Tr}_{\mathcal{H}_{lc}}[(-1)^F q^N] \tag{2.32}$$

vanishes identically, nevertheless we can compute the trace over the bosonic (or fermionic) sector shown in table (2.30). The trace breaks up into a sum given as a result

$$Z_{lc,bosonic}(q) = \mathbf{8}\prod_{n=1}^{\infty}\Big(\frac{1+q^n}{1-q^n}\Big)^{\mathbf{8}}. \tag{2.33}$$

Expanding this lasts expression (2.33) in powers of q we get

$$Z_{lc,bosonic}(q) = \mathbf{8} + \mathbf{128}q + \mathbf{1152}q^2 + \mathbf{7680}q^3 + \cdots\,, \tag{2.34}$$

and therefore the partition function is describing the number of superstring states at each mass level (2.30).

The fermionic part of the partition function is given by

$$Z_{lc,fermionic}(q) = -\mathbf{8}\prod_{n=1}^{\infty}\Big(\frac{1+q^n}{1-q^n}\Big)^{\mathbf{8}}, \tag{2.35}$$

and so the total partition function $Z_{lc,total}(q) = Z_{lc,bosonic}(q) + Z_{lc,fermionic}(q)$ vanishes identically as it was expected.

One aim of this thesis is to derive the lightcone degrees of freedom for the superstring states from the covariant partition function computed in the pure spinor formalism. Before going into this formalism, let us study a simple toy model.

Chapter 3

A simple constrained model

In order to understand the problem of computing the pure spinor partition function, we are going to study, in this chapter, a simple model based on reference [10].

As mentioned in the introduction, the main motivation for this investigation is to understand the proper Hilbert space of the pure spinor superstring by computing its partition function. We begin the study by defining and computing the partition function of gauge invariant polynomials in a simple model. Our main finding will be that, starting from the first mass level, the space of naive gauge invariants lacks the so-called field-antifield symmetry [9, 10] because some finite number of *fermionic* operators are missing.

On the other hand, in the BRST description of the model, the partition function of the BRST cohomology is found to satisfy the field-antifield symmetry. Furthermore, the BRST partition function is found to possess another discrete symmetry which we call "$*$-conjugation symmetry". Both field-antifield and $*$-conjugation symmetries reflect certain dualities of the cohomology, and their existence plays an important role in the consistency of the pure spinor formalism [9].

3.1 The model

The toy model is based on a pair of bosonic fields λ and $\tilde{\lambda}$ and their respective conjugate fields w and $\tilde{w}$, with action

$$S = \int d^2z(w\bar{\partial}\lambda + \tilde{w}\bar{\partial}\tilde{\lambda}) \; , \tag{3.1}$$

where the fields λ, $\tilde{\lambda}$, w, $\tilde{w}$ have conformal weights (0,0), (0,0), (1,0), (1,0) respectively, and additionally the fields λ and $\tilde{\lambda}$ are constrained to satisfy

$$\lambda\tilde{\lambda} = 0. \tag{3.2}$$

The action (3.1) has the following global $U(1)$ symmetry: $\delta\lambda = +\epsilon\lambda$, $\delta\tilde{\lambda} = +\epsilon\tilde{\lambda}$, $\delta w = -\epsilon w$, $\delta\tilde{w} = -\epsilon\tilde{w}$. The corresponding Noether current is

$$J = \lambda w + \tilde{\lambda}\tilde{w}. \tag{3.3}$$

Therefore the $U(1)$ charge for the fields $\lambda, \tilde{\lambda}$ and $w, \tilde{w}$ are given by $+1, +1$ and $-1, -1$ respectively. Note that due to the constraint (3.2) the action (3.1) has a gauge symmetry

$$\delta_\Lambda \lambda = \delta_\Lambda \tilde{\lambda} = 0 \quad , \quad \delta_\Lambda w = \Lambda\tilde{\lambda} \quad , \quad \delta_\Lambda \tilde{w} = \Lambda\lambda \; . \tag{3.4}$$

Now we are going to compute the partition function of this system by listing all possible gauge invariant states, i.e. operators which are compatible with the constraint (3.2) and the gauge invariance (3.4).

3.2 The partition function

We begin by describing the definition of our partition function. The characters of the states we are interested in are

- statistics (Grassmanity) measured by $(-1)^F$ (F: fermion number operator),
- weight (Virasoro level) measured by L_0, and
- t-charge measured by a $U(1)$ charge J_0.

By introducing formal variables (q, t) to keep track of the charges, the partition function is defined as

$$Z(q,t) = \sum_{J_0, h} (-1)^F q^h t^{J_0}, \tag{3.5}$$

for states: $|F, J_0, h\rangle$, where $(-1)^F = +1$ for bosonic and $(-1)^F = -1$ for fermionic states, h is the conformal weight and J_0 the respective $U(1)$ charge.

We now count the number of gauge invariant polynomials constructed out of λ, $\tilde{\lambda}$, w, $\tilde{w}$ and their derivatives, and compute the partition function (3.5). Similar counting of gauge invariant polynomials for the present and related models is given in [17].

Weight 0 At the lowest level $h = 0$, the states are exhausted by

$$\mathcal{H}_0 = \{1, \lambda^n, \tilde{\lambda}^n\} \; , \qquad n > 0 \; , \tag{3.6}$$

therefore the partition function (character) at this level reads

$$Z_0(t) = 1 + \frac{t}{1-t} + \frac{t}{1-t} = \frac{1-t^2}{(1-t)^2} \,. \tag{3.7}$$

Note that the level 0 partition function satisfies the following identity

$$Z_0(t) = -t^{-a_g} Z_0(1/t) \,. \tag{3.8}$$

We call this property as field-antifield symmetry. As explained in [11], the number a_g (which in our model is $a_g = 0$) on the exponent is the ghost number anomaly of the system. Since this symmetry plays an important role in our forthcoming discussions (as well as in the pure spinor superstring), let us explain the implication of its existence before going on to the weight 1 partition function.

Field-antifield symmetry Suppose one couples the system to free fermionic bc systems $(p, \tilde{p}; \theta, \tilde{\theta})$ of weight $(1, 1; 0, 0)$, and extends the definition of the t-charge to the new sector as $t(p, \tilde{p}; \theta, \tilde{\theta}) = (-1, -1; 1, 1)$. By analogy with the pure spinor superstring, one also defines the "physical" BRST operator as

$$Q = \int (\lambda p + \tilde{\lambda} \tilde{p}) \,. \tag{3.9}$$

Then the symmetry $Z_0(t) = -t^{-a_g} Z_0(1/t)$ implies that all Q-cohomology elements appear in "spacetime" field-antifield pairs [18]. Indeed, the total zero-mode partition function reads

$$\mathbf{Z}_0(t) = Z_{\lambda,0}(t) Z_{\theta,0}(t) = 1 - t^2 \,, \tag{3.10}$$

which is accounted for by a pair of "massless" cohomologies

$$1 \text{ at } t^0 \quad \leftrightarrow \quad (\lambda\theta) = \lambda\tilde{\theta} + \tilde{\lambda}\theta \text{ at } \; -t^2 \,. \tag{3.11}$$

The field-antifield symmetry implies the existence of a non-degenerate inner product that pairs every operator V to its antifield V_A

$$(V, V_A) \;\; = 1 \,. \tag{3.12}$$

For the case at hand, the inner product can be defined as the overlap

$$(V, W) \;\; = \lim_{z\to 0} \langle 0 | z^{2L_0} V(1/z) W(z) | 0 \rangle \,, \tag{3.13}$$

with the condition

$$\langle 0 | (\lambda\theta) | 0 \rangle = 1 \,. \tag{3.14}$$

It is easy to see that Q-exact states decouples from the inner product. Of course, this construction of the inner product is reminiscent of that of the pure spinor superstring [5] where one uses the rule

$$\langle 0|(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)|0\rangle = 1\,. \tag{3.15}$$

We will shortly observe that the space of gauge invariant polynomials at weight 1 and higher lacks the field-antifield symmetry. It might sound harmless but we stress the importance of having the field-antifield symmetry at all mass levels to define the "spacetime amplitude" appropriately. Otherwise, some "massive" vertex operators in the cohomology of $Q = \int(\lambda p + \tilde{\lambda}\tilde{p})$ would unfavorably decouple from the amplitude. In fact, in the pure spinor formulation of superstring, demonstrating the existence of field-antifield symmetry for the full cohomology of $Q = \int \lambda^\alpha d_\alpha$ was an unresolved challenge. This and related issues has been reported in [9].

Weight 1 Having explained the notion of field-antifield symmetry, let us go back to the construction of gauge invariant polynomials at weight 1. All possible states which are gauge invariant under (3.4) modulo the constraint (3.2) are given by

$$\mathcal{H}_1 = \{\tilde{\lambda}^n\tilde{\omega},\ \lambda^n\omega,\ \partial\lambda,\ \partial\tilde{\lambda},\ \lambda\partial\tilde{\lambda},\ \lambda^n\partial\lambda,\ \tilde{\lambda}^n\partial\tilde{\lambda}\}\,,\qquad n>0\,, \tag{3.16}$$

therefore the partition function at this level is given by

$$Z_{1,poly}(t) = \frac{2}{1-t} + 2t + t^2 + \frac{2t^2}{1-t} = \frac{2-t^2-2t^3+t^4}{(1-t)^2}\,. \tag{3.17}$$

Note that $Z_{1,poly}(t)$ as defined in (3.17) does not posses the field-antifield symmetry. However, it is easy to see that

$$Z_1(t) = Z_{1,poly}(t) - t^{-2} \tag{3.18}$$

satisfies the symmetry. This suggests that one needs an extra *fermionic* state with t-charge -2. As we are going to see in the next section, in the BRST cohomology, this extra state corresponds to the b ghost.

3.3 BRST description of the model

For the model with the irreducible constraint (3.2) the conventional BRST formalism provides a very simple way of describing it, compared to the elaborate language of the curved $\beta\gamma$ formulation [10]. (This is not necessarily the case for infinitely reducible constraints such as the ones for the pure spinors [9].) Here, a fermionic (b, c) ghost pair is introduced to impose the constraint effectively and the physical states are described as the cohomology of the BRST operator

$$D = \int b(\lambda\tilde{\lambda})\,. \tag{3.19}$$

The action in the BRST description is given by

$$S = \int d^2z(w\bar{\partial}\lambda + \tilde{w}\bar{\partial}\tilde{\lambda} + b\bar{\partial}c)\,, \tag{3.20}$$

where b and c are the usual Faddeev-Popov ghosts and they have conformal weights (1,0) and (0,0) respectively. Clearly, this action is invariant under the following ghost number symmetry $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$, the corresponding Noether current is

$$J_g = cb\,. \tag{3.21}$$

In the BRST framework, basic fields obey the following free field operator product expansions

$$\tilde{\lambda}(x)\tilde{w}(y) \sim -\frac{1}{x-y}\,, \tag{3.22}$$

$$\lambda(x)w(y) \sim -\frac{1}{x-y}\,, \tag{3.23}$$

$$b(x)c(y) \sim \frac{1}{x-y}\,. \tag{3.24}$$

The t-charge current is defined as

$$J = \lambda w + \tilde{\lambda}\tilde{w} + 2J_g\,. \tag{3.25}$$

The charges of the basic operators are

$$F(\omega,\lambda) = (0,0)\,, \quad h(\omega,\lambda) = (1,0)\,, \quad t(\omega,\lambda) = (-1,1)\,, \tag{3.26}$$

$$F(b,c) = (1,1)\,, \quad h(b,c) = (1,0)\,, \quad t(b,c) = (-2,2)\,. \tag{3.27}$$

3.4 BRST cohomology and symmetries of partition function

Since the BRST operator D carries t-charge 0, the partition function of D-cohomology coincides with that of the unconstrained space of (ω, λ, b, c) in which the cohomology is computed. This is because the elements not in the cohomology form BRST doublets and cancel out due to $(-1)^F$. Therefore, the partition function is simply given by [19]

$$Z(q,t) = \frac{1-t^2}{(1-t)^2}\prod_{h=1}^{\infty}\frac{(1-t^2q^h)(1-t^{-2}q^h)}{(1-tq^h)^2(1-t^{-1}q^h)^2}\,. \tag{3.28}$$

By expanding in q, partition functions (character formulas) at fixed Virasoro levels can be readily obtained.

The full partition function enjoys the following two symmetries, which turn out to be of fundamental importance [9]. First is the "field-antifield symmetry" we already encountered:

$$Z(q,t) \;\; = -t^{-a_g} Z(q, 1/t)\,. \tag{3.29}$$

As explained in section 3.2, this symmetry is important to have a nice inner product after coupling to the fermionic partners $(p, \tilde{p}; \theta, \tilde{\theta})$. The other is what we shall call "$*$-conjugation symmetry"

$$Z(q, q/t) \;\; = -q^1 t^{-2} Z(q, q/t)\,. \tag{3.30}$$

A little computation shows that this symmetry (3.30) relates the states at $q^m t^n$ and those at $q^{1+m+n} t^{-2-n}$.

In the previous paragraph, we found that the partition function of the BRST cohomology possesses the field-antifield symmetry while that of the gauge invariant polynomials does not. We here explicitly construct the elements of the BRST cohomology and identify the extra states that are responsible for the discrepancy.

Weight 0**:** The zero mode contributions to the full partition function (3.28) is simply

$$Z_0(t) = \frac{1-t^2}{(1-t)^2}\,, \tag{3.31}$$

and it coincides with the result obtained from counting the number of gauge invariant polynomials (3.7). Indeed, since functions of the form $c\lambda^n \tilde{\lambda}^m$ are never D-closed, and since the functions of the form $\lambda^n \tilde{\lambda}^m$ are D-exact, cohomology representatives can be taken as

$$\{1, \lambda^n, \tilde{\lambda}^n\}\,, \qquad n > 0\,, \tag{3.32}$$

but now with λ's unconstrained. Of course, this is expected from the outset as the BRST construction is designed to realize what we have just described.

Weight 1**:** From (3.28) one immediately finds

$$Z_1(t) = \frac{-t^{-2} + 2t^{-1} + 1 - t^2 - 2t^3 + t^4}{(1-t)^2}\,, \tag{3.33}$$

and it possesses the field-antifield symmetry unlike the level 1 partition function $Z_{1,poly}(t)$ of the gauge invariant polynomials. As expected, $Z_1(t)$ contains an extra fermionic state with respect to $Z_{1,poly}(t)$:

$$Z_1(t) - Z_{1,poly}(t) = -t^{-2} \,. \tag{3.34}$$

Clearly, the cohomology element responsible for $-t^{-2}$ is the BRST b ghost carrying charges $-q^1 t^{-2}$. For completeness, let us write the D-cohomology representatives at level $h = 1$

$$\{b,\, \partial\lambda,\, \partial\tilde{\lambda},\, \lambda^{n-1}(\lambda w + bc),\, \tilde{\lambda}^{n-1}(\tilde{\lambda}\tilde{w} + bc),\, \lambda\partial\tilde{\lambda},\, \lambda^n\partial\lambda,\, \tilde{\lambda}^n\partial\tilde{\lambda}\}\,, \qquad n > 0\,. \tag{3.35}$$

Let us make a final remark. As it was noted in references [9, 11, 13], the ghost-for-ghost multiplicities N_k in the BRST description for the constrained model can be obtained by writing the level zero character (3.7) as

$$Z_0(t) = \prod_{k=1}^{\infty} (1 - t^k)^{-N_k} \,. \tag{3.36}$$

The fields at t-charge k will have $|N_k|$ components, and will be bosons for $N_k > 0$ and fermions for $N_k < 0$. For our model for instance we get, by comparing (3.7) with (3.36), $N_1 = 2$, $N_2 = -1$ and $N_k = 0$ for $k > 2$.

In chapter 5, we are going to use the ideas given in this chapter to compute the partition function of pure spinors. The $U(1)$ charge adopted for the pure spinor fields λ^α and its conjugate w_α will be $+1$ and -1 respectively. Before going into the details of chapter 5, let us review some aspects of the pure spinor formalism for the superstring.

Chapter 4

Pure spinor formalism for the superstring

Since the problem of covariant quantization of the GS superstring was discovered, many attempts to solve this problem were done. For instance, one tentative was developed in the work of Siegel [20], he suggested the following action for the open superstring

$$\int d^2z(\frac{1}{2}\partial x^m\bar{\partial}x_m + p_\alpha\bar{\partial}\theta^\alpha)\,, \tag{4.1}$$

where the spinorial index α goes from 1 to 16, and the conjugate momentum of θ^α, p_α, is treated as an independent variable. Furthermore, Siegel added an appropriated set of first class constraints. Inside this set of constraints, it must be the Virasoro constraint, $T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha$ and the kappa symmetry generators of the GS formalism, given by $G^\alpha = \Pi^m(\gamma_m d)^\alpha$, where

$$\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta. \tag{4.2}$$

In this approach, the variable

$$d_\alpha = p_\alpha - \frac{1}{2}\Big(\partial x^m + \frac{1}{4}\theta\gamma^m\partial\theta\Big)(\gamma_m\theta)_\alpha \tag{4.3}$$

does not need to be constrained to be zero. Siegel's approach was applied with success to the superparticle [21]; but for the case of the superstring a set of first class constraints which closes the algebra at the quantum level and which reproduces the correct physical superstring spectrum was never found.

In the year 2000, a new formalism was proposed by Prof. Nathan Berkovits for quantizing the superstring in a manifestly ten-dimensional super-Poincaré covariant manner [5]. This formalism has been used for computing covariant multiloop amplitudes [6], leading to new insights into perturbative finiteness of superstring theory [22]. This formalism is going to be the topic of the next sections.

4.1 The action

Like the GS description of the superstring, the starting point of the pure spinor formalism is the employ of the superspace in $D = 10$ as the target space for the superstring. Nevertheless new ingredients are added such that covariant quantization is achieved.

The basic variables are the superspace coordinates (x^m, θ^α). As it is known one of the main problems in the GS formalism is the definition of the conjugate momentum for the θ^α variables, which gives constraints related to the kappa symmetry. In the pure spinor formalism the conjugate momentum p_α is directly introduced in the action

$$S = \int d^2z \Big(\frac{1}{2}\partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial}\theta^\alpha + w_\alpha \bar{\partial}\lambda^\alpha\Big). \tag{4.4}$$

The ghost λ^α is a pure spinor, i.e., $\lambda^\alpha \gamma^m_{\alpha\beta}\lambda^\beta = 0$, and w_α is its conjugate momentum. The pure spinor constraint implies that the action is invariant under the following transformation

$$\delta w_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha \,, \tag{4.5}$$

for any Λ^m. Therefore we can choose this freedom to eliminate 5 components of the conjugate momentum w_α.

Defining x^m, θ^α, λ^α as fields with conformal weight zero and p_α, w_α with conformal weight one, the action (4.4) turn to be conformal invariant. The fundamental OPES are easy to compute and they are given by

$$\begin{aligned} x^m(w)x^n(z) &\sim -\eta^{mn}\ln|w-z| \,, \\ p_\alpha(w)\theta^\beta(z) &\sim \frac{\delta_\alpha^\beta}{w-z} \,, \\ d_\alpha(w)d_\beta(z) &\sim -\frac{1}{w-z}\gamma^m_{\alpha\beta}\Pi_m(z) \,, \\ d_\alpha(w)\Pi^m(z) &\sim \frac{1}{w-z}\gamma^m_{\alpha\beta}\partial\theta^\beta(z). \end{aligned} \tag{4.6}$$

where Π^m and d_α are defined in (4.2) and (4.3).

The energy momentum tensor for the matter variables is given by

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha \,, \tag{4.7}$$

which has -22 central charge, the contribution for the central charge coming from the ghosts variables is $+22$. To see this, it is necessary to break Lorentz $SO(10)$ (after perform the Wick rotation) invariance to the subgroup $SU(5) \times U(1)$ [5].

The pure spinor constraint can be solved and we can express λ^α in terms of the 11 independent degrees of freedom by decomposing the 16 components of λ^α into $SU(5)\times U(1)$ representations as

$$\lambda^+ = \gamma, \quad \lambda_{[ab]} = \gamma u_{[ab]}, \quad \lambda_{[abcd]} = -\frac{1}{8}\gamma u_{[ab}u_{cd]}, \tag{4.8}$$

where γ is an $SU(5)$ scalar, and $u_{[ab]}$ is an $SU(5)$ antisymmetric two-form. Using this decomposition, and by bosonizing the (β,γ) fields as $(\beta = \partial\xi e^{-\phi}, \gamma = \eta e^{\phi})$, we can write the formulas for the currents [11]

$$\begin{aligned} J &= -\frac{5}{2}\partial\phi - \frac{3}{2}\eta\xi, \\ N^{ab} &= v^{ab}, \\ N_a^b &= -u_{ac}v^{bc} + \delta_a^b(\frac{5}{4}\eta\xi + \frac{3}{4}\partial\phi), \\ N_{ab} &= 3\partial u_{ab} + u_{ac}u_{bd}v^{cd} + u_{ab}(\frac{5}{2}\eta\xi + \frac{3}{2}\partial\phi), \\ T &= \frac{1}{2}v^{ab}\partial u_{ab} - \frac{1}{2}\partial\phi\partial\phi - \eta\partial\xi + \frac{1}{2}\partial(\eta\xi) - 4\partial(\partial\phi + \eta\xi), \end{aligned} \tag{4.9}$$

where T is the stress-energy tensor for the ghosts fields and the worldsheet fields satisfy the OPES

$$\eta(y)\xi(z) \sim (y-z)^{-1}, \quad \phi(y)\phi(z) \sim -\log(y-z), \quad v^{ab}u_{cd} \sim \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}. \tag{4.10}$$

Using these parameterizations of a pure spinor, the OPES of the currents in (4.9) can be computed to be

$$\begin{aligned} N_{mn}(y)\lambda^\alpha(z) &\sim \frac{1}{2}\frac{1}{y-z}(\gamma_{mn}\lambda)^\alpha, \quad J(y)\lambda^\alpha(z) \sim \frac{1}{y-z}\lambda^\alpha, \\ N^{kl}(y)N^{mn}(z) &\sim -\frac{3}{(y-z)^2}(\eta^{n[k}\eta^{l]m}) + \frac{1}{y-z}(\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}), \\ J(y)J(z) &\sim -\frac{4}{(y-z)^2}, \quad J(y)N^{mn}(z) \sim 0, \\ N_{mn}(y)T(z) &\sim \frac{1}{(y-z)^2}N_{mn}(z), \quad J(y)T(z) \sim -\frac{8}{(y-z)^3} + \frac{1}{(y-z)^2}J(z), \\ T(y)T(z) &\sim \frac{1}{2}\frac{22}{(y-z)^4} + \frac{2}{(y-z)^2}T(z) + \frac{1}{y-z}\partial T. \end{aligned} \tag{4.11}$$

Therefore, as we can see, the conformal central charge is $+22$, the ghost-number anomaly is -8, the Lorentz central charge is -3, and the ghost-number central charge is -4.

4.2 Physical states

In the pure spinor formalism, the superstring physical states are identified with states having ghost number one * in the cohomoly of the BRST operator

$$Q = \oint dz\, \lambda^\alpha d_\alpha\,. \tag{4.12}$$

Using the OPES (4.6), it is easy to show that this BRST Q operator is nilpotent thanks to the pure spinor constraint $\lambda\gamma^m\lambda = 0$.

The reader may have a question related to the definition of the physical states, why are the physical states defined to have ghost number one? To answer this question, let us write the most general vertex operator at the massless level

$$\Psi(x,\theta,\lambda) = C + \lambda^\alpha A_\alpha + \lambda\gamma^{mnpqr}\lambda A^*_{mnpqr} + \lambda^\alpha\lambda^\beta\lambda^\gamma C^*_{\alpha\beta\gamma} + \cdots\ , \tag{4.13}$$

where $C(x,\theta)$, $A_\alpha(x,\theta)$, $A^*_{mnpqr}(x,\theta)$ and $C^*_{\alpha\beta\gamma}(x,\theta)$ are superfields the $\cdots$ includes superfields with powers greater than three λ's. As we are going to see in more detail, in section 4.3, the superfield $C(x,\theta)$ has ghost number zero and contains the spacetime ghost, the superfield $A_\alpha(x,\theta)$ which has ghost number one contains the super Yang-Mills fields, the superfield $A^*_{mnpqr}(x,\theta)$ has ghost number two and contains the super Yang-Mills antifields, finally the superfield $C^*_{\alpha\beta\gamma}(x,\theta)$ which has ghost number three contains the spacetime antighost. It is possible to show that the superfields with ghost number greater than three will have trivial cohomology [23]. And therefore, it is consistent to take as physical states those states having ghost number one.

The pure spinor condition implies that the conjugate momentum of λ^α, w_α, can appear in combinations which are invariant under the transformation (4.5)

$$J = \omega_\alpha\lambda^\alpha\,, \quad N^{mn} = \frac{1}{2}\omega\gamma^{mn}\lambda\,,$$

where N^{mn} and J are defined in terms of the $U(5)$ fields as in the previous section. The open superstring vertex operators are constructed as an arbitrary combination of the fields $[x^m, \theta^\alpha, d_\alpha, \lambda^\alpha, N^{mn}, J]$, with ghost number one and conformal weight h. In our case h is related with the level of the state, i.e., $(\text{mass})^2 = h$.

In the case where the state has $(\text{mass})^2 = 0$, i.e., $h = 0$, the vertex operator is given by

$$U = \lambda^\alpha A_\alpha(x,\theta)\,, \tag{4.14}$$

*The ghost number is given by the number of λ's which appears in the vertex operators. For instance, the vertex operator $U = \lambda^\alpha A_\alpha(x,\theta)$ has ghost number one.

using the physical state condition $QU = 0$, $\delta U = Q\Omega$ and the OPE $d_\alpha(y) f(x(z), \theta(z)) \sim (y - z)^{-1} D_\alpha f(x(z), \theta(z))$, where

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} \theta^\beta \gamma^m_{\alpha\beta} \partial_m \,, \tag{4.15}$$

we obtain the super Maxwell equation of motion $\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0$ together with its gauge invariance $\delta A_\alpha = D_\alpha \Omega$. For massive states, the superspace description is more complicated [24], however, it has been proven that the cohomology of Q at ghost-number one correctly describes the open superstring spectrum [25]. One aim of this thesis is to obtain the open superstring spectrum from the pure spinor superstring partition function, this topic will be studied in the next chapters. Prescriptions to compute multiloop amplitudes using this pure spinor formalism can be found in [6, 7].

4.3 Massless pure spinor superstring state: fields and anti-fields

We have seen that the lowest open superstring excitation describes on-shell super-Yang-Mills theory. There is a Poincaré-covariant description of this theory using an $SO(9,1)$ vector field $a_m(x)$ and an $\chi^\alpha(x)$ $SO(9,1)$ spinor field which satisfy the equations of motion

$$\partial^m f_{mn} = 0 \,, \qquad \gamma^m_{\alpha\beta} \partial_m \chi^\beta = 0 \,, \tag{4.16}$$

and gauge invariance

$$\delta a_m = \partial_m s \,, \qquad \delta \chi^\alpha = 0 \,, \qquad \delta f_{mn} = 0 \,. \tag{4.17}$$

where f_{mn} is the Yang-Mills field strength. However, there is also a super-Poincaré covariant description using an $SO(9,1)$ spinor wavefunction $A_\alpha(x, \theta)$ defined in ten dimension superspace. As it will be explained below, on-shell super-Yang-Mills theory can be described by a spinor superfield $A_\alpha(x, \theta)$ which satisfies the superspace equation of motion

$$\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0 \,, \tag{4.18}$$

for any five-form direction $mnpqr$, with the gauge invariance

$$\delta A_\alpha = D_\alpha \Omega \,, \tag{4.19}$$

where $\Omega(x, \theta)$ is any scalar superfield and D_α is the supersymmetric derivative (4.15).

One can also define field strengths constructed from A_α by

$$\begin{aligned}
B_m &= \frac{1}{8}\gamma_m^{\alpha\beta} D_\alpha A_\beta \,, \\
W^\alpha &= \frac{1}{10}\gamma_m^{\alpha\beta}(D_\beta B^m - \partial^m A_\beta)\,, \\
F_{mn} &= \partial_{[m}B_{n]} = \frac{1}{8}(\gamma_{mn})_\alpha^\beta D_\beta W^\alpha \,.
\end{aligned} \tag{4.20}$$

Under the gauge transformation of (4.17),

$$\delta B_m = \partial_m \Omega \,, \qquad \delta W^\alpha = 0 \,, \qquad \delta F_{mn} = 0 \,. \tag{4.21}$$

To show that $A_\alpha(x,\theta)$ describes on-shell super-Yang-Mills theory, it will be useful to first note that in ten dimensions any symmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a vector and a five-form as $f_{\alpha\beta} = \gamma^m_{\alpha\beta} f_m + \gamma^{mnpqr}_{\alpha\beta} f_{mnpqr}$ and any antisymmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a three-form as $f_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta} f_{mnp}$. Since $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$, one can check that $\delta A_\alpha = D_\alpha \Omega$ is indeed a gauge invariance of (4.18).

Using $\Omega(x,\theta) = s(x) + h_\alpha(x)\theta^\alpha + j_{\alpha\beta}(x)\theta^\alpha\theta^\beta$; one can gauge away $(A_\alpha(x))|_{\theta=0}$ and the three-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$. Furthermore, eq. (4.18) implies that the five-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$ vanishes. So the lowest non-vanishing component of $A_\alpha(x,\theta)$ in this gauge is the vector component $(D\gamma_m A(x))|_{\theta=0}$ which will be defined as $8a_m(x)$. Continuing this type of argument to higher order in θ^α, one finds that there exists a gauge choice such that

$$A_\alpha(x,\theta) = \frac{1}{2}(\gamma^m\theta)_\alpha a_m(x) + \frac{1}{12}(\theta\gamma^{mnp}\theta)(\gamma_{mnp})_{\alpha\beta}\chi^\beta(x) + \cdots \,, \tag{4.22}$$

where $a_m(x)$ and $\chi^\beta(x)$ are $SO(9,1)$ vector and spinor fields satisfying (4.16) and where the component fields in $\cdots$ are functions of spacetime derivatives of $a_m(x)$ and $\chi^\beta(x)$. Furthermore, this gauge choice leaves the residual gauge transformations of (4.17) where $s(x) = (\Omega(x))|_{\theta=0}$. Also, one can check that the $\theta = 0$ components of the superfields B_m, W^α and F_{mn} of (4.20) are a_m, χ^α and f_{mn}, respectively. So the equations of motion and gauge invariances of (4.18) and (4.19) correctly describe on-shell super-Yang-Mills theory.

Now one would like to obtain this super-Poincaré covariant description of super-Yang-Mills theory by quantizing the open superstring. This covariant description can be done by using a BRST-like operator out of the fermionic constraints d_α in the pure spinor formalism for the superstring.

We have already seen that the most general super-Poincaré covariant wavefunction $\Psi(x,\theta,\lambda)$ (at the massless level) that can be constructed from the worldsheet pure spinor variables $(x^m, \theta^\alpha, \lambda^\alpha)$ was given by (4.13). Since $Q\Psi = \lambda^\alpha D_\alpha C +$

$\lambda^\alpha\lambda^\beta D_\alpha A_\beta + \cdots$, $Q\Psi = 0$ implies that $A_\alpha(x,\theta)$ satisfies the equation of motion $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$. But since $\lambda^\alpha\lambda^\beta$ is proportional to $(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$, this implies that $D\gamma^{mnpqr}A = 0$, which is the super-Yang-Mills equation of motion of (4.18). Furthermore, if one defines the gauge parameter $\Lambda = \Omega + \lambda^\alpha f_\alpha + \cdots$, the gauge transformation $\delta\Psi = Q\Lambda$ implies $\delta A_\alpha = D_\alpha\Omega$ which is the super-Yang-Mills gauge transformation of (4.19).

As it is well known, the only states at non-zero momentum in the cohomology of Q are the on-shell super-Yang-Mills gluon and gluino, $a_m(x)$ and $\chi^\alpha(x)$, and their antifields, $a^{*m}(x)$ and $\chi^*_\alpha(x)$. Since in the antifield formalism gauge invariances of the antifields correspond to equations of motion of the fields and vice versa, one expects $a^{*m}(x)$ and $\chi^*_\alpha(x)$ to satisfy the equation of motion $\partial_m a^{*m} = 0$ with the gauge invariances

$$\delta a^{*m} = \partial_n(\partial^n s^m - \partial^m s^n)\,, \qquad \delta\chi^*_\alpha(x) = \gamma^m_{\alpha\beta}\partial_m\kappa^\beta\,, \tag{4.23}$$

where s_m and κ^β are gauge parameters.

The fields a_m and χ^α appear in components of A_α as in (4.22), and the antifields a^{*m} and χ^*_α appear in components of the ghost-number $+2$ superfield A^*_{mnpqr} of (4.13). Using $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$, A^*_{mnpqr} satisfies the equation of motion $\lambda^\alpha(\lambda\gamma^{mnpqr}\lambda)D_\alpha A^*_{mnpqr} = 0$ with the gauge invariance $\delta A^*_{mnpqr} = \gamma^{\alpha\beta}_{mnpqr}D_\alpha f_\beta$. Expanding f_α and A^*_{mnpqr} in components, one learns that A^*_{mnpqr} can be gauged to the form

$$A^*_{mnpqr} = (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]})^\alpha\chi^*_\alpha(x) + (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]s}\theta)a^{*s}(x) + \cdots\,, \tag{4.24}$$

where χ^*_α and a^{*s} satisfy the equations of motion and residual gauge invariances of (4.23), and $\cdots$ involves terms with higher order in θ^α which depend on derivatives of χ^*_α and a^{*s}.

As it is shown in [23], there are also zero momentum states in the cohomology of Q. In addition to the states described by the zero-momentum gluon, gluino, antigluon, and antigluino, there are also zero-momentum ghost and antighost states c and c^* in the $\theta = 0$ component of the ghost-number zero superfield, $C(x,\theta) = c(x)+\cdots$, and in the $(\theta)^5$ component of the ghost-number $+3$ superfield, $C^*_{\alpha\beta\gamma}(x,\theta) = \cdots + c^*(x)(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta)+\cdots$. So although (4.13) contains superfields of arbitrarily high ghost number, only superfields with ghost-number between zero and three contain states in the cohomology of Q.

The linearized equations of motion and gauge invariances $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$ are easily generalized to the non-linear equations of motion and gauge invariances

$$Q\Psi + g\Psi\Psi = 0\,, \qquad \delta\Psi = Q\Lambda + g[\Psi,\Lambda]\,. \tag{4.25}$$

For the superfield $A_\alpha(x,\theta)$, (4.25) implies the super-Yang-Mills equations of motion and gauge transformations of (4.18) and (4.19). Furthermore, the equations of motion and gauge transformation of (4.25) can be obtained from the spacetime action

$$S = \int d^{10}x \left\langle \frac{1}{2}\Psi Q \Psi + \frac{1}{3} g \Psi\Psi\Psi \right\rangle , \tag{4.26}$$

using the normalization definition that

$$\left\langle (\lambda\gamma^m\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{mpq}\theta) \right\rangle = 1 \ . \tag{4.27}$$

After writing (4.27) in terms of component fields and integrating out auxiliary fields, it should be possible to show that (4.26) reduces to the standard Batalin-Vilovisky (BV) action for super-Yang-Mills,

$$S = \int d^{10}x \Big(\frac{1}{4} f_{mn} f^{mn} + \chi^\alpha \gamma^m_{\alpha\beta} \partial_m \chi^\beta + a^{*m}\partial_m c - gccc^* \Big) \ . \tag{4.28}$$

Remember that an essential ingredient of the BV formalism is the doubling of the complete set of fields [26]. To each field one associates an antifield with opposite statistics. We have seen that the massless state defined as the cohomology of the BRST operator Q contains field as well as their respective antifields.

So far we have analyzed the massless open superstring state which in the pure spinor formalism is defined as an element in the cohomology of the BRST operator Q. We have shown that the cohomology at the massless level is describing super-Yang-Mills fields together with their respective antifields. It would be nice, by explicitly computing the cohomology of the BRST operator Q, to see if massive states have this field and antifield structure. Covariant computation of the cohomology seems to be hard, nevertheless using the partition function of pure spinors one can argue that massive states will appear in fields and antifields with the same physical degrees of freedom.

Chapter 5

Naive partition function of pure spinors and missing states

To explain what we have just stated, as in the case of the toy model (see chapter 3), we are going to compute the partition function of globally defined gauge invariant operators by explicitly constructing them at lower Virasoro levels. It turns out that, starting from level $h = 2$, this space of gauge invariant polynomials by itself lacks some operators for having the field-antifield symmetry. The missing state turns out to be *fermionic* and so it cannot be a usual gauge invariant state. Indeed, it is related to a term (called b_3 [9]) which appears in the expression for the b-ghost used in multiloop amplitude computations [7].

5.1 Definition of the partition function

The characters of the states we wish to keep track of are their statistics, weights (Virasoro levels), t-charge (measured by $J = \omega\lambda + p\theta$) and the Lorentz spin. The Lorentz spin of a state can be labeled by five integers which we denote by

$$\begin{aligned} \mu &= (a_1 a_2 a_3 a_4 a_5) \quad \text{Dynkin basis}\,, \\ &= \frac{1}{2}[\mu_1\mu_2\mu_3\mu_4\mu_5] \quad \text{“five sign” basis}\,. \end{aligned} \tag{5.1}$$

Introducing formal variables $(q, t, \vec{\sigma})$ for each quantum numbers, we define the partition function (character) as

$$\begin{aligned} Z(q,t,\vec{\sigma}) &= \mathrm{Tr}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}\mathrm{t}^{\mathrm{J_0}}\mathrm{e}^{\mu\cdot\sigma} \\ &= \sum_{h\geq 0} Z_h(t,\vec{\sigma})q^h\,. \end{aligned} \tag{5.2}$$

The trace is taken over various states defined in the pure spinor Hilbert space. Characters of the basic operators ω and λ are

$$h(\omega,\lambda) = (1,0)\,, \quad t(\omega,\lambda) = (-1,1)\,, \tag{5.3}$$

$$
\begin{aligned}
\mu(\omega) &= e^{(00010)} = e^{\frac{1}{2}(\pm\sigma_1\pm\sigma_2\pm\sigma_3\pm\sigma_4\pm\sigma_5)} \quad (\text{odd } \# \text{ of } -\text{'s})\,, \\
\mu(\lambda) &= e^{(00001)} = e^{\frac{1}{2}(\pm\sigma_1\pm\sigma_2\pm\sigma_3\pm\sigma_4\pm\sigma_5)} \quad (\text{even } \# \text{ of } -\text{'s})\,.
\end{aligned}
$$

The relation between the Dynkin basis and the "five sign basis" can be found in appendix A.1.

Sometimes, it is convenient to ignore the spin characters and concentrate on the dimensions of the Hilbert space

$$
\begin{aligned}
Z(q,t) &= \mathrm{Tr}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}\mathrm{t}^{\mathrm{J_0}} \\
&= \textstyle\sum_{h\geq 0} Z_h(t)q^h \\
&= \textstyle\sum_{h\geq 0,n} N_{h,n}q^h t^n\,.
\end{aligned}
\tag{5.4}
$$

A list of partition functions at lower levels can be found in appendix B.

5.2 Cohomology via partition function

In chapter 9, we will relate the partition function of pure spinors to that of the cohomology of the physical BRST operator $Q = \int \lambda^\alpha d_\alpha$. Let us explain the basic idea behind this, which is also useful for the computation of the partition function itself.

Let $\mathcal{O}$ be a fermionic nilpotent operator that commutes with L_0, J_0 and the Lorentz charge, let $\mathcal{H}$ be the cohomology of $\mathcal{O}$, and let $\mathcal{F}$ be the Hilbert space in which the cohomology of $\mathcal{O}$ is computed. Then it can be shown that the traces over $\mathcal{H}$ and $\mathcal{F}$ coincide:

$$
\mathrm{Tr}_{\mathcal{H}}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}\mathrm{t}^{\mathrm{J_0}}\mathrm{e}^{\mu\cdot\sigma} = \mathrm{Tr}_{\mathcal{F}}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}\mathrm{t}^{\mathrm{J_0}}\mathrm{e}^{\mu\cdot\sigma}\,. \tag{5.5}
$$

To show this, first split $\mathcal{H}$ and $\mathcal{F}$ to even and odd parts:

$$
\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o\,, \quad \mathcal{F} = \mathcal{F}_e \oplus \mathcal{F}_o\,. \tag{5.6}
$$

(In our case, fermion numbers will be carried by θ's and the fermionic BRST ghosts.) Then, since

$$
\begin{aligned}
\mathcal{F}_e &= \mathcal{Z}_e + \mathcal{F}_e/\mathcal{Z}_e = (\mathcal{H}_e + \mathcal{B}_e) + \mathcal{B}_o\,, \\
(\mathcal{Z} &= \mathrm{Ker}\,\mathcal{O}\,, \quad \mathcal{B} = \mathrm{Im}\,\mathcal{O})\,,
\end{aligned}
\tag{5.7}
$$

and similar for $e \leftrightarrow o$, the trace over $\mathcal{B}_e$ and $\mathcal{B}_o$ do not contribute to the right-hand side of (5.5) due to the factor $(-1)^F$.

Thus, although we defined the partition function as the trace over the cohomology of some nilpotent operator $\mathcal{O}$, it could have been the trace over the space in which the cohomology is computed. Below, we use the formula (5.5) freely when computing the partition functions.

We will also use the formula (5.5) in chapter 9 when we relate the partition function of pure spinors to the cohomology of the physical BRST operator $Q = \int \lambda^\alpha d_\alpha$. Although Q does not commute with J_0, we will argue that one can twist the t-charge using the Lorentz current so that the twisted charge 0 piece of Q has the same cohomology as Q (except for the on-shell condition $L_0 = 0$). Then the cohomology of Q can be read off from the twisted partition function. It will be shown in chapter 9 that the cohomology thus obtained precisely reproduces the lightcone spectrum of the superstring.

5.3 Counting of gauge invariant polynomials and the missing states

The states we construct are polynomials of ω, λ and their derivatives, and are invariant under the "gauge transformation"

$$\delta_\Lambda \omega_\alpha \ = \Lambda^m (\gamma_m \lambda)_\alpha \, . \tag{5.8}$$

In the language of curved $\beta\gamma$ theory these correspond to globally defined operators. Basic invariants with a single ω are the $U(1)$ current, Lorentz currents, and the energy-momentum tensor for the pure spinor sector

$$J \ = \omega\lambda \, , \quad N^{pq} = \tfrac{1}{2}\omega\gamma^{pq}\lambda \, , \quad T = \omega\partial\lambda \, . \tag{5.9}$$

Of course, arbitrary products of these operators are again gauge invariant. Starting from level 2, there will be certain gauge invariant polynomials with negative J_0 charge meaning that the number of ω's is strictly larger than that of λ's. These, however, are perfectly normal gauge invariant operators and should not be confused with the "missing states" alluded to at the beginning of this chapter.

The true missing states, which first appear at level 2, are fermionic and are crucial for reproducing the massive spectrum of the superstring. The purpose of this section is to show that the Hilbert space of "naive" gauge invariants lacks field-antifield symmetry and hence is not the appropriate Hilbert space in the pure spinor formalism. Descriptions of gauge invariants at levels 0 and 1 can also be found in references [11][12].

5.3.1 Level 0 gauge invariants

At the lowest level, the Hilbert space is spanned by non-vanishing polynomials of λ. Due to the pure spinor constraint, λ's can only appear in the "pure spinor representations"

$$\lambda^{(\!(\alpha_1}\lambda^{\alpha_2}\cdots\lambda^{\alpha_n)\!)} \;= (0000n)t^n\,, \quad (n\geq 0)\,. \tag{5.10}$$

Here, we also indicated the t-charge of the state, and the symbol $(\!(\alpha_1\alpha_2\cdots\alpha_n)\!)$ signifies the "spinorial γ-traceless condition", which means that the expression is zero when any two indices $\alpha_i\alpha_j$ are contracted using $\gamma^m_{\alpha_i\alpha_j}$. Since the pure spinor representations have dimensions

$$\dim(0000n) = \frac{(n+7)(n+6)(n+5)^2(n+4)^2(n+3)^2(n+2)(n+1)}{7\cdot 6\cdot 5^2\cdot 4^2\cdot 3^2\cdot 2}\,, \tag{5.11}$$

the level 0 partition function is easily found to be [11]

$$Z_0(t) = \frac{1-10t^2+16t^3-16t^5+10t^6-t^8}{(1-t)^{16}} = \frac{(1+t)(1+4t+t^2)}{(1-t)^{11}}\,. \tag{5.12}$$

5.3.2 Field-antifield symmetry

Before proceeding to the next level, let us explain an important symmetry possessed by the zero-mode partition function. Looking at (5.12), one immediately notices that $Z_0(t)$ has the following symmetry:

$$Z_0(t) \;= -t^{-8}Z_0(1/t)\,. \tag{5.13}$$

As we shall explain shortly, this symmetry is related to *field-antifield symmetry* in the pure spinor superstring. The symmetry is important for having a non-degenerate inner product on the physical states and the value -8 is related to the ghost number anomaly of the pure spinor system [11].

In order to explain how the field-antifield symmetry is related to the inner product structure of pure spinor superstring, let us compute the total weight 0 partition function for the pure spinor superstring, by including the contribution from θ^α. Assigning t-charge 1 to θ^α, the partition function for θ^α is easily computed and reads

$$Z_{\theta,0}(t) = \mathrm{Tr}_\theta(-1)^{\mathrm{F}}\mathrm{t}^{\mathrm{J_0}} = (1-\mathrm{t})^{16}\,. \tag{5.14}$$

Hence, the total weight 0 partition function is

$$\begin{aligned}\mathbf{Z_0(t)} &= Z_{\lambda,0}(t)Z_{\theta,0}(t) \\ &= 1-10t^2+16t^3-16t^5+10t^6-t^8\,.\end{aligned} \tag{5.15}$$

Now $\mathbf{Z_0(t)}$ is nothing but the partition function for the cohomology of $Q_0 = \int \lambda^\alpha p_\alpha$ carrying t-charge zero. For the massless sector, the cohomology of Q_0 coincides with the zero-momentum cohomology of $Q = \int \lambda^\alpha d_\alpha$. The cohomology representatives can be explicitly identified as follows:

$$
\begin{aligned}
t^0 &: & &1\,, \\
-10t^2 &: & &(\lambda\gamma^m\theta)\,, \\
16t^3 &: & &(\lambda\gamma^m\theta)(\gamma_m\theta)_\alpha\,, \\
-16t^5 &: & &(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\gamma_{pq}\theta)^\alpha\,, \\
10t^6 &: & &(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{mpq}\theta)\,, \\
-t^8 &: & &(\lambda\gamma^m\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{mpq}\theta)\,.
\end{aligned}
\tag{5.16}
$$

It is then easy to see that an appropriate inner product (V, W) can be defined on the cohomology using the zero-mode prescription

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{mpq}\theta)\rangle \;\; = 1\,. \tag{5.17}$$

Every cohomology element V has its conjugate (antifield) V_A such that

$$(V, V_A) \;\; = \langle V^\dagger V_A\rangle = 1\,, \tag{5.18}$$

where $V^\dagger$ denotes the BPZ conjugate of V [27]. Since λ^α has t-charge anomaly -8 while θ^α has 16, the rule (5.17) precisely saturates the anomaly. It is analogous to the rule for the bosonic string, $\langle c\partial c\partial^2 c\rangle = 1$, and can be derived from functional integration methods after including an appropriate BRST-invariant measure factor [6].

Below, we shall argue that the partition function of pure spinors has the field-antifield symmetry (5.13) at each Virasoro level, and therefore all physical states in the pure spinor superstring appear in field-antifield pairs.

5.3.3 Level 1 gauge invariants

The weight 1 can be saturated either by one ω or one $\partial\lambda$, and we wish to count the states that do not vanish due to the pure spinor constraints

$$\lambda\gamma^m\lambda = 0\,, \quad \partial(\lambda\gamma^m\lambda) = 2\lambda\gamma^m\partial\lambda = 0\,. \tag{5.19}$$

For the states with ω, one must also require invariance under the gauge transformation $\delta_\Lambda\omega_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha$. For the level 1 operators, the latter condition implies that

ω must appear in the form of the gauge invariant currents J and N^{mp}. Hence, all the possible states with a single ω are $(n \geq 0)$

$$\begin{aligned} \omega_\alpha \lambda^{((\alpha} \lambda^{\beta_1} \cdots \lambda^{\beta_n))} &= (0000n)t^n \,, \\ \omega_\alpha (\gamma^{mp})^\alpha{}_\beta \lambda^{((\beta} \lambda^{\beta_1} \cdots \lambda^{\beta_n))} &= (0100n)t^n \,. \end{aligned} \tag{5.20}$$

The states involving $\partial\lambda$ are described by $(n \geq 0)$

$$\begin{aligned} \partial\lambda^{((\alpha} \lambda^{\beta_1} \cdots \lambda^{\beta_n))} &= (0000, n+1)t^{n+1} \,, \\ \partial\lambda^\alpha \gamma^{mpq}_{\alpha\beta} \lambda^{((\beta} \lambda^{\beta_1} \cdots \lambda^{\beta_n))} &= (0010n)t^{n+2} \,. \end{aligned} \tag{5.21}$$

Note that while $\lambda\gamma^m\partial\lambda = 0$ due to the pure spinor constraint, the 3-form $\lambda\gamma^{mpq}\partial\lambda$ is non-vanishing.

Adding up all four contributions, one finds [12]

$$Z_1(t) = \frac{46 - 144t + 116t^2 + 16t^3 - 16t^5 - 116t^6 + 144t^7 - 46t^8}{(1-t)^{16}} \tag{5.22}$$

$$= \frac{2(1+t)(23 + 20t + 23t^2)}{(1-t)^{11}} \,. \tag{5.23}$$

This satisfies the same field-antifield symmetry as $Z_0(t)$:

$$Z_1(t) = -t^{-8} Z_1(1/t) \,. \tag{5.24}$$

5.3.4 Level 2 gauge invariants and a missing state

Explicit constructions of the gauge invariant polynomials at level 2 can be obtained using similar methods. But at this level we encounter several new features. Most importantly, we will find that the space of gauge invariant polynomials does not posses the field-antifield symmetry. This implies the space has to be augmented by some finite number of terms. We shall explain the symmetries of the partition function in chapter 6. For now, however, let us focus on the space of gauge invariant polynomials and enumerate them.

First of all, there are polynomials with two ω's. One might expect that these ω's only appear in the form of N^{mp} or J, but there is in fact a gauge invariant polynomial with negative t-charge

$$f_\alpha = 3J\omega_\alpha + N^{mp}(\gamma_{mp}\omega)_\alpha \,. \tag{5.25}$$

Appearance of f_α is interesting, but we stress that it is a perfectly normal gauge invariant polynomial and has nothing to do with the "missing states". Of course,

f_α multiplied by some function of λ is again gauge invariant, but this carries non-negative t-charge and can be expressed in terms of operators constructed from N^{mp} and J.

The states with two N's, two J's, and one N and one J are $(n \geq 0)$

$$\begin{aligned} N_{[\![mp}N_{qr]\!]}\lambda^{(n)} &= (\gamma_{[\![mp}\omega)_{\alpha_1}(\gamma_{qr]\!]}\omega)_{\alpha_2}\lambda^{((\alpha_1}\lambda^{\alpha_2}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0200n)t^n\,, \\ N_{[mp}N_{qr]}\lambda^{(n)} &= (\gamma_{[mp}\omega)_{\alpha_1}(\gamma_{qr]}\omega)_{\alpha_2}\lambda^{((\alpha_1}\lambda^{\alpha_2}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0001,n+1)t^n\,, \\ N_{mp}J\lambda^{(n)} &= (\gamma_{mp}\omega)_{\alpha_1}\omega_{\alpha_2}\lambda^{((\alpha_1}\lambda^{\alpha_2}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0100n)t^n\,, \\ JJ\lambda^{(n)} &= \omega_{\alpha_1}\omega_{\alpha_2}\lambda^{((\alpha_1}\lambda^{\alpha_2}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0000n)t^n\,. \end{aligned} \tag{5.26}$$

Here, we left the γ-traceless conditions implicit, and the indices in $[\![mp,qr]\!]$ are traceless, block-symmetric, and antisymmetric within each blocks. In fact, the 4-form piece of $NN\lambda^{(n)}$, $NJ\lambda^{(n)}$ and $JJ\lambda^{(n)}$ can be written as $(n \geq 0)$

$$\begin{aligned} f_\alpha\lambda^{(n)} &= (3\omega_{\alpha_0}\omega_\alpha + (\gamma^{mp}\omega)_{\alpha_0}(\gamma_{mp}\omega)_\alpha)\lambda^{((\alpha_0}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} \\ &= (00010)\otimes(0000n)t^{n-1}\,, \end{aligned} \tag{5.27}$$

so one must be careful not to double count.

As for the polynomials with a single derivative, the following states are independent $(n \geq 0)$:

$$\begin{aligned} \partial N^{mp}\lambda^{(n)} &= \partial(\omega_{\alpha_0}\gamma^{mp\alpha_0}{}_{\alpha_1}\lambda^{((\alpha_1})\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0100n)t^n\,, \\ \partial J\lambda^{(n)} &= \partial(\omega_{\alpha_1}\lambda^{((\alpha_1})\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} = (0000n)t^n\,, \\ N^{mp}\partial\lambda\lambda^{(n)} &= ((\omega\gamma^{mp})_{\alpha_1}\partial\lambda^{((\alpha_0}\lambda^{\alpha_1}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} + (\gamma - traces)) \\ &\qquad +(\omega\gamma^{[\![mp})_{\alpha_0}(\partial\lambda\gamma^{qrt]\!]})_{\alpha_1}\lambda^{((\alpha_0}\lambda^{\alpha_1}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} \\ &= ((0100,n+1)+(1001n)+(0000,n+1))t^{n+1}+(0110n)t^{n+2}\,, \\ J\partial\lambda\lambda^{(n)} &= \omega_{\alpha_1}\partial\lambda^{((\alpha_0}\lambda^{\alpha_1}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} + \omega_{\alpha_0}(\partial\lambda\gamma_{mpq})_{\alpha_1}\lambda^{((\alpha_0}\lambda^{\alpha_1}\lambda^{\beta^1}\cdots\lambda^{\beta_n}))} \\ &= (0000,n+1)t^{n+1}+(0010n)t^{n+2}\,, \\ T &= \omega_\alpha\partial\lambda^\alpha = (00000)t^0\,. \end{aligned} \tag{5.28}$$

Note that $T\lambda^{(n+1)}$ and $\omega_{\alpha_1}\partial\lambda^{((\alpha_0}\lambda^{\alpha_1}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))}$ are not independent.

Finally, there are two types of polynomials with two derivatives, $\partial^2\lambda\lambda^{(n)}$ and $(\partial\lambda)^2\lambda^{(n)}$, and some of them are related by the level 2 pure spinor condition

$$\lambda\gamma^m\partial^2\lambda + \partial\lambda\gamma^m\partial\lambda = 0\,. \tag{5.29}$$

The independent states are $(n \geq 0)$

$$\begin{aligned} \partial^2\lambda^\alpha\lambda^{((\beta_1}\cdots\lambda^{\beta_n}))} &= (00001)\otimes(0000n)t^{n+1}\,, \\ \partial\lambda^{((\alpha_1}\partial\lambda^{\alpha_2}\lambda^{\beta_1}\cdots\lambda^{\beta_n}))} &= (0000,n+2)t^{n+2}\,, \\ (\partial\lambda\gamma^{mpq})_{\beta_1}\partial\lambda^{((\alpha}\lambda^{\beta_1}\lambda^{\beta_2}\cdots\lambda^{\beta_{n+1}}))} &= (0010,n+1)t^{n+3}\,, \\ (\partial\lambda\gamma^{[\![mpq})_{\beta_1}(\partial\lambda\gamma^{rst]\!]})_{\beta_2}\lambda^{((\beta_1}\lambda^{\beta_2}\cdots\lambda^{\beta_{n+2}}))} &= (0020n)t^{n+4}\,. \end{aligned} \tag{5.30}$$

Adding up all the contributions, (5.25), (5.26), (5.28) and (5.30), one finds

$$\begin{aligned} Z_{2,poly}(t) &= \frac{1}{(1-t)^{16}}\Big\{16t^{-1}+817-3840t+7794t^2-10848t^3+12870t^4-12032t^5 \\ &\quad +8222t^6-4896t^7+2823t^8-1136t^9+240t^{10}-32t^{11}+2t^{12}\Big\}. \end{aligned} \tag{5.31}$$

The missing state As already mentioned, $Z_{2,poly}$ we just computed does not posses the field-antifield symmetry. However, one finds that

$$\begin{aligned} Z_2(t) &= Z_{2,poly}(t) - t^{-4} \\ &= \frac{1}{(1-t)^{16}}\Big\{-t^{-4}+16t^{-3}-120t^{-2}+576t^{-1}-1003+528t-214t^2+592t^3 \\ &\quad -592t^5+214t^6-528t^7+1003t^8-576t^9+120t^{10}-16t^{11}+t^{12}\Big\} \end{aligned} \tag{5.32}$$

does have the desired symmetry

$$Z_2(t) \ = -t^{-8}Z_2(1/t)\,. \tag{5.33}$$

Therefore, we expect to have an extra *fermionic* singlet with t-charge -4 at level 2. Because it is fermionic, it cannot be a usual gauge invariant state. Indeed, it is related to a term (called b_3 [9]) which appears in the expression for the b-ghost used in multiloop amplitude computations [7].

Remember that the b ghost satisfying $\{Q,b\}=T$ is a composite operator constructed from both the matter variables $(x^m,\theta^\alpha,p_\alpha)$ and ghost variables $(\lambda^\alpha,w_\alpha)$. This composite operator cannot be globally defined on all patches (see chapter 8), and in the non-minimal pure spinor formalism, is described by the sum of a zero-form, one-form, two-form and three-form [14]. The three-form in the b ghost is independent of the matter variables $(x^m,\theta^\alpha,p_\alpha)$

$$b_3 = \frac{(r\gamma^{\mu\nu\rho}r)(\bar\lambda\gamma_{\sigma\tau\rho}r)N_{\mu\nu}N^{\sigma\tau}}{512(\lambda\bar\lambda)^4}\,. \tag{5.34}$$

Chapter 6

Partition function of pure spinors and its symmetries

In this and in the next chapter, we are going to present two independent methods for computing the partition function of pure spinors. The first method utilizes Chesterman's ghost-for-ghost description of pure spinors [8], while the second method uses a fixed point formula extending the zero mode result of [11]. Neither method gives the complete partition function in closed form, but the partition functions can be computed level by level unambiguously once one imposes the requirements of field-antifield and "$*$-conjugation" symmetries.

We present the ghost-for-ghost method first because the two symmetries of the partition function are (formally) manifest in this formalism. However, since the ghost-for-ghost description of the pure spinor requires an infinite tower of ghosts-for-ghosts, the expression for the partition function is ill-defined and one has to invoke an analytic continuation in order to maintain the two symmetries. Also, using this method, it is difficult to compute the partition function keeping the spin dependence of the states.

6.1 Ghost-for-ghost description of pure spinors

In this section, we analyze the (infinite) reducibility conditions for the pure spinor constraint using the BRST formalism. The resulting BRST operator D was first introduced by Chesterman [8]. As already mentioned, we sometimes refer to D as the mini-BRST operator to avoid confusion with the physical BRST operator $Q = \int \lambda^\alpha d_\alpha$.

Chesterman's ghost-for-ghost construction is designed so that it reproduces the space of gauge invariant functions of the constrained system. Indeed, the partition functions of D-cohomology in weight 0 and 1 sectors precisely describe the number of gauge invariant objects described in section 5.3. However, starting at weight 2, we find extra cohomology elements which do not correspond to the naive gauge invariants. We shall claim that those extra states are as important part of the Hilbert space of the pure spinor system as the naive gauge invariants.

Let us present the prescription to find the ghosts in order to describe the pure spinor constraint $\lambda\gamma^m\lambda = 0$. Following the usual strategy of the ghost-for-ghost construction, we introduce a fermionic ghost c^m to 'kill' the pure spinor part of λ^α and define the D-action

$$D = \int \lambda\gamma^m\lambda\, b_m \quad \leftrightarrow \quad Dc^m = \lambda\gamma^m\lambda\,, \tag{6.1}$$

where b_m is the conjugate field of c^m. Then, a function $f(\lambda)$ proportional to $\lambda\gamma^m\lambda$ is D-exact and does not contribute to the D-cohomology. However, because the pure spinor constraint is *reducible*, this is not the end of the story, using the identity

$$(\lambda\gamma^m\lambda)(\gamma_m\lambda)_\alpha = 0\,, \tag{6.2}$$

one can construct a D-closed state

$$c^m(\gamma_m\lambda)_\alpha\,, \tag{6.3}$$

which must be killed by introducing another generation of ghost. For the case at hand, we introduce a bosonic ghost χ_α and define $D\chi_\alpha = c^m(\gamma_m\lambda)_\alpha$. So that $c^m(\gamma_m\lambda)_\alpha$ becomes trivial, and the operator D becomes

$$D = \int \Big(\lambda\gamma^m\lambda\, b_m + c^m(\lambda\gamma_m\zeta)\Big)\,, \tag{6.4}$$

where ζ^α is the conjugate field of χ_α. Now we repeat the above argument, there is a D-closed state of (6.4), and it is given by $c^mc^n + \frac{1}{2}\chi\gamma^{mn}\lambda$, again we introduce a new ghost C^{mn} and define $DC^{mn} = c^mc^n + \frac{1}{2}\chi\gamma^{mn}\lambda$. We can continue this path, getting an infinite set of unconstrained fields: λ^α, c^m, χ_α, C^{mn}, ... with their respective conjugate fields: w_α, b_m, ζ^α, B_{mn}, ... with free action

$$S = \int d^z(w_\alpha\bar\partial\lambda^\alpha + b_m\bar\partial c^m + \zeta^\alpha\bar\partial\chi_\alpha + B_{mn}\bar\partial C^{mn} + ...)\,, \tag{6.5}$$

and mini-BRST operator D given by [9]

$$D = \int \Big(\lambda\gamma^m\lambda\, b_m + c^m(\lambda\gamma_m\zeta) + \frac{1}{2}B^{mn}(c_mc_n + \frac{1}{2}\chi\gamma_{mn}\lambda) + \cdots\Big)\,. \tag{6.6}$$

To compute the partition function, we need to assign t-charges to the fields. In order that D carries zero t-charge, we can easily see that the values of the t-charges for the fields: λ^α, c^m, χ_α, C^{mn}, ... are $1, 2, 3, 4, \cdots$, and for their conjugate fields $w_\alpha, b_m, \zeta^\alpha, B_{mn}, \cdots$ are $-1, -2, -3, -4, \ldots$ respectively. Furthermore, note that, the multiplicities of the ghosts λ^α, c^m, χ_α, C^{mn}, ... are 16, -10, 16, -45, ... respectively, where the sign in from of each numbers tell us whether the ghost is fermionic $(-)$ or bosonic $(+)$.

The multiplicities N_k of the ghosts can be also obtained by writing the level zero character (5.12) as [9, 11]

$$Z_0(t) = \prod_{k=1}^{\infty} (1-t^k)^{-N_k} \,. \tag{6.7}$$

The fields at t-charge k will have $|N_k|$ components, and will be bosons for $N_k > 0$ and fermions for $N_k < 0$. The multiplicities N_k contain the information about the Virasoro central charge, as well as the ghost current algebra:

$$\frac{1}{2} c_{\text{vir}} = \sum_{k=1}^{\infty} N_k, \qquad a_{\text{ghost}} = -\sum_{k=1}^{\infty} k N_k, \qquad c_{\text{ghost}} = -\sum_{k=1}^{\infty} k^2 N_k. \tag{6.8}$$

We can easily deduce from (5.12) and (6.7):

$$N_1 = 16,\ N_2 = -10,\ N_3 = 16,\ N_4 = -45,\ N_5 = 144,\ N_6 = -456,\ \ldots \tag{6.9}$$

For the computations to be done in the appendix C, we will need to know the value of the moments of the N_k's, i.e. we want: $\sum_{k=1}^{\infty} k^{s+1} N_k$. This was analyzed in [11]. The moments of (6.9) are given by

$$\begin{aligned} \sum_{k=1}^{\infty} k^{s+1} N_k &= 12 - 2^{s+1} - \frac{1}{\zeta(-s)} \sum_{k=1}^{\infty} k^s ((-2-\sqrt{3})^k + (-2+\sqrt{3})^k) \qquad (6.10) \\ &= 12 - 2^{s+1} - \frac{Li_{-s}(-2-\sqrt{3}) + Li_{-s}(-2+\sqrt{3})}{\zeta(-s)} \,, \end{aligned}$$

where $Li_s(z)$ is the so-called polylogarithm (also known as de Jonquière's function), it is a special function defined by the sum

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \,.$$

Another way to get $\sum_{k=1}^{\infty} k^{s+1} N_k$ is by considering the general expression of the form we analyzed in (5.12) and (6.7):

$$\prod_{k=1}^{\infty} (1-t^k)^{-N_k} = \frac{P(t)}{Q(t)},$$

where P and Q are some polynomials. We have

$$\sum_{k=1}^{\infty} N_k \log(1-\mathrm{e}^{kx}) = -\log \frac{P(\mathrm{e}^x)}{Q(\mathrm{e}^x)}.$$

Since

$$\log(1-\mathrm{e}^x) = \log(-x) + \frac{x}{2} + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g},$$

where B_k are Bernoulli numbers, we have:

$$\log(x)\sum_{k=1}^{\infty} N_k + \sum_{k=1}^{\infty}\log(-k)N_k + \frac{x}{2}\sum_{k=1}^{\infty} kN_k + \sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g}\sum_{k=1}^{\infty}k^{2g}N_k = -\log\frac{P(\mathrm{e}^x)}{Q(\mathrm{e}^x)}. \tag{6.11}$$

Using (5.12) and expanding the right hand side RHS of (6.11), we obtain the following value for the moments

$$\sum_{k=1}^{\infty} N_k = 11\,, \quad \sum_{k=1}^{\infty} kN_k = 8\,, \quad \sum_{k=1}^{\infty} k^2 N_k = 4\,, \quad \sum_{k=1}^{\infty} k^4 N_k = -4\,, \tag{6.12}$$
$$\sum_{k=1}^{\infty} k^6 N_k = 4\,, \quad \sum_{k=1}^{\infty} k^8 N_k = \frac{68}{3} \quad \text{and} \quad \sum_{k=1}^{\infty} k^{10} N_k = -396\,.$$

The first three moments of N_k's given in (6.12) contain the information about the conformal central charge c_{vir}, the ghost-number anomaly a_{ghost}, and the ghost-number central c_{ghost} (6.8). As we can check, the value of these current central charges are in agreement with the non-covariant calculation (4.11).

6.2 Partition function of the mini-BRST cohomology

In the previous section, we resolved the pure spinor constraint using the infinite chain of free-field ghosts, and constructed the BRST operator D. Since D carries zero t-charge, the partition function of its cohomology is equal to that of the total Hilbert space of (now unconstrained) pure spinors and the ghosts. Therefore, the full partition function of pure spinors can be formally written as

$$\begin{aligned} Z(q,t) &= \prod_{k=1}^{\infty}[(1-t^k)^{-N_k}\prod_{h=1}^{\infty}(1-q^h t^k)^{-N_k}(1-q^h t^{-k})^{-N_k}] \\ Z(q,t) &= \prod_{k=1}^{\infty}[\prod_{h=0}^{\infty}(1-q^h t^k)^{-N_k}\prod_{h=1}^{\infty}(1-q^h t^{-k})^{-N_k}] = \sum_{h=0}^{\infty} Z_h(t) q^h. \end{aligned} \tag{6.13}$$

It may seem difficult to extract useful information from this formal expression. In fact, on the contrary, once the moments of N_k's are known, the two important symmetries of the partition function—the field-antifield symmetry and the $*$-conjugation symmetry—can be easily deduced from (6.13). Also, by expanding in q, and employing Padé approximants [13], one can obtain from (6.13) a well-defined expression at each Virasoro level. In section 6.4, we shall demonstrate this by computing the partition function up to level $h = 12$.

6.3 Symmetries of the partition function

Elementary calculations show that $Z(q,t)$ defined in (6.13) has the following symmetries:

$$
\begin{aligned}
\text{field-antifield symmetry:}\quad Z(q,t) &= \prod_{k\geq 0}\Big((-1)^{N_k} t^{-(k+1)N_k}\Big) Z(q,1/t) \qquad (6.14)\\
&= -t^{-8} Z(q,1/t)\,,
\end{aligned}
$$

$*$-conjugation symmetry:

$$
\begin{aligned}
Z(q,t) &= \prod_{k\geq 0}\Big((-1)^{-kN_k} q^{-\frac{1}{2}k(k+1)N_k} t^{k(k+1)N_k}\Big) Z(q,q/t) \qquad (6.15)\\
&= -q^2 t^{-4} Z(q,q/t)\,.
\end{aligned}
$$

Imposing those symmetries on the formal expression for $Z(q,t)$ means that one has made an analytical continuation

$$
Z(q,t) \; = \prod\nolimits_{k\geq 0}(-iq^{-1/12}\eta(q)^{-1}\sqrt{t^k}\,\vartheta_{11}(q,t^k))^{-N_k}\,, \qquad (6.16)
$$

where the elliptic functions are defined as

$$
\begin{aligned}
\vartheta_{11}(q,t) &= i\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{1}{2}(k-1/2)^2} t^{k-1/2} \qquad (6.17)\\
&= -iq^{1/12}\eta(q)(t^{1/2}-t^{-1/2})\prod_{h\geq 1}(1-q^h t)(1-q^h t^{-1})\,,\\
\eta(q) &= q^{1/24}\prod_{h\geq 1}(1-q^h)\,. \qquad (6.18)
\end{aligned}
$$

The symmetries above follow from the well-known transformation properties of the theta function:

$$
\vartheta_{11}(q,t) \; = -\vartheta_{11}(q,1/t) = -q^{1/2}t\,\vartheta_{11}(q,qt)\,. \qquad (6.19)
$$

$*$-conjugation symmetry and the higher cohomology As in the case of the toy model (see chapter 3 and [10]), the $*$-conjugation symmetry suggests that there are non-trivial fermionic elements in the higher D-cohomology. The element with charges $-q^2t^{-4}$ generalizing the state b of the toy model is of particular importance. Unfortunately, the construction of this state in the BRST framework is not straightforward, obstructed by the complexity of the infinite ghosts-for-ghosts. However, the state has a particularly nice interpretation in the Čech/Dolbeault cohomologies. In fact, it turns out to be nothing but the tail term b_3 of the composite reparameterization b-ghost [9].

In the next section, starting from the formal expression of the partition function (6.13), we are going to describe a method for computing higher level character formulas $Z_h(t)$. We refer this method as Padé approximants. The method is mainly based in the knowledge of the zero mode part $Z_0(t)$ of the partition function. From this level zero character formula, we extract the ghosts multiplicities N_k, and using the moments ($\sum_k k^{s+1}N_k$) of those multiplicities, by employing a novel application of Padé approximants, we are able to compute higher level character formulas $Z_h(t)$ of pure spinors (up to the twelfth mass-level $h = 12$). We find that our results are in agreement with the results found in [9] (up to the fifth mass-level $h = 5$) where the fixed point technique was used (the fixed point technique will be discussed in the next chapter).

6.4 Padé approximants

The Padé approximation seeks to approximate the behavior of a function by a ratio of two polynomials. This ratio is referred to as the Padé approximant. This approximation works nicely even for functions containing poles, because the use of rational functions allows them to be well-represented. Recently, the Padé approximation has been applied to string field theory to analyze the tachyon condensation [28, 29, 30].

Let us now consider the general equations of the Padé approximation. Given some function $f(t)$, its $[M/N]$ Padé approximant denoted by $f^{[M/N]}(t)$ is a rational function of the form [31]

$$f^{[M/N]}(t) = \frac{1 + \sum_{j=1}^{M} p_j t^j}{\sum_{j=0}^{N} q_j t^j}\,, \tag{6.20}$$

where the coefficients $p_1, p_2, \cdots, p_M, q_0, q_1, \cdots, q_N$, are obtained by solving a system of $M + N + 1$ algebraic equations

$$\frac{d^n f^{[M/N]}}{dt^n}(a) = f^{(n)}(a)\,, \qquad\qquad n = 0,\, 1,\, 2,\, \cdots,\, M + N\,. \tag{6.21}$$

The equations (6.21) come from equating the coefficients of $(t-a)^n$ (up to the order $n = M + N$) in the Taylor expansion of the functions $f(t)$ and $f^{[M/N]}(t)$ around some point $t = a$ (which usually is taken at $t = 0$).

Having sketched briefly the method to approximate functions by means of rational functions. Next, we are going to use this method for computing higher level character formulas of pure spinors. Let us start by writting the formal expression (6.13) for the partition function of pure spinors like the following

$$Z(q,t) = Z_0(t)\Big[1 + \sum_{h=1}^{\infty} f_h(t) q^h\Big]\,, \tag{6.22}$$

where the level h function $f_h(t)$ is defined by

$$f_h(t) = \frac{1}{h!}\frac{\partial^h}{\partial q^h}\tilde{Z}(q,t)|_{q=0} \quad \text{where} \quad \tilde{Z}(q,t) = \prod_{k=1}^{\infty}\prod_{h=1}^{\infty}(1-q^h t^k)^{-N_k}(1-q^h t^{-k})^{-N_k} \tag{6.23}$$

As we know by a previous work [9], up to the level $h = 5$, these level h functions are given by rational functions. Therefore, this result is an indication that these level h functions can be computed by means of Padé approximants. In fact this is the case as it is shown in the appendix C, the functions $f_1(t)$, $f_2(t)$, $f_3(t)$, $\cdots$ can be calculated using Padé approximants. As our main result, we have noted that these functions can be written like the following

$$f_h(t) = \frac{\sum_{i=0}^{2h+6} C_{i,h} t^i}{t^{h+2}(1+4t+t^2)}\,, \tag{6.24}$$

where the value of the coefficients $C_{i,h}$ up to the level $h = 12$ are given in the following tables

i	$C_{i,0}$	$C_{i,1}$	$C_{i,2}$	$C_{i,3}$	$C_{i,4}$	$C_{i,5}$	$C_{i,6}$	$C_{i,7}$
0	0	0	-1	-16	-126	-672	-2772	-9504
1	0	0	12	146	920	3996	13440	37224
2	1	0	-67	-536	-2411	-7616	-18358	-35184
3	4	46	248	822	1852	3270	7752	33356
4	1	40	319	1200	1745	-5944	-48147	-179648
5	0	46	628	4114	17000	48206	91948	87730
6	0	0	319	3720	21767	82112	210717	326760
7	0	0	248	4114	32356	162662	585464	1575690
8	0	0	-67	1200	21767	162552	778424	2706944
9	0	0	12	822	17000	162662	977032	4215020
10	0	0	-1	-536	1745	82112	778424	4454624
11	0	0	0	146	1852	48206	585464	4215020
12	0	0	0	-16	-2411	-5944	210717	2706944
13	0	0	0	0	920	3270	91948	1575690
14	0	0	0	0	-126	-7616	-48147	326760
15	0	0	0	0	0	3996	7752	87730
16	0	0	0	0	0	-672	-18358	-179648
17	0	0	0	0	0	0	13440	33356
18	0	0	0	0	0	0	-2772	-35184
19	0	0	0	0	0	0	0	37224
20	0	0	0	0	0	0	0	-9504

(6.25)

i	$C_{i,8}$	$C_{i,9}$	$C_{i,10}$	$C_{i,11}$	$C_{i,12}$
0	−28314	−75504	−184041	−416416	−884884
1	87912	180180	320892	484770	565136
2	−55368	−77968	−130185	−342472	−1118117
3	145512	513680	1480688	3596898	7511244
4	−467078	−900256	−1189750	−468240	2940853
5	−112192	−651084	−1221496	23356	8349688
6	−14878	−1971392	−7447790	−17913424	−30692216
7	3130008	3975312	77136	−15954844	−51076344
8	7136292	14067968	18009420	783936	−74353372
9	13953544	36499868	75237248	115010006	94216072
10	18453761	59453552	153557340	318143976	504911177
11	21308252	82467920	255938464	651539178	1363603964
12	18453761	88624848	330202624	999457424	2512598839
13	13953544	82467920	366326624	1301105402	3825279040
14	7136292	59453552	330202624	1398947880	4802902081
15	3130008	36499868	255938464	1301105402	5216743428
16	−14878	14067968	153557340	999457424	4802902081
17	−112192	3975312	75237248	651539178	3825279040
18	−467078	−1971392	18009420	318143976	2512598839
19	145512	−651084	77136	115010006	1363603964
20	−55368	−900256	−7447790	783936	504911177
21	87912	513680	−1221496	−15954844	94216072
22	−28314	−77968	−1189750	−17913424	−74353372
23	0	180180	1480688	23356	−51076344
24	0	−75504	−130185	−468240	−30692216
25	0	0	320892	3596898	8349688
26	0	0	−184041	−342472	2940853
27	0	0	0	484770	7511244
28	0	0	0	−416416	−1118117
29	0	0	0	0	565136
30	0	0	0	0	−884884

(6.26)

We have defined the values of the $C_{i,0}$ coefficients such that the level zero function is defined as $f_0(t) = 1$. It is interesting to note that the coefficients $C_{i,h}$ satisfy the following identities

$$C_{i,h} \quad = \quad C_{2h+6-i,h} \, , \tag{6.27}$$

$$\sum_{i=0}^{h'} \phi_{h'-i} C_{i,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,h'} , \tag{6.28}$$

which can be derived by using the two symmetries of the partition function (6.14), (6.15) and verified by using the coefficients shown in tables (6.25) and (6.26). The coefficient ϕ_m is generated by

$$\frac{Z_0(t)}{1+4t+t^2} = \sum_{n=0}^{\infty} \phi_n t^n , \tag{6.29}$$

and it is given explicitly by the formula

$$\phi_n = \frac{(1+n)(2+n)(3+n)(4+n)(5+n)^2(6+n)(7+n)(8+n)(9+n)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7} . \tag{6.30}$$

The importance of the identity (6.28) is as follows. If we know the coefficients $C_{i,0}$, $C_{i,1}$, $\cdots$, $C_{i,h'}$, it is possible to compute explicitly the coefficients $C_{0,h}$, $C_{1,h}$, $\cdots$, $C_{h',h}$. For instance, setting $h' = 0$ in equation (6.28), we get

$$C_{0,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,0} , \tag{6.31}$$

using (6.30) and the value of the coefficients $C_{i,0}$ given in the table (6.25) into the equation (6.31), we obtain

$$C_{0,h} = \frac{(1-h)h(1+h)^2(2+h)^2(3+h)^2(4+h)(5+h)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7} . \tag{6.32}$$

By employing the same steps given above, setting $h' = 1$ in equation (6.28), we arrive to the following expression for the $C_{1,h}$ coefficient

$$C_{1,h} = \frac{(h-1)h(1+h)^2(2+h)(3+h)(4+h)(108+10h+12h^2-h^3)}{2^5 \cdot 3^3 \cdot 5 \cdot 7} . \tag{6.33}$$

Finally, it would be important to find an explicit expression for a general coefficient $C_{i,h}$ (for all $h \geq 0$ and $i \geq 0$). It is clear that if we know explicitly $C_{i,h}$, it should be possible to write a compact expression for the complete pure spinor partition function

$$Z(q,t) = \frac{1+t}{(1-t)^{11}} \sum_{h=0}^{\infty} \sum_{i=0}^{2h+6} C_{i,h} t^{i-h-2} q^h , \tag{6.34}$$

where the factor $(1+t)/(1-t)^{11}$, in front of our formula (6.34), comes from substitution of equations (5.12) and (6.24) into the equation (6.22).

So far, we have computed the partition function without the spin dependence on the states. Spin dependence is crucial if we want to prove that the full partition function (including the contribution of the worldsheet matter sector) correctly reproduces the light cone superstring spectrum [9]. Therefore, it would be interesting to know the character formula with the spin dependence. In the next chapter, we are going to compute the character formulas including the spin dependence at each Virasoro level by using the so-called fixed point technique.

Chapter 7

Fixed point formulas for the full partition function

In the previous chapter, we have presented a formula for the full partition function of pure spinors using an infinite tower of ghosts-for-ghosts. The formula is natural and convenient for motivating the two important symmetries of the partition function, i.e. the field-antifield symmetry and the $*$-conjugation symmetry. Also, we were able to compute the partition function level by level, respecting those two symmetries. However, the computation using (6.13) is not easy if one wishes to keep the spin information of the states [13]. We here present a very simple fixed point formula for the partition function including the spin character, extending the zero mode formula given in [11]. Although our formulas (7.11) and (7.12) miss some finite number of states at each Virasoro level, these missing states can be recovered by imposing the two symmetries (6.14) and (6.15) of the full partition function.

7.1 Review of fixed point formula for level zero partition function

For convenience, we briefly review the fixed point formula for the zero mode partition function [11]. (See also [32].)

Geometric preliminary Let us begin by refining our description of the space of pure spinors

$$\begin{aligned} X_{10} &= \{\lambda^\alpha \mid \lambda^\alpha\gamma^\mu_{\alpha\beta}\lambda^\beta = 0\,,\ \lambda \neq 0\} \\ &= (\mathcal{C}^*\text{-bundle over } \mathcal{X}_{10})\,, \quad (\mathcal{X}_{10} = SO(10)/U(5))\,. \end{aligned} \tag{7.1}$$

With the removal of the origin understood, X_{10} can be covered by 16 patches, where in each patch at least one component of λ is non-vanishing. It is convenient to use the "five sign" notation to describe the components of λ. (See appendix A.1 for explanations.) In this notation, the character of 16 components are

$$\lambda_{\pm\pm\pm\pm\pm} = \mathrm{e}^{\frac{1}{2}(\pm\sigma_1\pm\sigma_2\pm\sigma_3\pm\sigma_4\pm\sigma_5)} \quad \text{(even number of $-$'s)}\,, \tag{7.2}$$

and X_{10} can be covered by 16 patches

$$U_{\pm\pm\pm\pm\pm} \ = \{\lambda \in X_{10} \mid \lambda_{\pm\pm\pm\pm\pm} \neq 0\}\,. \tag{7.3}$$

In a given patch, a pure spinor can be parameterized using eleven parameters (g, u_{ab}) which are in the $(\mathbf{1}, \mathbf{10})$ of $U(5)$. u_{ab} is the "angular" coordinate parameterizing the base $\mathcal{X}_{10}$ and g is the coordinate for the fiber $\mathcal{C}^*$. For example in U_{+++++},

$$\lambda \ = (\lambda_+, \lambda_{ab}, \lambda^a) = (g, g u_{ab}, \tfrac{1}{8} g \epsilon^{abcde} u_{bc} u_{de})\,. \tag{7.4}$$

where

$$\begin{aligned} \lambda_+ &= \lambda_{+++++} \neq 0\,, \\ \lambda_{ab} &= \{\lambda_{+++--} \text{ and permutations }\}\,, \\ \lambda^a &= \{\lambda_{+----} \text{ and permutations }\}\,. \end{aligned} \tag{7.5}$$

The characters of g and u_{ab} in this patch are

$$g = \mathrm{e}^{\frac{1}{2}(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5)}\,, \quad u_{ab} = \mathrm{e}^{-(\sigma_a+\sigma_b)}\,,\ (1 \le a < b \le 5)\,. \tag{7.6}$$

In other patches $U_{\epsilon(+++++)}$, the characters will be

$$g_\epsilon = \mathrm{e}^{\frac{1}{2}\epsilon(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5)} = \mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma}\,, \quad u_{\epsilon,ab} = \mathrm{e}^{-\epsilon(\sigma_a+\sigma_b)} = \mathrm{e}^{-(\epsilon_{\mathrm{a}}\sigma_{\mathrm{a}}+\epsilon_{\mathrm{b}}\sigma_{\mathrm{b}})}\,, \tag{7.7}$$

where ϵ acts by even number of sign changes.

The fixed point formula By constructing the symmetry generators explicitly, one finds the action of $N_{\mu\nu}$ on X_{10} which commutes with the J-rescaling of the $\mathcal{C}^*$-fiber. A generic action of the maximal torus of $SO(10)$ has 16 fixed points which are nothing but the "origins" ($u_{ab} = 0$) of 16 patches. The spin character of pure spinors can then be written as a sum of the contributions at the fixed points [32][33]

$$Z_0(t, \vec{\sigma}) = \sum_{\epsilon=1}^{16} \frac{1}{1 - t\mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1 - \mathrm{e}^{-(\epsilon_{\mathrm{a}}\sigma_{\mathrm{a}}+\epsilon_{\mathrm{b}}\sigma_{\mathrm{b}})}} \tag{7.8}$$

where we use the notation of [11]. The sum over ϵ describes the sum over 16 fixed points. The first term of the summand is the character of the non-vanishing component g_ϵ, and the second term is the character of the rest $u_{\epsilon,ab}$, both at a given fixed point ϵ. Summation over the fixed points in (7.8) is straightforward and one gets [11]

$$Z_0(t, \vec{\sigma}) = \frac{\mathbf{1} - \mathbf{10}t^2 + \overline{\mathbf{16}}t^3 - \mathbf{16}t^5 + \mathbf{10}t^6 - \mathbf{1}t^8}{(1-t)^{\mathbf{16}}}\,. \tag{7.9}$$

7.2 Fixed point formulas for the full partition function

Now, let us introduce two ways to extend the zero-mode fixed point formula (7.8). The two utilize different parameterizations for the non-zero modes and lead to partition functions that differ by a finite number of terms at each level. Also, both miss a finite number of terms with respect to the fully symmetric partition function. However, the missing states can be unambiguously recovered by imposing the field-antifield and $*$-conjugation symmetries, and the two formulas then give the same symmetric partition function.

The first way of extending is to simply include the non-zero modes of λ at each patch $(g_\epsilon, u_{\epsilon,ab}) \in U_\epsilon$ together with the modes of their conjugates $(h_\epsilon, v_\epsilon^{ab})$:

$$Z(q,t,\vec{\sigma}) = \sum_{\epsilon=1}^{16} Z_\epsilon(q,t,\vec{\sigma}) \,, \tag{7.10}$$

$$\begin{aligned} Z_\epsilon(q,t,\vec{\sigma}) &= \prod_{h\geq 0} \frac{1}{1-q^h t \mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1-q^h \mathrm{e}^{-(\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b})}} \\ &\quad \times \prod_{h\geq 1} \frac{1}{1-q^h t^{-1} \mathrm{e}^{-\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1-q^h \mathrm{e}^{\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b}}} \,. \end{aligned} \tag{7.11}$$

The first line represents the modes of (g, u_{ab}) and the second line represents the modes of (h, v^{ab}).

To obtain another way of parameterizing the non-zero modes, one observes that the constraints for the non-zero modes are essentially linear $\lambda_0\gamma^\mu\lambda_{-h}+\cdots=0$ ($\lambda_{-h} \sim \partial^h\lambda$) while the constraint for the zero mode is quadratic $\lambda_0\gamma^\mu\lambda_0 = 0$. Therefore, the 11 components of non-zero modes λ_{-h} (and their conjugates) can be thought as carrying different characters from the zero-mode λ_0, and the contribution from a fixed point is

$$\begin{aligned} Z_\epsilon(q,t,\vec{\sigma}) &= \frac{1}{1-t\mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1-\mathrm{e}^{-(\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b})}} \\ &\quad \times \prod_{h\geq 1} \frac{1}{1-q^h t \mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1-q^h t \mathrm{e}^{\frac{1}{2}\epsilon\cdot\sigma-(\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b})}} \\ &\quad \times \prod_{h\geq 1} \frac{1}{1-q^h t^{-1} \mathrm{e}^{-\frac{1}{2}\epsilon\cdot\sigma}} \prod_{(ab)=1}^{10} \frac{1}{1-q^h t^{-1} \mathrm{e}^{-\frac{1}{2}\epsilon\cdot\sigma+(\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b})}} \,. \end{aligned} \tag{7.12}$$

The second line describes the contributions of the λ non-zero modes and the third line describes the contributions of the ω non-zero modes. As mentioned above, it carries essentially the same information as the first formula (7.11).

By expanding either (7.11) or (7.12) in q, the level h partition function with spin information is expressed in a simple form for all $h \geq 1$. The summation over 16 fixed points is straightforward, and one gets a result of the form

$$Z_h(t,\vec{\sigma}) = \frac{P'_h(t,\vec{\sigma})}{(1-t)^{\mathbf{16}}}\,, \tag{7.13}$$

where $P'_h(t,\vec{\sigma})$ is some polynomial in t with coefficients taking values in the representations of $SO(10)$, and $(1-t)^{\mathbf{16}} = \prod_{\mu\in\mathbf{16}}(1-t\mathrm{e}^{\mu\cdot\sigma})$. We put a prime on $P'_h(t,\vec{\sigma})$ as it lacks field-antifield symmetry as of yet:

$$P'_h(t,\vec{\sigma}) \neq P'_h(1/t,-\vec{\sigma})\,. \tag{7.14}$$

We now turn to our results on $P'_h(t,\vec{\sigma})$ and explain how to improve them so that they respect the field-antifield and $*$-conjugation symmetries.

7.3 Partition functions for non-zero modes with spin character

Although the summation over 16 fixed points is straightforward, it is not obvious how to combine local $U(5)$ characters into $SO(10)$ characters in a simple manner. A convenient computational trick is to utilize the Weyl character formula to take care of the combinatorics. To do this, one first augments the factor for the zero-mode character $\prod_{(ab)=1}^{10}(1-\mathrm{e}^{-(\epsilon_\mathrm{a}\sigma_\mathrm{a}+\epsilon_\mathrm{b}\sigma_\mathrm{b})})^{-1}$ representing the 10 "positive roots of $SO(10)/U(5)$" by the character of the remaining 10 positive roots of $SO(10)$, i.e. those of $U(5)$, $\prod_{(ab)=1}^{10}(1-\mathrm{e}^{-(\epsilon_\mathrm{a}\sigma_\mathrm{a}-\epsilon_\mathrm{b}\sigma_\mathrm{b})})^{-1}$, and then extends the summation over the 16 fixed points ϵ to 1920 elements of the $SO(10)$ Weyl group W. Using the first parameterization of (7.11), 1920 "local" contributions are given by

$$\begin{aligned} Z_w(q,t,\vec{\sigma}) \;=\; & \frac{1}{1-t\mathrm{e}^{\frac{1}{2}w\cdot\sigma}}\prod_{(ab)=1}^{10}\frac{1}{(1-\mathrm{e}^{-(w_a\sigma_a+w_b\sigma_b)})(1-\mathrm{e}^{-(w_a\sigma_a-w_b\sigma_b)})} \\ & \times\prod_{h\geq1}\frac{1}{1-q^h t\mathrm{e}^{\frac{1}{2}w\cdot\sigma}}\prod_{(ab)=1}^{10}\frac{1}{1-q^h\mathrm{e}^{-(w_a\sigma_a+w_b\sigma_b)}} \\ & \times\prod_{h\geq1}\frac{1}{1-q^h t^{-1}\mathrm{e}^{-\frac{1}{2}w\cdot\sigma}}\prod_{(ab)=1}^{10}\frac{1}{1-q^h\mathrm{e}^{w_a\sigma_a+w_b\sigma_b}}\,. \end{aligned} \tag{7.15}$$

(Using the second parameterization of (7.12), the formula is the same except for the last two lines representing non-zero modes.) An element $w\in W$ acts on the five-sign basis by permutations and an even number of sign changes. The two modifications "cancel" each other and simply gives $\sum_w Z_w = \sum_\epsilon Z_\epsilon$.

Now multiplying $\mathrm{e}^{w\cdot\rho}$ (where ρ is the half sum of positive roots) to both the numerator and denominator of $Z_w(q,t,\vec{\sigma})$, and denoting the $SO(10)$ Weyl denominator by e^ρR, the sum over w reads

$$Z(q,t,\vec{\sigma}) = \sum_w^{1920} Z_w(q,t,\vec{\sigma}) \;=\; \frac{1}{(1-t)^{\mathbf{16}}}\frac{1}{e^\rho R}\sum_w (-1)^w e^{w\rho}\prod_{\epsilon\neq 1}^{15}(1-te^{\frac{1}{2}w(\epsilon\cdot\sigma)}) \quad (7.16)$$
$$\times \prod_{h\geq 1}\{\text{non-zero modes}\}\,.$$

Using the Weyl character formula, the summation over $w \in W$ is readily done leading to the expressions of the form (7.13). This trick also explains why one gets $SO(10)$ representations as the coefficients of t.

Level 1 Using the computational trick just mentioned at this level, the second parameterization of (7.12) yields

$$Z_{1,2nd}(t,\vec{\sigma}) = \frac{1}{(1-t)^{\mathbf{16}}}\quad \{(\mathbf{45}+\mathbf{1})t^0 - \overline{\mathbf{144}}t^1 + (\mathbf{126}^- - \mathbf{10})t^2 + \overline{\mathbf{16}}t^3 \quad (7.17)$$
$$-\mathbf{16}t^5 - (\mathbf{126}^+ - \mathbf{10})t^6 + \mathbf{144}t^7 - (\mathbf{45}+\mathbf{1})t^8\}\,,$$

while the first parameterization (7.11) yields

$$Z_{1,1st} = Z_{1,2nd} - \mathbf{1}\,. \quad (7.18)$$

The singlet missing from $Z_{1,1st}$ is the gauge invariant current $J = \omega\lambda$, and the only way to make $Z_{1,1st}$ consistent with field-antifield symmetry and $*$-conjugation symmetry up to this level is to add $\mathbf{1}$ to it. So we conclude

$$Z_1(t,\vec{\sigma}) = Z_{1,2nd}(t,\vec{\sigma}) = Z_{1,1st}(t,\vec{\sigma}) + \mathbf{1}\,, \quad (7.19)$$

where $Z_{1,2nd}$ obtained from our second parameterization is defined in (7.17).

Higher levels An important point to notice is that although Z_{2nd} reproduces the fully symmetric partition function at level 1, neither Z_{1st} nor Z_{2nd} reproduce the fully symmetric partition function at higher levels. In particular, they both miss the fermionic singlet at $-q^2t^{-4}$ discussed above, and (a part of) analogous states at higher levels. Also, both Z_{1st} and Z_{2nd} miss some gauge-invariant operators. For example, at level 3, the numerator $P_3'(t,\vec{\sigma})$ in Z_{2nd} starts as

$$P_3'(t,\vec{\sigma}) \;= -\mathbf{10}t^{-2} + (\mathbf{144} + \mathbf{560} + 3\cdot\overline{\mathbf{16}})t^{-1} + \cdots\,, \quad (7.20)$$

and the correction required includes both bosonic and fermionic operators. Nevertheless, at least up to the fifth level, the difference between the t-expansions of the fully symmetric partition function and the result from the fixed point formulas is always finite. Therefore, the fixed point result can be unambiguously improved to the symmetric one using the method described for the ghost-for-ghost partition function. (A list of the improved numerator $P_h(t,\vec{\sigma})$ can be found in appendix B.2.)

Towards fixed point formula for the fully symmetric partition function Since we use the field-antifield and $*$-conjugation symmetries as guiding principles for computing the complete partition function, it will be useful to build them into the fixed point formula itself. Although we do not have an answer to this problem at the present time, organizing the complete partition function into a character of $\widehat{SO}(10)$ affine Lie algebra seems to be promising. This should also be useful for extending our result to all mass levels. However, we leave the study of these issues for future research, and we now turn into the explanation of the structure of the pure spinor cohomology using Čech and Dolbeault descriptions.

Chapter 8

Pure spinor cohomology: Čech and Dolbeault descriptions

In this chapter, we explain the structure of the Hilbert space of the pure spinor system. We are going to give an explanation of the states which do not correspond to the usual gauge invariant polynomials. As mentioned those "missing" states are essential for the partition function to have the symmetries

$$Z(q,t) = -t^{-8} Z(q, 1/t) = -q^2 t^{-4} Z(q, q/t)\,. \tag{8.1}$$

Let us begin by studying certain aspects of the Čech type description of the pure spinors and indicating the results we have established in this thesis. This is not intended to be a complete overview of the formalism, as we only cover issues which are relevant for explaining the missing states appearing in the computation of the pure spinor partition function.

8.1 Pure spinor sector as a curved $\beta\gamma$ system

A standard way to construct a general curved $\beta\gamma$ system on a complex manifold X is to start with a set of free conformal field theories taking values in the coordinate patches $\{U_A\}$ of X, and try to glue them together [32, 34, 35]. The field content of each conformal field theory is described by the (holomorphic) coordinates of a patch u^a and its conjugate v_a satisfying the free field operator product expansion

$$u^a(z) v_b(w) \sim \frac{\delta^a{}_b}{z-w}\,. \tag{8.2}$$

Not all manifolds X, however, lead to a consistent worldsheet theory. A basic requirement is that one must be able to consistently glue the operator products (8.2) on overlaps. Gluing on double overlaps $U_A \cap U_B$ can always be done (though they are not quite unique), but the gluing on $U_A \cap U_B$, $U_A \cap U_C$ and $U_B \cap U_C$ must be consistent on the triple overlap $U_A \cap U_B \cap U_C$ (cocycle condition). In order that there is no topological obstruction for this, the first Pontryagin class $p_1(X)$ must be vanishing. Also, to be able to define the energy-momentum tensor T globally (i.e. to have a *conformal* field theory), X must possess a nowhere vanishing holomorphic top-form and hence the first Chern class $c_1(X)$ must also be vanishing.

In the case of pure spinors all these obstructions turn out to be absent [32]. The target space is basically the space of $SO(10)$ pure spinors, with the origin removed:

$$X_{10} = \{\lambda^\alpha \mid \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0\,, \lambda \neq 0\}\,, \tag{8.3}$$

which is a complex cone over a compact projective space $\mathcal{X}_{10}$. It is well known that $\mathcal{X}_{10}$ is the homogeneous space

$$\mathcal{X}_{10} = SO(10)/U(5)\,, \tag{8.4}$$

and has ten (complex) dimensions. The origin $\lambda = 0$ is removed from the space of all solutions to the equations $\lambda\gamma^m\lambda = 0$ in order to meet the general criteria above, $p_1 = c_1 = 0$. With this removal of the origin understood, X_{10} can be covered by 16 patches $\{U_A\}$ $(A = 1, \ldots, 16)$ where in each patch at least one component of λ (which we denote λ^A) is non-vanishing. Very explicit formulas for the gluing of operator products, symmetry currents J and N^{mn}, and the energy-momentum tensor T can be found in [32].

8.2 Čech description

Given a space X on which the curved $\beta\gamma$ system can be consistently defined, the space of observables, or simply the Hilbert space of the model, is defined as the cohomology of the difference operator $\check{\delta}$, also known as Čech operator.

To understand what Čech cohomology is, we need to grasp a few concepts first. Let X be a complex manifold, and $\mathcal{U} = \{U_A\}$ an open cover of X. This means that we have open sets U_A such that $\cup_A U_A = X$. Let F be a *presheaf** of Abelian groups on X, that is, a map that assigns to every set in X an Abelian group. This group can, in principle, vary from set to set. For instance, the Abelian group $F(U_A)$

*In mathematics, a presheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space. The data can be restricted to smaller open sets, and the data assigned to an open set is equivalent to all collections of compatible data assigned to collections of smaller open sets covering the original one. For example, such data can consist of continuous or smooth functions defined on each open set. Presheaves are by design quite general and abstract objects, and their correct definition is rather technical [36].

assigned to a set $U_A \subset X$ could be the group of continuous functions defined on U_A, where the group operation is simply the sum of functions.

A presheaf also assumes that if we have two sets $U_A, U_B \subset X$ such that $U_A \subset U_B$, then there exists a *restriction* map $r_{U_A,U_B} : F(U_B) \to F(U_A)$. We see that r_{U_A,U_B} maps, or restricts, the Abelian group corresponding to the bigger set U_B to the Abelian group corresponding to the smaller set U_A.

Now, consider ordered families of $(n+1)$ sets U_{A_i}, $i = 0, 1, \cdots, n$ from the cover. Ordered in this context means that the order of the given sets is to be taken into account. An *n-cochain* $\psi = (\psi^{A_0 A_1 \cdots A_n})$ assigns to a family $\{U_{A_0}, U_{A_1}, \ldots, U_{A_n}\}$ an element of the Abelian group $F(U_{A_0} \cap U_{A_1} \cap \ldots \cap U_{A_n})$. For instance, if F assigns to the group of continuous functions, a 1-cochain ψ assigns to $\{U_{A_0}, U_{A_1}\}$ a function whose domain is $U_{A_0} \cap U_{A_1}$. In other words, we have an object of the form $\psi(\{U_{A_0}, U_{A_1}\})(x) = f(x)$, with $x \in U_{A_0} \cap U_{A_1}$.

Since n-cochains take values in the Abelian groups (determined by the presheaf), one can add two different n-cochains to obtain another n-cochain. Thus, n-cochains themselves form an Abelian group. Suppose we have an n-cochain ψ, as we saw, this acts on families of $(n+1)$ sets. Let us see how we can use ψ to construct an $(n+1)$-cochain, that is, a map acting on families of $(n+2)$ sets. Let $\{U_{A_0}, \ldots, U_{A_{n+1}}\}$ be an ordered family of $(n+2)$ sets from the cover $\mathcal{U}$. If we remove a set U_{A_j} from the family, we are left with an ordered family of $(n+1)$ sets, $\{U_{A_0}, \ldots, U_{A_{j-1}}, U_{A_{j+1}}, \ldots, U_{A_{n+1}}\}$. Then, the n-cochain ψ can act on this reduced family to give an element of the Abelian group $F(U_{A_0} \cap \ldots \cap U_{A_{j-1}} \cap U_{A_{j+1}} \cap \ldots \cap U_{A_{n+1}})$. Now, we want to define an $(n+1)$-cochain, whose output should be an element of the Abelian group $F(U_{A_0} \cap \ldots\ U_{A_{n+1}})$, which corresponds to the intersection of all $(n+2)$ sets. To get such an element, we can use the restriction map. Indeed, since $U_{A_0} \cap \ldots\ U_{A_{n+1}} \subset U_{A_0} \cap \ldots \cap U_{A_{j-1}} \cap U_{A_{j+1}} \cap \ldots \cap U_{A_{n+1}}$ (notice that the intersection of $(n+2)$ sets is smaller than the intersection of $(n+1)$ sets), there exists a restriction map $r_j : F(U_{A_0} \cap \ldots \cap U_{A_{j-1}} \cap U_{A_{j+1}} \cap \ldots \cap U_{A_{n+1}}) \to F(U_{A_0} \cap \ldots\ U_{A_{n+1}})$. Then, $r_j\psi$ would be a $(n+1)$ cochain.

Now, the $(n+1)$-cochain we are really interested in constructing is a combination of the $(n+1)$-cochains we just defined. Specifically, we define the $(n+1)$-cochain $\check{\delta}\psi$ by

$$(\check{\delta}\psi)(U_{A_0}, \ldots, U_{A_{n+1}}) = \sum_{j=0}^{n+1} (-1)^j r_j \psi(U_{A_0}, \ldots, U_{A_{j-1}}, U_{A_{j+1}}, \ldots, U_{A_{n+1}}) \tag{8.5}$$

Note that the $(-1)^j$ makes this an alternating sum. Since the output of $r_j\psi$ is an element of an Abelian group, the negative sign would mean simply the inverse element in the group sense. The symbol $\check{\delta}$ is called the Čech differential which is nilpotent $\breve{\delta}^2 = 0$, and it takes n-cochains into an $(n+1)$-cochain.

An n-cochain such that $\breve{\delta}\psi = 0$ is called an n-cocycle. An n-cochain ψ that can be written as $\psi = \breve{\delta}\phi$, where ϕ is an $(n-1)$-cochain, will satisfy $\breve{\delta}\psi = 0$ trivially, i.e., it will be an n-cocycle. These trivial n-cocycles are called *n-coboundaries*. The nth Čech cohomology $H^n(\breve{\delta})$ is defined as the space of $\breve{\delta}$-closed n-cochains (n-cocycles) modulo $\breve{\delta}$-exact elements (n-coboundaries). In particular, the zeroth cohomology $H^0(\breve{\delta})$ is simply the space of "gauge invariant" operators defined globally on X. It turns out that the missing states (appearing in the computation of the pure spinor partition function) correspond to elements in the third Čech cohomology $H^3(\breve{\delta})$. A remarkable reason why $H^3(\breve{\delta})$ is important is that a nontrivial element in $H^3(\breve{\delta})$ is essential for the construction of the composite reparameterization b-ghost.

8.3 Čech/Dolbeault descriptions and the operator b_3

The description of the curved $\beta\gamma$ system in the previous section was done using the Čech language by patching together a collection of free conformal field theories. There is a closely related formulation which uses the Dolbeault language. The two are related in the same manner as the standard Čech and Dolbeault cohomologies of a complex manifold are related. In the Čech description, only the holomorphic local coordinates u_a of X were used, but the Dolbeault description utilizes the antiholomorphic variable $\overline{u}^{\overline{a}}$ as well. This allows the construction of a partition of unity on X and, by considering the cohomology of an extension of the Dolbeault operator $\overline{\partial}_X$, one can deal exclusively with globally defined objects [32, 35].

In the pure spinor formalism, the so-called non-minimal formulation corresponds to this Dolbeault formulation [7][14]. There, one introduces another set of pure spinor variables $\overline{\lambda}_\alpha$ and its (target space) differential $r_\alpha = \mathrm{d}\overline{\lambda}_\alpha$ which are constrained to satisfy

$$\overline{\lambda}_\alpha \gamma^{m\alpha\beta} \overline{\lambda}_\beta \ = 0\,, \quad \overline{\lambda}_\alpha \gamma^{m\alpha\beta} r_\beta = 0\,. \tag{8.6}$$

The conjugate momenta for the non-minimal fields are denoted by $\overline{\omega}^\alpha$ and s^α, and they must appear in combinations which are invariant under the non-minimal gauge transformations

$$\begin{aligned} \delta_{\overline{\Lambda}} \overline{\omega}^\alpha = \overline{\Lambda}^m (\gamma_m \overline{\lambda})^\alpha\,, \quad \delta_\Psi \overline{\omega}^\alpha &= \Psi^m (\gamma_m r)^\alpha\,, \\ \delta_\Psi s^\alpha &= \Psi^m (\gamma_m \overline{\lambda})^\alpha\,, \end{aligned} \tag{8.7}$$

with $\overline{\Lambda}_m$ and Ψ_m being bosonic and fermionic gauge parameters.

The Dolbeault operator $\overline{\partial}_X$ can be defined as a natural extension of the Dolbeault differential in complex geometry:

$$\overline{\partial}_X = -r_\alpha \overline{\omega}^\alpha \sim \mathrm{d}\overline{\lambda}_\alpha \frac{\partial}{\partial \overline{\lambda}_\alpha} \,. \tag{8.8}$$

Note that $\overline{\partial}_X$ is gauge invariant under (8.7). If one wishes to be more rigorous, the expression for $\overline{\partial}_X$ should be understood in terms of its local expressions that are consistently glued. Also, note that only the zero-modes for the non-minimal sector are relevant for the $\overline{\partial}_X$-cohomology due to the relation

$$\overline{\partial}_X(s\partial\overline{\lambda}) = \overline{\omega}\partial\overline{\lambda} + s\partial r = -T_{non-min} \,. \tag{8.9}$$

Whenever there is a $\overline{\partial}_X$-closed operator F with positive weight h carried by the non-minimal sector, it can be written as $\overline{\partial}_X$ of itself multiplied by the zero-mode of $s\partial\overline{\lambda}$:

$$-\frac{1}{h}\overline{\partial}_X((s\partial\overline{\lambda})_0 F) = F \,. \tag{8.10}$$

The minimal (Čech) and non-minimal (Dolbeault) formulations can be related by imitating the argument that establishes the usual Čech-Dolbeault isomorphism. That is, the cohomologies of $\check{\delta}$ and $\overline{\partial}_X$ are related using the partition of unity $\{\rho_A\}$ "subordinate to" the coordinate patches $\{U_A\}$:

$$\rho_A = \frac{\overline{\lambda}_A \lambda^A}{\lambda\overline{\lambda}} \quad \to \quad \textstyle\sum_A \rho_A = 1 \text{ and } \rho_A = 0 \text{ outside } U_A \,, \tag{8.11}$$
$$\mathrm{d}\rho_\mathrm{A} = \overline{\partial}_\mathrm{X}(\rho_\mathrm{A}) = \frac{(\lambda\overline{\lambda})\mathrm{r}_\mathrm{A}\lambda^\mathrm{A} - (\lambda\mathrm{r})\overline{\lambda}_\mathrm{A}\lambda^\mathrm{A}}{(\lambda\overline{\lambda})^2} \,.$$

(Here and hereafter, Einstein summation convention does not apply for the index A; when needed, we will always write the summation over A explicitly.) A Čech n-cochain $\check{\psi} = (\psi^{A_0\cdots A_n})$ is described in the Dolbeault language by an n-form

$$\bar{\psi} = \frac{1}{(n+1)!} \sum_{A_0,\cdots,A_n} \psi^{A_0\cdots A_n} \rho_{A_0} \mathrm{d}\rho_{\mathrm{A}_1} \wedge \cdots \wedge \mathrm{d}\rho_{\mathrm{A}_\mathrm{n}} \,. \tag{8.12}$$

Since $\check{\psi}$ is holomorphic (i.e. $\overline{\partial}_X \psi^{A_0\cdots A_n} = 0$), the usual argument relating the Čech and Dolbeault cohomologies can be applied (provided one uses a good cover so that $\overline{\partial}_X$-cohomology is locally trivial).

Since the non-minimal variable r_A (or consequently $\mathrm{d}\rho_\mathrm{A}$) is a fermionic variable, it is clear that using the map (8.12) which relates the minimal (Čech) and non-minimal (Dolbeault) formulations, elements in the Čech cohomology $H^n(\check{\delta})$ will be bosonic for n even and fermionic for n odd. In particular the operator b_3 which bellows to $H^3(\breve{\delta})$ is a fermionic one. The identification of b_3 is easier in the Čech/Dolbeault cohomologies. Indeed, it can be identified as the "tail term" of the reparameterization b-ghost of the pure spinor formalism. Recall that in the non-minimal pure spinor formalism the b-ghost is written as [6, 14]

$$\begin{aligned}
b &= b_0 + b_1 + b_2 + b_3 \,, \\
b_0 &= -s^\alpha \partial \overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha G^\alpha}{(\lambda\overline{\lambda})} = -s^\alpha \partial \overline{\lambda}_\alpha - \frac{\overline{\lambda}_\alpha \{\pi^m (\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha\}}{4(\lambda\overline{\lambda})} \,, \\
b_1 &= \frac{\overline{\lambda}_\alpha r_\beta H^{[\alpha\beta]}}{(\lambda\overline{\lambda})^2} = \frac{(\overline{\lambda}\gamma^{mnp} r)\{(d\gamma_{mnp} d) - 48 N_{mn}\pi_p\}}{768(\lambda\overline{\lambda})^2} \,, \\
b_2 &= \frac{\overline{\lambda}_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\lambda\overline{\lambda})} = -\frac{(r\gamma^{mnp} r) N_{mn} (\overline{\lambda}\gamma_p d)}{64(\lambda\overline{\lambda})^3} \,, \\
b_3 &= \frac{\overline{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\lambda\overline{\lambda})^4} = \frac{(r\gamma^{mnp} r)(\overline{\lambda}\gamma_{lkp} r) N_{mn} N^{lk}}{512(\lambda\overline{\lambda})^4} \,,
\end{aligned} \tag{8.13}$$

and satisfies

$$\{Q\,, b_0\} = T\,, \quad \{\overline{\partial}_X\,, b_i\} + \{Q\,, b_{i+1}\} = 0\,,\ (i = 0,1,2)\,, \quad \{\overline{\partial}_X\,, b_3\} = 0\,. \tag{8.14}$$

Being the tail of the b-ghost, b_3 is clearly in the Dolbeault cohomology of intrinsic, or gauge invariant operators. It is independent of (x, p, θ) and carries charges $-q^2 t^{-4}$. So this is the "missing state" we were looking for.

Chapter 9

Derivation of the lightcone spectrum

Finally in this chapter, we derive the Green-Schwarz lightcone spectrum by combining the pure spinor partition function with those of the matter variables x^m and $(p_\alpha, \theta^\alpha)$. The lightcone spectrum we are to derive is the Fock space spanned by the transverse oscillators

$$\alpha^i_{-n}\,, \quad S^a_{-n}\,, \quad (i \in \mathbf{8}_v\,,\ a \in \mathbf{8}_s\,,\ n \geq 1)\,, \tag{9.1}$$

on the super-Maxwell ground states

$$|i\rangle + |\dot{a}\rangle = \mathbf{8}_v + \mathbf{8}_a\,. \tag{9.2}$$

Their partition function is simply

$$\begin{aligned} Z_{lc}(q,\vec{\sigma}) &= \mathrm{Tr}_{\mathrm{lc}}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L}_0}\mathrm{e}^{\mu\cdot\sigma} \\ &= (\mathbf{8}_v - \mathbf{8}_a)\prod_{h\geq 1}\frac{(1-q^h)^{\mathbf{8}_s}}{(1-q^h)^{\mathbf{8}_v}}\,. \end{aligned} \tag{9.3}$$

Now, since the physical BRST operator of the pure spinor formalism Q contains pieces with *non-zero* t-charge, the total partition function of the pure spinor superstring $\mathbf{Z}(q,t,\vec{\sigma})$ (which includes x and (p,θ) sectors) is not directly related to the cohomology of Q. Moreover, $\mathbf{Z}(q,t,\vec{\sigma})$ differs from the lightcone partition function. However, it will be shown in this chapter that if the t-charge is twisted appropriately using the lightcone boost charge ($t \to \tilde{t}$), $\mathbf{Z}(q,t,\vec{\sigma})$ can be related to the lightcone partition function as

$$\tilde{\mathbf{Z}}(q,\tilde{t},\vec{\sigma}) \quad \leftrightarrow \quad -\tilde{t}^2 Z_{lc}(q,\vec{\sigma}) + \tilde{t}^6 Z_{lc}(q,\vec{\sigma})\,. \tag{9.4}$$

The first term at $\tilde{t}^2$ represents the usual lightcone spectrum and the second term at $\tilde{t}^6$ represents the spectrum of the antifields. If one writes $Q = Q_0 + Q_1 + \cdots$ where Q_n carries t-charge n, it is obvious that the twisted total partition function $\tilde{\mathbf{Z}}(q,\tilde{t},\vec{\sigma})$ represents the cohomology of $\tilde{t}$-charge 0 piece Q_0 of Q. One might think that the cohomology of Q_0 has nothing to do with that of Q, but it will be shown that Q_0 and Q have the same cohomology, *except* that the on-shell condition ($L_0 = 0$) is not implied for the former.

Let us begin by first illustrating the analogous result for the bosonic string.

9.1 Lightcone spectrum of bosonic string from covariant partition function

The BRST operator of the bosonic string takes the form

$$\begin{aligned} Q &= cT_x + bc\partial c \\ &= \sum_{n\in\mathcal{Z}} c_{-n}L_n - \frac{1}{2}\sum_{m,n\in\mathcal{Z}}(m-n)c_{-m}c_{-n}b_{m+n}\,, \end{aligned} \tag{9.5}$$

where we leave the normal orderings implicit and the Virasoro operators are given by

$$\begin{aligned} L_0 &= \frac{1}{2}k^2 + \sum_{m\geq 1}\alpha^{\mu}_{-m}\alpha_{\mu,m} - 1 \\ L_n &= \frac{1}{2}\sum_{m\in\mathcal{Z}}\alpha^{\mu}_{n-m}\alpha_{\mu,m} \quad (n\neq 0)\,. \end{aligned} \tag{9.6}$$

Because of the ghost zero-mode oscillators $\{b_0,\, c_0\} = 1$, the cohomology of Q consists of two identical copies of the lightcone spectrum—those without c_0 (fields) and those with c_0 (antifields). Thus, the partition function defined by $\mathrm{Tr}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}$ vanishes identically due to field-antifield cancellation. One way to get a non-zero result is to impose an additional condition $b_0 = 0$ which drops all the antifields from the trace, but it is difficult to perform an analogous operation in the pure spinor formalism. Another way to get a non-zero result is to introduce a charge that distinguishes fields from antifields. Clearly, the ghost number (t-charge) measured by

$$J \;= -bc \quad (\rightarrow \quad t(b,c) = (-1,1)\,) \tag{9.7}$$

does the job. The (lightcone) partition function would then be

$$Z_{lc}(q,t) = \mathrm{Tr}(-1)^{\mathrm{F}}\mathrm{q}^{\mathrm{L_0}}\mathrm{t}^{\mathrm{J_0}} = -\mathrm{q}^{-1}(\mathrm{t}-\mathrm{t}^2)\prod_{\mathrm{h}\geq 1}\frac{1}{(1-\mathrm{q}^{\mathrm{h}})^{24}}\,, \tag{9.8}$$

where the prefactor represents the ground state tachyon ($c = -q^{-1}t$) and its antifield ($c\partial c = q^{-1}t^2$).

In obtaining the expression (9.8), we used the well-known fact that the physical spectrum is spanned by the transverse oscillators α^i_{-n} ($i = 1,\ldots,24$, $n > 0$). Now, let us explain how it can be obtained from the covariant partition function

$$\begin{aligned} \mathbf{Z}(q,t,\vec{\sigma}) &= Z_x(q,t,\vec{\sigma}) Z_{bc}(q,t,\vec{\sigma})\,, \\ Z_x &= \prod_{h\geq 1} \frac{1}{(1-q^h)^{\mathbf{26}}}\,, \\ Z_{bc} &= \prod_{h\geq 2} (1-q^h t^{-1})^{\mathbf{1}} \prod_{h\geq -1} (1-q^h t)^{\mathbf{1}}\,. \end{aligned} \tag{9.9}$$

If the BRST operator Q carried ghost number (t-charge) 0, the total partition function $\mathbf{Z}(q,t,\vec{\sigma})$ would represent its cohomology. But since Q carries ghost number 1, $\mathbf{Z}(q,t,\vec{\sigma})$ is not directly related to the cohomology. Nevertheless, there is a simple way to obtain the partition function of cohomology (9.8) from $\mathbf{Z}(q,t,\vec{\sigma})$. The procedure is simple and one only has to twist the t-charge by the lightcone boost charge for the non-zero modes $\alpha^{\pm}_{n'}$

$$J \to \tilde{J} = J + \frac{1}{2} N^{+-}_{x'}\,. \tag{9.10}$$

(The zero-modes $k^{\pm}$ are kept intact.) Then, the twisted $\tilde{t}$-charges read

$$\tilde{t}(k^{\pm}, \alpha^{\pm}_{n'}, \alpha^{i}_{n}, b, c) \;\; = (0, \pm 1, 0, -1, 1)\,, \tag{9.11}$$

and the twisted partition function becomes identical to (9.8) representing lightcone fields and antifields:

$$\tilde{\mathbf{Z}}(q,\tilde{t},\vec{\sigma}) = -q^{-1}(\tilde{t} - \tilde{t}^2) \prod_{h\geq 1} \frac{1}{(1-q^h)^{\mathbf{24}}}\,. \tag{9.12}$$

Of course, $\tilde{\mathbf{Z}}(q,\tilde{t},\vec{\sigma})$ represents the cohomology of the $\tilde{t}$-charge 0 piece of Q,

$$\begin{aligned} Q &= Q_0 + Q_1 + Q_2\,, \\ Q_0 &= -\frac{1}{2} k^{+} \sum_{n\neq 0} c_{-n} \alpha^{-}_{n}\,, \end{aligned} \tag{9.13}$$

and not necessarily that of Q itself. However, as is apparent from (9.13) the cohomologies of Q and Q_0 are identical, except that the on-shell conditions are not implied for the latter. (Recall that we are not imposing the $b_0 = 0$ condition.) So the twisted partition function (9.12) in fact represents the lightcone spectrum but *without* the on-shell condition. In the previous paragraph, we recovered the well-known fact that the BRST cohomology reproduces the lightcone spectrum [37].

9.2 Lightcone spectrum from pure spinor partition function

As explained earlier, physical states in the pure spinor formalism are defined as the cohomology of Q. We now wish to define an operator that is the analog of

$Q_0 = -(1/2)k^+ \sum_{n\neq 0} c_{-n}\alpha_n^-$ of the bosonic string as the twisted $\tilde{t}$-charge 0 piece of the physical BRST operator. To find the appropriate twisting of the t-charge current $J_t = \omega\lambda + p\theta$, let us study the massless vertex operators and see where the lightcone degrees of freedom reside.

9.2.1 Twisting of t-charge

The super-Poincaré covariant vertex operator for the super-Maxwell fields is given by

$$\begin{aligned} V &= \lambda^\alpha A_\alpha(x,\theta) \\ &= \lambda^\alpha \psi_\alpha(x) + (\lambda\gamma^m\theta)a_m(x) + (\lambda\gamma^m\theta)(\theta\gamma_m)_\alpha \chi^\alpha(x) + \cdots \end{aligned} \tag{9.14}$$

with $a^m(x)$ and $\chi^\alpha(x)$ the photon and photino wave functions. (The first term $\lambda^\alpha\psi_\alpha$ is pure gauge and the ellipsis involve spacetime derivatives of a^m and χ^α.) In this form, the lightcone degrees of freedom $(a^i, \chi^{\dot{a}})$ are contained in the terms at t^2 and t^3. But if the t-charges of λ and θ are twisted by the lightcone boost charge as

$$\begin{aligned} J \quad \to \qquad & \tilde{J} = \omega\lambda + p\theta + N^{+-}_{\omega\lambda} + N^{+-}_{p\theta}\,, \\ & (\tilde{t}(\gamma^+\lambda, \gamma^-\lambda, \gamma^+\theta, \gamma^-\theta) = (2,0,2,0))\,, \end{aligned} \tag{9.15}$$

both are brought to $\tilde{t}^2$

$$\tilde{t}^2: \qquad (\lambda\gamma^i\theta)a_i(x)\,, \quad (\lambda\gamma^-\theta)(\theta\gamma^+)_{\dot{a}}\chi^{\dot{a}}(x)\,. \tag{9.16}$$

Similar analysis shows that the lightcone degrees of freedom for the antiphoton and antiphotino are brought to $\tilde{t}^6$, which explains our expectation (9.4):

$$\tilde{\mathbf{Z}}(q,\tilde{t},\vec{\sigma}) = -\tilde{t}^2 Z_{lc}(q,\vec{\sigma}) + \tilde{t}^6 Z_{lc}(q,\vec{\sigma})\,. \tag{9.17}$$

The analysis here does not tell us how the t-charges of (the non-zero modes of) ∂x^m should be twisted, but it turns out that the appropriate definition of $\tilde{J}$ is

$$\tilde{J} = \omega\lambda + p\theta + K\,, \quad K = N^{+-}_{\omega\lambda} + N^{+-}_{p\theta} + 2N^{+-}_{x'}\,. \tag{9.18}$$

Note that we twisted the lightcone coordinates $\partial x^\pm$ twice as much as others, and we indicate by the prime in $N^{+-}_{x'}$ that the zero-mode of $\partial x^\pm = k^\pm$ are kept intact. In our convention, the boost charge K of the basic operators are

$$\begin{aligned} & K(\gamma^\pm\omega, \gamma^\pm\lambda) = (\pm 1, \pm 1)\,, \quad K(\gamma^\pm p, \gamma^\pm\theta) = (\pm 1, \pm 1)\,, \\ & K(k^\pm, \partial x'^\pm, \partial x^i) = (0, \pm 4, 0)\,. \end{aligned} \tag{9.19}$$

It will now be argued that the $\tilde{t}$-charge 0 piece of the physical BRST operator plays a role analogous to the $Q_0 = -(1/2)k^+\sum_{n\neq 0} c_{-n}\alpha_n^-$ of the bosonic string. As a first step, let us see how the total partition function $\mathbf{Z}(q,t,\vec{\sigma})$ is twisted at several lower mass levels.

9.2.2 Massless states

It is easy to see that the twisted partition function for the zero modes $\tilde{\mathbf{Z}}_0(\tilde{t},\vec{\sigma})$ represents the lightcone super-Maxwell ground state. The twisted partition function can be easily computed from the original spin partition function:

$$\begin{aligned} & \mathbf{Z}_0(t,\vec{\sigma}) &=& \ \mathbf{1}-\mathbf{10}t^2+\overline{\mathbf{16}}t^3-\mathbf{16}t^5+\mathbf{10}t^6-\mathbf{1}t^8\,, \\ \to\ & \tilde{\mathbf{Z}}_0(\tilde{t},\vec{\sigma}) &=& \ -(\mathbf{8}_v-\mathbf{8}_a)\tilde{t}^2+(\mathbf{8}_v-\mathbf{8}_a)\tilde{t}^6\,. \end{aligned} \tag{9.20}$$

At the level of vertex operators, this formula can be understood as follows. Covariant vertex operators for the super-Maxwell antifields (a^*,χ^*_α), for the ghost c, and for the antighost c^* are similar to that of the super-Maxwell field (9.14) but have different numbers of λ:

$$\begin{aligned} V^* &= \lambda^\alpha\lambda^\beta A_{\alpha\beta}(x,\theta) \\ &= \cdots+(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\gamma^{np}\theta)^\alpha\chi^*_\alpha(x)+(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta)a^*_m(x)+\cdots \\ U &= A(x,\theta)=\mathbf{1}c(x)+\cdots\,, \\ U^* &= \lambda^\alpha\lambda^\beta\lambda^\gamma A_{\alpha\beta\gamma}(x,\theta)=\cdots+(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)c^*(x)+\cdots\,. \end{aligned} \tag{9.21}$$

The terms of the covariant partition function $\mathbf{Z}_0$ corresponds to the (vertex operators of) component fields as

$$\mathbf{1}-\mathbf{10}t^2+\overline{\mathbf{16}}t^3-\mathbf{16}t^5+\mathbf{10}t^6-\mathbf{1}t^8 \quad\leftrightarrow\quad (c,a_m,\chi^\alpha,\chi^*_\alpha,a^*_m,c^*)\,. \tag{9.22}$$

Under the twisting (9.15) one finds that only the lightcone degrees of freedom survives as in

	$\tilde{t}^0$	$\tilde{t}^1$	$\tilde{t}^2$	$\tilde{t}^3$	$\tilde{t}^4$	$\tilde{t}^5$	$\tilde{t}^6$	$\tilde{t}^7$	$\tilde{t}^8$
$\mathbf{1}$	c								
$-\mathbf{10}t^2$	a^+		a^i		a^-				
$\mathbf{16}t^3$			$\chi^{\dot{a}}$		χ^a				
$-\mathbf{16}t^5$					χ^*_a		$\chi^*_{\dot{a}}$		
$\mathbf{10}t^6$					a^{*+}		a^{*i}		a^{*-}
$-\mathbf{1}t^8$									c^*

Spurious degrees of freedom and the ghosts ($a^\pm$, χ^a, c etc.) are brought outside $\tilde{t}^{2,6}$ and get cancelled by components with the opposite statistics.

9.2.3 First massive states

The lightcone partition function at level 1 can be derived in a similar manner. A new feature here is the appearance of non-zero modes of x^m which have to be twisted twice as much (9.18). The total partition function before the twisting is

$$\begin{aligned}
\mathbf{Z}_1(t,\vec{\sigma}) &= Z_{\omega\lambda,1}Z_{p\theta,0} + Z_{\omega\lambda,0}Z_{p\theta,1} + Z_{\omega\lambda,0}Z_{p\theta,0}Z_{x,1} \\
&= \big((\mathbf{45}+\mathbf{1}) - \mathbf{144}t + (\overline{\mathbf{126}} - \mathbf{10})t^2 + \overline{\mathbf{16}}t^3 \\
&\qquad -\mathbf{16}t^5 - (\mathbf{126}-\mathbf{10})t^6 + \mathbf{144}t^7 - (\mathbf{45}+\mathbf{1})t^8\big)_{\omega\lambda} \\
&\quad +((\mathbf{1} - \mathbf{10}t^2 + \overline{\mathbf{16}}t^3 - \mathbf{16}t^5 + \mathbf{10}t^6 - \mathbf{1}t^8)_\lambda \otimes (\mathbf{10}_x - \overline{\mathbf{16}}_p t^{-1} - \mathbf{16}_\theta t))\,,
\end{aligned}$$

and after the twisting (9.18), it becomes

$$\begin{aligned}
\tilde{\mathbf{Z}}_1(\tilde{t},\vec{\sigma}) &= (\mathbf{8}_v - \mathbf{8}_a)\tilde{t}^{-2} + (-\mathbf{56}_{va} + \mathbf{35}_{aa} + \mathbf{28} - \mathbf{8}_s + \mathbf{1})\tilde{t}^0 + (-\mathbf{56}_{vs} + \mathbf{56}_{sa})\tilde{t}^2 \\
&\quad -(-\mathbf{56}_{vs} + \mathbf{56}_{sa})\tilde{t}^6 - (-\mathbf{56}_{va} + \mathbf{35}_{aa} + \mathbf{28} - \mathbf{8}_s + \mathbf{1})\tilde{t}^9 - (\mathbf{8}_v - \mathbf{8}_a)\tilde{t}^{10} \\
&\quad +\tilde{Z}_0(\tilde{t},\vec{\sigma}) \otimes ((\mathbf{1}_x - \mathbf{8}_{p,a})\tilde{t}^{-2} + (\mathbf{8}_{p,s} + \mathbf{8}_{\theta,a})\tilde{t}^0 + (\mathbf{1}_x + \mathbf{8}_{\theta,s})\tilde{t}^2)\,. \qquad (9.23)
\end{aligned}$$

A little algebra shows that again only the terms at $\tilde{t}^{2,6}$ survives:

$$\begin{aligned}
\tilde{\mathbf{Z}}_1(\tilde{t},\vec{\sigma}) &= -\tilde{t}^2(\mathbf{35} + \mathbf{28} + \mathbf{1} + \mathbf{56}_{sa} + \mathbf{8}_v - \mathbf{56}_{va} - \mathbf{8}_s - \mathbf{56}_{vs} - \mathbf{8}_a) \qquad (9.24) \\
&\quad +\tilde{t}^6(\mathbf{35} + \mathbf{28} + \mathbf{1} + \mathbf{56}_{sa} + \mathbf{8}_v - \mathbf{56}_{va} - \mathbf{8}_s - \mathbf{56}_{vs} - \mathbf{8}_a)\,.
\end{aligned}$$

The cancellation among spurious states and ghosts occurs as indicated in figure 9.1.

9.2.4 Higher massive states

The very same twisting procedure leads to the lightcone spectrum for the higher massive states. The computations are straightforward once the spin partition functions $Z_h(t,\vec{\sigma})$ of the pure spinor are obtained. We list the latter up to level $h=5$ in appendix B.2, so the interested reader can readily check the emergence of the lightcone spectrum.

9.3 Q_0-cohomology and absence of on-shell condition

Finally, let us study the relation between the cohomologies of Q_0 and Q. It will be argued that Q_0 is an analog of $k^+\sum_{n\neq 0}\alpha_n^- c_{-n}$ of the bosonic string, and in particular that it does not imply the on-shell condition.

Under the twisted $\tilde{t}$-charge, Q splits into three pieces

$$\begin{aligned}
Q &= Q_0 + Q_2 + Q_4\,, \qquad (9.25) \\
Q_0 &= \lambda^\alpha d'_\alpha\,, \quad (d'_a = p_a + k^+\theta_a\,,\ d'_{\dot{a}} = p_{\dot{a}} + \theta_{\dot{a}}\partial x'^-)\,, \\
Q_2 &= (\lambda^{\dot{a}}\theta^{\dot{a}})k^- + (\lambda\gamma^i\theta)\partial x^i\,, \\
Q_4 &= (\lambda^a\theta^a)\partial x'^+ - \frac{1}{2}(\lambda\gamma^m\theta)(\theta\gamma_m\partial\theta)\,,
\end{aligned}$$

	$\tilde{t}^{-4}$	$\tilde{t}^{-2}$	$\tilde{t}^{0}$	$\tilde{t}^{2}$	$\tilde{t}^{4}$	$\tilde{t}^{6}$	$\tilde{t}^{8}$	$\tilde{t}^{10}$	$\tilde{t}^{12}$
$-\overline{\mathbf{16}}t^{-1}$		$-\mathbf{8}_a$	$-\mathbf{8}_s$						
$\mathbf{45}t^0$		$\mathbf{8}_v$	$\mathbf{28}+\mathbf{1}$	$\mathbf{8}_v$					
$\mathbf{10}_x t^0$	$\mathbf{1}$		$\mathbf{8}_v$		$\mathbf{1}$				
$\mathbf{1}t^0$			$\mathbf{1}$						
$-\mathbf{120}t^2$			$-\mathbf{28}$	$\underline{-\mathbf{56}_{sa}-\mathbf{8}_v}$	$-\mathbf{28}$				
$-\mathbf{100}_x t^2$	$-\mathbf{1}$	$-\mathbf{8}_v$	$-\mathbf{1}$	$\underline{-\mathbf{35}}$	$-\mathbf{1}$	$-\mathbf{8}_v$	$-\mathbf{1}$		
			$-\mathbf{8}_v$	$\underline{-\mathbf{28}}$	$-\mathbf{8}_v$				
				$\underline{-\mathbf{1}}$					
$-2\cdot\mathbf{10}t^2$			$-2\cdot\mathbf{1}$	$-2\cdot\mathbf{8}_v$	$-2\cdot\mathbf{1}$				
$\mathbf{144}t^3$			$\mathbf{8}_s$	$\underline{\mathbf{56}_{vs}+\mathbf{8}_a}$	$\mathbf{56}_{va}+\mathbf{8}_s$	$\mathbf{8}_a$			
$\overline{\mathbf{144}}_x t^3$		$\mathbf{8}_a$		$\underline{\mathbf{56}_{va}+\mathbf{8}_s}$	$\mathbf{56}_{vs}+\mathbf{8}_a$		$\mathbf{8}_s$		
$\mathbf{16}_x t^3$			$\mathbf{8}_s$			$\mathbf{8}_a$			
$2\cdot\overline{\mathbf{16}}t^3$				$2\cdot\mathbf{8}_a$	$2\cdot\mathbf{8}_s$				
					$\star$				
$-2\cdot\mathbf{16}t^5$					$-2\cdot\mathbf{8}_s$	$-2\cdot\mathbf{8}_a$			
$-\overline{\mathbf{16}}_x t^5$				$-\mathbf{8}_a$			$-\mathbf{8}_s$		
$-\mathbf{144}_x t^5$			$-\mathbf{8}_s$		$-\mathbf{56}_{vs}-\mathbf{8}_a$	$\underline{-\mathbf{56}_{va}-\mathbf{8}_s}$		$-\mathbf{8}_a$	
$-\overline{\mathbf{144}}t^5$				$-\mathbf{8}_a$	$-\mathbf{56}_{va}-\mathbf{8}_s$	$\underline{-\mathbf{56}_{vs}-\mathbf{8}_a}$	$-\mathbf{8}_s$		
$2\cdot\mathbf{10}t^6$					$2\cdot\mathbf{1}$	$2\cdot\mathbf{8}_v$	$2\cdot\mathbf{1}$		
$\mathbf{100}_x t^6$						$\underline{\mathbf{1}}$			
					$\mathbf{8}_v$	$\underline{\mathbf{28}}$	$\mathbf{8}_v$		
			$\mathbf{1}$	$\mathbf{8}_v$	$\mathbf{1}$	$\underline{\mathbf{35}}$	$\mathbf{1}$	$\mathbf{8}_v$	$\mathbf{1}$
$\mathbf{120}t^6$					$\mathbf{28}$	$\underline{\mathbf{56}_{sa}+\mathbf{8}_v}$	$\mathbf{28}$		
$-\mathbf{1}t^8$							$-\mathbf{1}$		
$-\mathbf{10}_x t^8$					$-\mathbf{1}$		$-\mathbf{8}_v$		$-\mathbf{1}$
$-\mathbf{45}t^8$						$-\mathbf{8}_v$	$-\mathbf{28}-\mathbf{1}$	$-\mathbf{8}_v$	
$\mathbf{16}t^9$							$\mathbf{8}_s$	$\mathbf{8}_a$	
	0	0	0	$-(\mathbf{35},\mathbf{28},\mathbf{1})$	0	$(\mathbf{35},\mathbf{28},\mathbf{1})$	0	0	0
				$-(\mathbf{56}_{sa},\mathbf{8}_v)$		$(\mathbf{56}_{sa},\mathbf{8}_v)$			
				$(\mathbf{56}_{va},\mathbf{8}_s)$		$-(\mathbf{56}_{va},\mathbf{8}_s)$			
				$(\mathbf{56}_{vs},\mathbf{8}_a)$		$-(\mathbf{56}_{vs},\mathbf{8}_a)$			

Figura 9.1: Lightcone first massive states from level 1 twisted character

where the notation $\partial x'^{\pm}$ signifies the omission of zero-modes $k^{\pm}$. Q_0 is certainly nilpotent, but since it only contains the k^+ component of the momentum, setting $Q_0 = 0$ cannot imply the on-shell condition.

In order to see that Q_0 indeed works as $k^+ \sum_{n\neq 0} \alpha_n^- c_{-n}$ of the bosonic string, we study its cohomology directly, by employing the method utilized in [38] to derive the (on-shell) lightcone spectrum from Q.

9.3.1 Ghost-for-ghost method with an $SO(8)$ parameterization of pure spinor

In section 6.1, we analyzed the reducibility conditions of the pure spinor constraint in an $SO(10)$ covariant manner. As was noted in [38], there is a simpler version of this analysis if one breaks the covariance down to $SO(8)$.

First, parameterize $SO(8)$ antichiral and chiral components of λ^α as

$$\lambda^{\dot{a}} = s^{\dot{a}}\,, \quad \lambda^a = v^i \gamma_i^{a\dot{b}} s^{\dot{b}}\,. \tag{9.26}$$

λ^α satisfies the pure spinor condition provided $s^{\dot{a}}$ is constrained to be *null*, $s^{\dot{a}}s^{\dot{a}} = 0$. However, half of v^i is spurious because of the gauge invariance

$$\delta_\Lambda v^i \;= \Lambda^a(\gamma^i s)^a \quad \to \quad \delta_\Lambda \lambda^a = \Lambda^a(ss) = 0\,. \tag{9.27}$$

Repeating the BRST construction (section 6.1) in an $SO(8)$ covariant manner, one obtains a chain of free-field ghosts-for-ghosts*

$$(B_n, C_n) \;:\; (b_1^a, c_1^a)\,,\;(\rho_2^i, \sigma_2^i)\,,\;(b_3^a, c_3^a)\,,\;(\rho_4^i, \sigma_4^i)\,,\;\cdots\,, \tag{9.28}$$

where as before (b_{2n-1}^a, c_{2n-1}^a) are fermionic and $(\rho_{2n}^i, \sigma_{2n}^i)$ are bosonic. Introducing a fermionic ghost pair (b, c) for the remaining constraint $s^{\dot{a}}s^{\dot{a}} = 0$ (and denoting the conjugate to $s^{\dot{a}}$ and v^i by $t^{\dot{a}}$ and w^i), the mini-BRST operator reads [38]

$$D = \int (b s^{\dot{a}} s^{\dot{a}} + s^{\dot{a}} \mathcal{G}_{gh}^{\dot{a}} + c\mathcal{T}_{gh})\,, \tag{9.29}$$

where

$$\begin{aligned} \mathcal{G}_{gh}^{\dot{a}} &= -w^i \gamma_i^{\dot{a}b} c_1^b + \sigma_2^i \gamma_i^{\dot{a}b} b_1^b - \rho_2^i \gamma_i^{\dot{a}b} c_3^b + \cdots\,, \\ \mathcal{T}_{gh} &= (w^i \sigma_2^i) + (b_1 c_3) + (\rho_2^i \sigma_4^i) + \cdots\,. \end{aligned} \tag{9.30}$$

Using a regularization $1 - 1 + 1 - \cdots = \lim_{x\to 1}(1+x)^{-1} = 1/2$ familiar in covariant treatments of the κ-symmetry, it is straightforward to check that the combined

*We departed from [38] in notation to match the notation of the present thesis. In [38], the initial parameterization was chosen oppositely (i.e. $\lambda^a = s^a$) and the ghosts $(b_{2n-1}, c_{2n-1}, \rho_{2n}, \sigma_{2n})_{n\geq 1}$ were denoted by (u_n, t_n, w_n, v_n).

system of $(t^{\dot{a}}, s^{\dot{a}}; w^i, v^i; B_n, C_n, b, c)$ has the desired central charge 22. Moreover, one can construct a set of generators for the full $SO(10)$ Lorentz current algebra (with appropriate level -3), under which D and the physical BRST operator Q' (to be defined shortly) are invariant [38, 39].

In [38], the $SO(8)$ mini-BRST operator D was used to construct the ghost extended physical BRST operator $Q' = D + \int \lambda^\alpha d_\alpha + \cdots$, whose cohomology is equivalent to that of $Q = \int \lambda^\alpha d_\alpha$. The operator can be written in the same form as D,

$$Q' = \int (b s^{\dot{a}} s^{\dot{a}} + s^{\dot{a}} \mathcal{G}^{\dot{a}} + 2c\mathcal{T}) , \tag{9.31}$$

provided one defines

$$\begin{aligned} \mathcal{G}^{\dot{a}} &= d^{\dot{a}} + v^i \gamma_i^{\dot{a}b} d^b + \mathcal{G}^{\dot{a}}_{gh} , \\ \mathcal{T} &= -(\pi^- + 2v^i\pi^i + v^2\pi^+) + 2c_1^a d^a + \mathcal{T}_{gh} , \end{aligned} \tag{9.32}$$

with π^m being the superinvariant momentum $\partial x^m - \theta\gamma^m\partial\theta$. The combinations $(\mathcal{G}^{\dot{a}}, \mathcal{T})$ and $(\mathcal{G}^{\dot{a}}_{gh}, \mathcal{T}_{gh})$ satisfy the same algebra

$$\mathcal{G}^{\dot{a}}(z)\mathcal{G}^{\dot{b}}(w) = \frac{-2\delta^{\dot{a}\dot{b}}\mathcal{T}}{z-w} , \quad \mathcal{G}^{\dot{a}}(z)\mathcal{T}(w) = \mathcal{T}(z)\mathcal{T}(w) = \text{regular} . \tag{9.33}$$

This algebra appears repeatedly in the pure spinor formalism, and is related to the algebra generated by the first-class part of the Green-Schwarz-Siegel constraint d_α [20].

Now that the ghost extended physical BRST operator (9.31) is written entirely in terms of free fields, the analysis of its cohomology is straightforward as explained in [38]. Let us apply the argument to the case at hand, where the full operator Q is replaced by its $\tilde{t}$-charge 0 piece Q_0.

9.3.2 Lightcone "off-shell" spectrum from Q_0-cohomology

By coupling the $SO(8)$ mini-BRST operator D to Q_0, one concludes that the cohomology of Q_0 is equivalent to that of the $\tilde{t}$-charge 0 contribution to Q' which is

$$Q_0' = \int (b s^{\dot{a}} s^{\dot{a}} + s^{\dot{a}} \mathcal{G}_0^{\dot{a}} + 2c\mathcal{T}_0) , \tag{9.34}$$

where

$$\begin{aligned} \mathcal{G}_0^{\dot{a}} &= d'^{\dot{a}} + v^i \gamma_i^{\dot{a}b} d'^b + \mathcal{G}^{\dot{a}}_{gh} , \\ \mathcal{T}_0 &= -\frac{1}{2}\partial x'^- + v^2 k^+ + 2c_1^a d'^a + \mathcal{T}_{gh} . \end{aligned} \tag{9.35}$$

To study the cohomology of Q_0', it is convenient to introduce the grading defined by

$$l(p_{\dot a}, \theta_{\dot a}, \partial x'^{\pm}) = (1, -1, \pm 1)\,. \tag{9.36}$$

Under the l-grading, Q_0' splits to

$$\begin{aligned} Q_0' &= Q_{0,1}' + Q_{0,0}'\,, \\ Q_{0,1}' &= \int (s^{\dot a} p_{\dot a} - c'\partial x'^{-})\,, \\ Q_{0,0}' &= \text{(rest)} \\ &= \int (b s^{\dot a} s^{\dot a} + s^{\dot a}\theta_{\dot a}\partial x'^{-} + s^{\dot a} v^i \gamma_i^{\dot a b} d'^b + s^{\dot a}\mathcal{G}_{gh}^{\dot a} + 2cv^2 k^+ + 4c(c_1^a d'^a) + 2c\mathcal{T}_{gh})\,. \end{aligned} \tag{9.37}$$

It immediately follows that two quartets

$$(p_{\dot a}, \theta_{\dot a}, t^{\dot a}, s^{\dot a})\,, \quad (\partial x'^{\pm}, b', c')\,, \tag{9.38}$$

decouple from the cohomology. Furthermore, the conditions implied by Q_0' on the remaining fields

$$(k^{\pm}, \partial x^i)\,, \quad (p_a, \theta^a)\,, \quad (b_0, c_0)\,, \quad (w^i, v^i)\,, \quad (B_n, C_n)\,, \tag{9.39}$$

are the cohomology condition of

$$\begin{aligned} Q'' &= c_0 \int (k^+ v^2 + 2(c_1 d') + \mathcal{T}_{gh}) \\ &= c_0 (k^+ v^2 + 2(d' c_1) + w^i \sigma_2^i + b_1^a c_3^a + \cdots)_0\,. \end{aligned} \tag{9.40}$$

Remembering $d_a' = p_a + k^+\theta_a$, it then follows that the cohomology of Q' (and hence that of Q_0) is spanned by

$$\partial x^i\,, \quad (p_a - k^+\theta_a)\,, \tag{9.41}$$

on the super-Maxwell ground states $(1, \lambda\gamma^m\theta, (\lambda\gamma^m\theta)(\gamma_m\theta)_\alpha, \cdots, \lambda^{(3)}\theta^{(5)})$ (with appropriate BRST ghost extensions).

If the full operator Q was used in place of Q_0 one would find $c_0 k^-$ (among other terms) in the final form of Q'', and this leads to the on-shell condition [38]. Summing up, we have learned that the physical BRST operator Q of the pure spinor formalism contains a piece Q_0 which plays an analogous role as $k^+ \sum_{n\neq 0} \alpha_n^- c_{-n}$ of the bosonic string, and the role of the rest of Q is to impose the on-shell condition on the "off-shell" lightcone spectrum. This was what we wanted to explain.

Chapter 10

Summary and future applications

Taking aim at clarifying the Hilbert space for the pure spinors, we have started by given a brief review of the pure spinor formalism for the superstring and then we have analyzed the Hilbert space of a simple model defined by a quadratic constraint. We have used the partition function as a guide to study the structure of the Hilbert space.

The Hilbert space of this simple model was studied using two different approach, namely the intrinsic curved $\beta\gamma$ description and the BRST description. Although there are slight mismatches between the two descriptions due to the quantum ordering problem, we found that their partition functions agree. Since the partition functions in both descriptions are insensitive to quantum corrections, the agreement of the partition functions can be explained by classically relating the elements of the cohomologies of the two formalisms [10].

In the BRST description of the model, the full partition function of the BRST cohomology can be easily computed and it manifestly possesses two important symmetries that we have called "field-antifield" and "$*$-conjugation" symmetries. The field-antifield symmetry implies that, after coupling to matter variables, the cohomology of the physical BRST operator Q comes in field-antifield pairs.

There, however, are several points in the present model that require further clarifications. One of them is to understand the discrepancy between the extrinsic (BRST) and intrinsic (curved $\beta\gamma$) descriptions more precisely. At the quantum level, a source for the discrepancy between the BRST and curved $\beta\gamma$ descriptions arises from the different normal ordering prescriptions used in the two. A pair of the elements of the classical BRST cohomology can drop out from the quantum cohomology by forming a BRST doublet. In the curved $\beta\gamma$ framework, similar phenomenon occurs when the quantum effect spoils the gluing property of a classical cohomology. In this case, the failure of gluing is represented by a higher cochain which is also in the classical cohomology. Since the two frameworks use different normal ordering prescriptions, there are discrepancies between the two phenomena. It would be useful to study if this type of discrepancy can be remedied, for example, by appropriately bosonizing the BRST ghosts.

Another clarification that should be attempted is to explore the one-loop path integral expression for our partition function. When properly understood, it should be useful for unconvering the origin of the field-antifield and $*$-conjugation symmetries.

After we have introduced the study of the simple model, we extend the result to the more interesting case of pure spinors. Despite the fact that the pure spinor constraint is infinitely reducible, it will be argued that the structures above carry over almost literally. For the case of pure spinors its BRST description is more complicated than the simple model, because the pure spinors constraint turn out to be infinitely reducible and so an infinite chain of ghosts are required. Nevertheless its partition function can be written at least formally and it posses the two important "field-antifield" and "$*$-conjugation" symmetries.

For computing the character of pure spinors, another method, called as fixed-point technique, has been developed [9, 11]. In this thesis, we have computed the partition function of pure spinors up to the fifth mass level using the fixed-point method and up to the twelfth mass level using the ghost-for-ghost method. After including the partition function of the matter variables, we showed agreement with the light-cone open superstring spectrum.

In the ghost-for-ghost scheme, the method for computing higher level pure spinor character formulas is based on the formal expression for the full partition function of pure spinors [9]

$$Z(q,t) = \prod_{k=1}^{\infty} \left[(1-t^k)^{-N_k} \prod_{h=1}^{\infty} (1-q^h t^k)^{-N_k} (1-q^h t^{-k})^{-N_k} \right], \tag{10.1}$$

where N_k are the multiplicities of the ghost fields. The use of these ghosts comes from the resolution of the pure spinor constraint, and the necessity of infinite many of them is because the pure spinor constraint is infinitely reducible [9, 40]. Although it may seem difficult to extract useful information from this formal expression. In fact, on the contrary, once the moments of N_k's are known, the two important symmetries of the partition function, the field-antifield symmetry and the $*$-conjugation symmetry, have been easily deduced from (10.1).

By expanding (10.1) in powers of q, we obtain character formulas $Z_h(t)$ at each Virasoro level

$$Z(q,t) = \sum_{h=0}^{\infty} Z_h(t) q^h \,. \tag{10.2}$$

To derive explicit expressions for these character formulas, we have required to appeal some regularization procedure in order to guarantee the convergence of the infinite product over k. We have shown that the method based on Padé approximants does this job of regularization and in fact we have obtained character formulas up to level $h = 12$ [13]. The method was mainly based on the knowledge of the zero mode part $Z_0(t)$ of the partition function. From this level zero character formula, we have extracted the ghosts multiplicities N_k, and using the moments $(\sum_k k^{s+1} N_k)$ of those multiplicities, by employing Padé approximants, we were able to compute higher level character formulas $Z_h(t)$ of pure spinors. We have found that our results are in agreement with the results found in [9] (up to the fifth mass-level $h = 5$) where the fixed point technique was used. The good advantage of this method based on Padé approximants is that it preserves level by level the two important symmetries of the partition function.

Methods based on Padé approximants have many applications in the issue of tachyon condensation in string field theory [28, 29, 30], as it is a good tool for summing numerically divergent series.

We hope that computations beyond the level $h = 12$ will help us to guess the explicit form of the complete pure spinor partition function. Using Padé approximants, we can implement a computer code in order to obtain higher level character formulas and the time consuming by the computer is much less compared with computations involving the fixed point method. In general for $SO(2d)$ pure spinors the number of fixed points is 2^{d-1} and the complexity of summing over these fixed points (as it was also noted in [11]) grows exponentially with $N = 2^{d-1}$. On the other hand the computations shown by means of Padé approximants are less complicated, so this technique can be used as an alternative (to the fixed point technique) easier (computationally) way to get the character formulas for higher-dimensional pure spinors.

So far, using the ghost-for-ghost technique, we have computed the partition function without the spin dependence on the states. Spin dependence is crucial if we want to prove that the full partition function (including the contribution of the worldsheet matter sector) correctly reproduces the light cone superstring spectrum [9]. Therefore, it would be interesting to know the character formula with the spin dependence in the ghosts-for-ghosts scheme. We leave this issue as a future work.

The main surprise we have found in the computation of the pure spinor partition function is the appearance of fermionic states starting at the second mass level. These fermionic states all correspond to three-forms on the pure spinor space, and are related to a term in the b ghost in the pure spinor formalism.

There are several possible applications of these results for amplitude computations and for superstring field theory. Using the RNS formalism, scattering amplitudes can be computed either using conformal field theory techniques or using the operator method. Although conformal field theory techniques are more convenient for multiloop amplitudes, the operator method is convenient for one-loop computations where one expresses the amplitude as a trace over states in the Hilbert space.

In this thesis, the pure spinor partition function was only computed up to some mass level, but it might be possible to extend our results and construct an explicit formula for the complete pure spinor partition function. One could then use the operator method in the pure spinor formalism, which might simplify the computations of one-loop amplitudes.

Knowing that there can be no Čech cohomologies with degree greater than 3 is nice for the pure spinor multiloop amplitudes, because it implies that one need not worry about the poles coming from the fusion of many reparameterization b-ghosts. The troublesome poles are necessarily in Čech cohomologies with degree greater than 3 and, modulo the subtleties coming from the divergences at the boundary of moduli spaces, they can be ignored without requiring the regularization introduced in [7]. It would be interesting to work out how it is actually realized, and the present work might be useful to clarify some aspects of this issue.

Another topic that can be addressed is the one we have mentioned at the end of section 4.3, namely the covariant computation of the second mass-level state contained in the cohomology of the BRST operator Q. This problem should be and interesting issue since, as we have seen, the b ghost (which appears at the second mass-level) has a non-trivial dependence on the non-minimal variables. In the Dolbeault like description of the pure spinor formalism, it would be nice to see how the level two massive vertex operator depends on the b ghost.

A fourth possible application of these results is for superstring field theory. In [14], a cubic open superstring field theory action was constructed using the pure spinor formalism. However, the correct definition of the Hilbert space was unclear because of the possibility of states diverging when $(\lambda\bar{\lambda}) \to 0$. Using the results of this thesis, one now knows that the Hilbert space must at least allow states which diverge as $(\lambda\bar{\lambda})^{-3}$ in order to reproduce the correct massive spectrum [9]. But it is an open question if one can consistently define a multiplication rule for string fields in such a manner that states diverging like $(\lambda\bar{\lambda})^{-11}$ are never produced [7, 14].

Appendix A

$SO(10)$ conventions and formulas

A.1 Dynkin labels

As is well known, all the irreducible representations of $SO(10)$ can be labeled by five integers called Dynkin labels. Those are nothing but the highest weights of the representations in an appropriate basis. In our convention,

$$\begin{aligned}
&\text{vector}: \quad (10000) = \mathbf{10}\,, \\
&\text{2-form}: \quad (01000) = \mathbf{45}\,, \\
&\text{3-form}: \quad (00100) = \mathbf{120}\,, \\
&\text{antichiral spinor}: \quad (00010) = \overline{\mathbf{16}}\,, \\
&\text{chiral spinor}: \quad (00001) = \mathbf{16}\,.
\end{aligned} \tag{A.1}$$

When computing the partition functions, it is sometimes more convenient to introduce an orthogonal basis for the Cartan subalgebra, e_a $(a = 1, \dots, 5)$ such that the fundamental roots are

$$e_1 - e_2, \quad e_2 - e_3, \quad e_3 - e_4, \quad e_4 \pm e_5\,. \tag{A.2}$$

We then denote the character of e_a by e^{σ_a} where σ_a is a formal variable for book-keeping. Also the weight vectors in this basis are denoted by square bracket:

$$\mu \quad = \textstyle\sum_a \mu_a e_a \quad \leftrightarrow \quad [\mu_1\mu_2\mu_3\mu_4\mu_5] \quad \leftrightarrow \quad \mathrm{e}^{\mu\cdot\sigma}\,. \tag{A.3}$$

The components μ_a's take values in half integers and are related to the (integer valued) Dynkin labels $(a_1a_2a_3a_4a_5)$ by

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1/2 & 1/2 \\ 0 & 1 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}\,. \tag{A.4}$$

We refer to this basis as the "five sign basis" because the weights and characters of chiral spinors are expressed as

$$\mu = \frac{1}{2}[\pm 1, \pm 1, \pm 1, \pm 1, \pm 1] \quad \leftrightarrow \quad \mathrm{e}^{\frac{1}{2}(\pm\sigma_1 \pm\sigma_2 \pm\sigma_3 \pm\sigma_4 \pm\sigma_5)}, \tag{A.5}$$

with even number of minus signs.

A.2 Some dimension formulas

Dimensions of the $SO(10)$ irreducible representations are given by

$$\begin{aligned}
&\dim(a\,b\,c\,d\,e) \\
&= \frac{1}{2^4 \cdot 3^4 \cdot 4^3 \cdot 5^2 \cdot 6 \cdot 7} \Big\{ (a+1)(b+1)(c+1)(d+1)(e+1) \\
&\quad (a+b+2)(b+c+2)(c+d+2)(c+e+2) \\
&\quad (a+b+c+3)(b+c+d+3)(b+c+e+3)(c+d+e+3) \\
&\quad (a+b+c+d+4)(a+b+c+e+4)(b+c+d+e+4)(b+2c+d+e+5) \\
&\quad (a+b+c+d+e+5)(a+b+2c+d+e+6)(a+2b+2c+d+e+7) \Big\}.
\end{aligned} \tag{A.6}$$

Of special interest are the '(chiral) pure spinor representations' $(0000n)$, which have the following dimensions

$$\dim(0000n) = \frac{(n+7)(n+6)(n+5)^2(n+4)^2(n+3)^2(n+2)(n+1)}{7 \cdot 6 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2}. \tag{A.7}$$

Appendix B

Table of partition functions

B.1 Partition functions without spin: number of states

List of coefficients $N_{m,n}$ present in the expansion $Z(q,t) = \sum_{m\geq 0}\sum_n N_{m,n} q^m t^n$ of the pure spinors partition function. We include the usual gauge invariant states ($N_{m,n} > 0$) as well as the extra states ($N_{m,n} < 0$).

n	$N_{0,n}$	$N_{1,n}$	$N_{2,n}$	$N_{3,n}$	$N_{4,n}$	$N_{5,n}$	$\cdots$
-8	0	0	0	0	0	0	
-7	0	0	0	0	0	-672	
-6	0	0	0	0	-126	-4068	
-5	0	0	0	-16	-592	-11408	
-4	0	0	-1	-46	-1073	-16974	
-3	0	0	0	-16	-592	-11408	
-2	0	0	0	0	0	0	
-1	0	0	16	592	11408	152736	
0	1	46	1073	16974	205373	2031130	
1	16	592	11408	153408	1617344	14228752	
2	126	4068	70522	868012	8479364	69771888	
3	672	19824	320304	3716208	34489920	271222800	
4	2772	76824	1180602	13125484	1173525227	892615196	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

(B.1)

B.2 Spin partition functions

For convenience, we here list the partition functions with spin dependence up to fifth Virasoro levels. Partition functions at each level are of the form

$$Z_h(t,\vec{\sigma}) = \frac{P_h(t,\vec{\sigma})}{(1-t)^S} \quad \text{where} \quad (1-t)^S \equiv \prod_{\mu\in S}(1-te^{\mu\cdot\sigma}), \quad \mathrm{S} = (00001) = \mathbf{16} \tag{B.2}$$

and $P_h(t,\vec{\sigma})$ is a polynomial of t with coefficients taking values in the representations of $SO(10)$. For brevity, we only write the numerator $P_h(t)$. Again, formulas include the extra states.

Level 0**:**

$$\begin{aligned}P_0(t,\vec{\sigma}) \quad &= (00000)_1-(10000)_{10}t^2+(00010)_{16}t^3\\ &-(00001)_{16}t^5+(10000)_{10}t^6-(00000)_1t^8\end{aligned} \tag{B.3}$$

Level 1**:**

$$\begin{aligned}P_1(t,\vec{\sigma}) \quad &= ((01000)_{45}+(00000)_1)-(10010)_{144}t^1+((00020)_{126}-(10000)_{10})t^2\\ &+(00010)_{16}t^3-(00001)_{16}t^5-((00002)_{126}-(10000)_{10})t^6\\ &+(10001)_{144}t^7-((01000)_{45}+(00000)_1)t^8\end{aligned} \tag{B.4}$$

Level 2**:**

$$\begin{aligned}P_2(t,\vec{\sigma}) = &-(00000)_1t^{-4}+(00001)_{16}t^{-3}-(00100)_{120}t^{-2}\\ &+(+(01010)_{560}+(00010)_{16})t^{-1}\\ &+(-(10020)_{1050}+(01000)_{45}+2(00000)_1)t^0+(+(00030)_{672}-(10010)_{144})t^1\\ &+(-(11000)_{320}+(00020)_{126}-2(10000)_{10})t^2+(+(01010)_{560}+2(00010)_{16})t^3\\ &+(-(01001)_{560}-2(00001)_{16})t^5\\ &+(+(11000)_{320}-(00002)_{126}+2(10000)_{10})t^6\\ &+(-(00003)_{672}+(10001)_{144})t^7+(+(10002)_{1050}-(01000)_{45}-2(00000)_1)t^8\\ &+(-(01001)_{560}-(00001)_{16})t^9\\ &+(00100)_{120}t^{10}-(00010)_{16}t^{11}+(00000)_1t^{12}\end{aligned} \tag{B.5}$$

Level 3**:**

$$\begin{aligned}P_3(t,\vec{\sigma}) = &-(00010)_{16}t^{-5}+(00011)_{210}t^{-4}-(00110)_{1200}t^{-3}+(01020)_{3696}t^{-2}\\ &+(-(10030)_{5280}+(01010)_{560}+2(00010)_{16})t^{-1}\\ &+(+(00040)_{2772}-(10020)_{1050}+(02000)_{770}+3(01000)_{45}+3(00000)_1)t^0\\ &+(-(11010)_{3696}+(00030)_{672}-3(10010)_{144})t^1\\ &+(+(01020)_{3696}-2(11000)_{320}+3(00020)_{126}-3(10000)_{10})t^2\\ &+(+2(01010)_{560}+3(00010)_{16})t^3\\ &+(-2(01001)_{560}-3(00001)_{16})t^5\end{aligned}$$

$$+(-(01002)_{3696}+2(11000)_{320}-3(00002)_{126}+3(10000)_{10})t^{6}$$
$$+(+(11001)_{3696}-(00003)_{672}+3(10001)_{144})t^{7}$$
$$+(-(00004)_{2772}+(10002)_{1050}-(02000)_{770}-3(01000)_{45}-3(00000)_{1})t^{8}$$
$$+(+(10003)_{5280}-(01001)_{560}-2(00001)_{16})t^{9}$$
$$-(01002)_{3696}t^{10}+(00101)_{1200}t^{11}-(00011)_{210}t^{12}+(00001)_{16}t^{13} \qquad \text{(B.6)}$$

Level 4**:**

$$P_4(t,\vec{\sigma})=-(00020)_{126}t^{-6}+(+(00021)_{1440}-(00010)_{16})t^{-5}$$
$$+(-(00120)_{6930}+(00011)_{210}-(00000)_{1})t^{-4}$$
$$+(+(01030)_{17280}-(00110)_{1200}+(00001)_{16})t^{-3}$$
$$+(-(10040)_{20790}+(01020)_{3696}+(00020)_{126}-(00100)_{120})t^{-2}$$
$$+(+(00050)_{9504}-(10030)_{5280}+(02010)_{8064}$$
$$+(10001)_{144}+3(01010)_{560}+4(00010)_{16})t^{-1}$$
$$+(-(11020)_{23040}+(00040)_{2772}-3(10020)_{1050}$$
$$+2(02000)_{770}+5(01000)_{45}+6(00000)_{1})t^{0}$$
$$+(+(01030)_{17280}-(20001)_{720}-2(11010)_{3696}+3(00030)_{672}-5(10010)_{144})t^{1}$$
$$+(-(12000)_{4410}+2(01020)_{3696}-4(11000)_{320}+5(00020)_{126}-6(10000)_{10})t^{2}$$
$$+(+(02010)_{8064}+(00021)_{1440}+4(01010)_{560}+6(00010)_{16})t^{3}$$
$$+(-(02001)_{8064}-(00012)_{1440}-4(01001)_{560}-6(00001)_{16})t^{5}$$
$$+(+(12000)_{4410}-2(01002)_{3696}+4(11000)_{320}-5(00002)_{126}+6(10000)_{10})t^{6}$$
$$+(-(01003)_{17280}+(20010)_{720}+2(11001)_{3696}-3(00003)_{672}+5(10001)_{144})t^{7}$$
$$+(+(11002)_{23040}-(00004)_{2772}+3(10002)_{1050}$$
$$-2(02000)_{770}-5(01000)_{45}-6(00000)_{1})t^{8}$$
$$+(-(00005)_{9504}+(10003)_{5280}-(02001)_{8064}$$
$$-(10010)_{144}-3(01001)_{560}-4(00001)_{16})t^{9}$$
$$+(+(10004)_{20790}-(01002)_{3696}-(00002)_{126}+(00100)_{120})t^{10}$$
$$+(-(01003)_{17280}+(00101)_{1200}-(00010)_{16})t^{11}$$
$$+(+(00102)_{6930}-(00011)_{210}+(00000)_{1})t^{12}$$
$$+(-(00012)_{1440}+(00001)_{16})t^{13}+(00002)_{126}t^{14} \qquad \text{(B.7)}$$

Level 5**:**

$$P_5(t,\vec{\sigma})=-(00030)_{672}t^{-7}+(+(00031)_{6930}-(00020)_{126}-(00100)_{120})t^{-6}$$

$$
\begin{aligned}
&+(-(00130)_{29568}+(00021)_{1440}+(00101)_{1200}-2(00010)_{16})t^{-5}\\
&+(+(01040)_{64350}-(00120)_{6930}-(00200)_{4125}+2(00011)_{210}-(00000)_{1})t^{-4}\\
&+(-(10050)_{68640}+(01030)_{17280}-2(00110)_{1200}+(00001)_{16})t^{-3}\\
&+(+(00060)_{28314}-(10040)_{20790}+(02020)_{46800}+3(01020)_{3696}+2(00020)_{126})t^{-2}\\
&+(-(11030)_{102960}+(00050)_{9504}-3(10030)_{5280}+(11001)_{3696}\\
&\quad+2(02010)_{8064}+2(10001)_{144}+6(01010)_{560}+8(00010)_{16})t^{-1}\\
&+(+(01040)_{64350}-(20011)_{8085}-2(11020)_{23040}+3(00040)_{2772}\\
&\quad-5(10020)_{1050}+(03000)_{7644}+4(02000)_{770}+10(01000)_{45}+9(00000)_{1})t^{0}\\
&+(-(12010)_{43680}+2(01030)_{17280}-2(20001)_{720}-5(11010)_{3696}\\
&\quad+5(00030)_{672}-10(10010)_{144})t^{1}\\
&+(+(02020)_{46800}+(00031)_{6930}-(20100)_{4312}-2(12000)_{4410}\\
&\quad+5(01020)_{3696}-8(11000)_{320}+10(00020)_{126}-9(10000)_{10})t^{2}\\
&+(+(10110)_{8800}+2(02010)_{8064}+2(00021)_{1440}+8(01010)_{560}+9(00010)_{16})t^{3}\\
&+(-(10101)_{8800}-2(02001)_{8064}-2(00012)_{1440}-8(01001)_{560}-9(00001)_{16})t^{5}\\
&+(-(02002)_{46800}-(00013)_{6930}+(20100)_{4312}+2(12000)_{4410}\\
&\quad-5(01002)_{3696}+8(11000)_{320}-10(00002)_{126}+9(10000)_{10})t^{6}\\
&+(+(12001)_{43680}-2(01003)_{17280}+2(20010)_{720}+5(11001)_{3696}\\
&\quad-5(00003)_{672}+10(10001)_{144})t^{7}\\
&+(-(01004)_{64350}+(20011)_{8085}+2(11002)_{23040}-3(00004)_{2772}\\
&\quad+5(10002)_{1050}-(03000)_{7644}-4(02000)_{770}-10(01000)_{45}-9(00000)_{1})t^{8}\\
&+(+(11003)_{102960}-(00005)_{9504}+3(10003)_{5280}-(11010)_{3696}\\
&\quad-2(02001)_{8064}-2(10010)_{144}-6(01001)_{560}-8(00001)_{16})t^{9}\\
&+(-(00006)_{28314}+(10004)_{20790}-(02002)_{46800}-3(01002)_{3696}-2(00002)_{126})t^{10}\\
&+(+(10005)_{68640}-(01003)_{17280}+2(00101)_{1200}-(00010)_{16})t^{11}\\
&+(-(01004)_{64350}+(00102)_{6930}+(00200)_{4125}-2(00011)_{210}+(00000)_{1})t^{12}\\
&+(+(00103)_{29568}-(00012)_{1440}-(00110)_{1200}+2(00001)_{16})t^{13}\\
&+(-(00013)_{6930}+(00002)_{126}+(00100)_{120})t^{14}+(00003)_{672}t^{15}
\end{aligned}
\tag{B.8}
$$

Appendix C

Computation of higher level character formulas

Higher level character formulas $Z_h(t)$ can be obtained from the formal expression (6.13) as follows. Performing a Taylor expansion of the expression (6.13) around $q = 0$, we have

$$\begin{aligned} Z(q,t) &= Z_0(t) + \sum_{h=1}^{\infty} \frac{q^h}{h!} \frac{\partial^h}{\partial q^h} Z(q,t)|_{q=0} \\ &= Z_0(t)\Big[1 + \sum_{h=1}^{\infty} f_h(t) q^h\Big] , \end{aligned} \tag{C.1}$$

where the level h function $f_h(t)$ has been defined as the expression (6.23). To obtain these level h functions, we are going to use a method based on Padé approximants. Let us explain our method by computing in detail the level one function $f_1(t)$.

From the expression (6.23), we derive the following expression for the level one function

$$f_1(t) = \sum_{k=1}^{\infty} N_k (t^k + t^{-k}) , \tag{C.2}$$

expanding the RHS of (C.2) around $t = 1$ and keeping terms up to some order (relevant for the computations to be done next), we get

$$f_1(t) \;=\; 2\sum_{k=1}^{\infty} N_k + (t-1)^2 \sum_{k=1}^{\infty} k^2 N_k - (t-1)^3 \sum_{k=1}^{\infty} k^2 N_k + \cdots \tag{C.3}$$

Applying the formula (6.10) to find the even moments $\sum_k N_k$, $\sum_k k^2 N_k$ and replacing them into the equation (C.3), we obtain

$$f_1(t) = 22 + 4(t-1)^2 - 4(t-1)^3 + \cdots \tag{C.4}$$

Using Padé approximants, we express the function $f_1(t)$ as a rational function

$$f_1(t) \cong f_1^{[M/N]}(t) = \frac{1 + \sum_{j=1}^{M} p_j t^j}{\sum_{j=0}^{N} q_j t^j} , \tag{C.5}$$

for instance, as a pedagogical illustration let us compute explicitly the [2/1] Padé approximant of $f_1(t)$

$$f_1^{[2/1]}(t) = \frac{1 + p_1 t + p_2 t^2}{q_0 + q_1 t}\,, \tag{C.6}$$

expanding the RHS of (C.6) around $t = 1$, we get

$$\begin{aligned} f_1^{[2/1]}(t) &= \frac{1 + p_1 + p_2}{q_0 + q_1} + \frac{p_1 q_0 + 2p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2}(t-1) \\ &+ \frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3}(t-1)^2 - \frac{p_2 q_1 q_0^2 - p_1 q_0 q_1^2 + q_1^3}{(q_0 + q_1)^4}(t-1)^3 + \cdots \end{aligned} \tag{C.7}$$

Equating the coefficients of $(t-1)^0$, $(t-1)^1$, $(t-1)^2$, $(t-1)^3$ in equations (C.4) and (C.7), we get 4 equations for the unknown coefficients p_1, p_2, q_0, q_1

$$\begin{aligned} \frac{1 + p_1 + p_2}{q_0 + q_1} &= 22\,, \\ \frac{p_1 q_0 + 2p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2} &= 0\,, \\ \frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3} &= 4\,, \\ \frac{p_2 q_1 q_0^2 - p_1 q_0 q_1^2 + q_1^3}{(q_0 + q_1)^4} &= 4\,, \end{aligned} \tag{C.8}$$

solving these system of equations (C.8) we obtain

$$p_1 = \frac{7}{2}\,, \quad p_2 = 1\,, \quad q_0 = 0\,, \quad q_1 = \frac{1}{4}\,. \tag{C.9}$$

Computations of higher Padé approximants follows in the same way as it was shown above. The results of these computations are given in table (C.10).

$[M/N]$	$p_1,\ p_2,\ \cdots,\ p_M$	$q_0,\ q_1,\ \cdots,\ q_N$
[2/1]	7/2, 1	0, 1/4
[2/2]	20/23, 1	1/46, 2/23, 1/46
[3/1]	1, 18/25, −2/25	1/50, 1/10
[1/3]	43/23	75/5566, 875/5566, −135/2783, 1/121
[3/2]	20/23, 1, 0	1/46, 2/23, 1/46
[2/3]	20/23, 1	1/46, 2/23, 1/46, 0
[3/3]	20/23, 1, 0	1/46, 2/23, 1/46, 0
[4/4]	20/23, 1, 0, 0	1/46, 2/23, 1/46, 0, 0

(C.10)

As we can see by explicit computations, the Padé approximants are approaching to the rational function $(46 + 40t + 46t^2)/(1 + 4t + t^2)$, and therefore we take this function as being the level one function $f_1(t)$

$$f_1(t) = \frac{46 + 40t + 46t^2}{1 + 4t + t^2}\,. \tag{C.11}$$

By multiplying this function (C.11) with the level zero character $Z_0(t)$, we get

$$Z_1(t) = \frac{46 - 144t + 116t^2 + 16t^3 - 16t^5 - 116t^6 + 144t^7 - 46t^8}{(t-1)^{16}}\,, \tag{C.12}$$

and therefore, we correctly reproduce the level one character formula given in [9, 12].

For the next level $h = 2$, by using the same strategy shown above, we have found that the Padé approximant computation gives the following result for the level two function

$$f_2(t) = \frac{-1 + 12t - 67t^2 + 248t^3 + 319t^4 + 628t^5 + 319t^6 + 248t^7 - 67t^8 + 12t^9 - t^{10}}{t^4(1 + 4t + t^2)}.$$

By multiplying this level two function $f_2(t)$ with the level zero character $Z_0(t)$, we correctly reproduce the level two character formula found in [9].

Computation of higher level functions $f_h(t)$ by means of Padé approximants, suggest us that these functions can be written like

$$f_h(t) = \frac{\sum_{i=0}^{2h+6} C_{i,h} t^i}{t^{h+2}(1 + 4t + t^2)}\,. \tag{C.13}$$

We have computed the $C_{i,h}$ coefficients up to the level $h = 12$, the results are given in the tables (6.25) and (6.26) of chapter 6. Multiplying the functions $f_h(t)$ with the level zero character formula $Z_0(t)$, we obtain the characters $Z_h(t)$. We have compared our first five character formulas with the formulas given in [9] and we have found agreement.

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