



Involutions fixing $F^n \cup F^3$

Evelin M. Barbaresco^a, Pedro L.Q. Pergher^{b,*}

^aDepartamento de Matemática, Universidade Estadual Paulista - Ibilce, São José do Rio Preto, SP 15054-000, Brazil

^bDepartamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, São Carlos, SP 13565-905, Brazil

Received 10 October 2017; received in revised form 12 January 2018; accepted 12 January 2018

Communicated by J. van Mill

Abstract

Let M^m be a closed smooth manifold equipped with a smooth involution having fixed point set of the form $F^n \cup F^3$, where F^n and F^3 are submanifolds with dimensions n and 3, respectively, where $3 < n < m$ and with the normal bundles over F^n and F^3 being nonbounding. The authors of this paper, together with Patricia E. Desideri, previously showed that, when n is even, then $m \leq n + 4$, which we call a *small codimension phenomenon*. Further, they showed that this small bound is best possible. In this paper we study this problem for n odd, which is much more complicated, requiring more sophisticated techniques involving characteristic numbers. We show in this case that $m \leq \mathcal{M}(n - 3) + 6$, where $\mathcal{M}(n)$ is the *Stong–Pergher number* (see the definition of $\mathcal{M}(n)$ in Section 1). Further, we show that this bound is *almost best possible*, in the sense that there exists an example with $m = \mathcal{M}(n - 3) + 5$, which means that for n odd the small codimension phenomenon does not occur and the bound in question is meaningful. The existence of these bounds is guaranteed by the famous *Five Halves Theorem* of J. Boardman, which establishes that, under the above hypotheses, $m \leq \frac{5}{2}n$.

© 2018 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.

Keywords: Involution; Fixed-data; Whitney number; Wu formula; Steenrod operation; Stong–Pergher number

* Corresponding author.

E-mail addresses: evelin@ibilce.unesp.br (E.M. Barbaresco), pergher@dm.ufscar.br (P.L.Q. Pergher).

1. Introduction

Let F be a disjoint (finite) union of smooth and closed manifolds and M be a smooth and closed m -dimensional manifold equipped with a smooth involution $T : M \rightarrow M$ whose fixed point set is F . Write $F = \cup_{j=0}^n F^j$, where F^j denotes the union of those components of F having dimension j . Up to equivariant cobordism, each F^j can be supposed connected; also, if the normal bundle over some F^j bounds as a bundle, then after an equivariant surgery an appropriate tubular neighborhood over F^j can be removed, and thus again, up to equivariant cobordism, we can suppose that F has not the j -dimensional component (see [4]). Then suppose that the normal bundle over each F^j occurring does not bound. If n is the dimension of the component of F of maximal dimension, then $m \leq \frac{5}{2}n$; this follows from the famous Five Halves Theorem of J. Boardman, announced in [2], and its strengthened version of [12]. In fact, the Five Halves Theorem asserts that this is valid when M is not a boundary, and in [12] R.E. Stong and C. Kosniowski established the same conclusion under the weaker hypothesis that (M, T) is a nonbounding involution. The assertion then follows from the fact that the equivariant cobordism class of (M, T) is determined by the cobordism class of the normal bundle of F in M (see [4]). The generality of this result, which is valid for every $n \geq 1$, allows the possibility that fixed components of all dimensions j , $0 \leq j \leq n$, occur; in this way, it is natural to ask whether there exist better bounds for m when we omit some components of F and restrict the set of the involved maximal dimensions n . This class of problems was introduced by P. Pergher in [13], with its general formulation and with the following particular result: if F has the form $F = F^n \cup \{point\}$, where $n = 2p$ with p odd, then $m \leq 3p + 3$ (which is better than $\frac{5}{2}n = 5p$). This case ($F = F^n \cup \{point\}$) was completed by R. Stong and P. Pergher in [17], where they introduced the *mysterious* number $\mathcal{M}(n)$: writing $n = 2^p q$, where $p \geq 0$ and q is odd, then $\mathcal{M}(n) = 2n + p - q + 1$ if $p \leq q$ and $\mathcal{M}(n) = 2n + 2^{p-q}$ if $p > q$. They proved that, for every $n \geq 1$, $m \leq \mathcal{M}(n)$ and this bound is best possible.

Remark. In [12], R. Stong and C. Kosniowski proved the following relevant result: if $F = F^n$ has constant dimension n and $m = 2n$, then (M, T) is equivariantly cobordant to the *twist* involution $(F^n \times F^n, S)$, $S(x, y) = (y, x)$; further, if $m > 2n$, then (M, T) bounds equivariantly. Thus, if the normal bundle over F^n does not bound, $m \leq 2n$ (which is better than $m \leq \frac{5}{2}n$), and for each fixed n , with the exception of the dimensions $n = 1$ and $n = 3$, the maximal value $m = 2n$ is achieved by taking the involution $(F^n \times F^n, twist)$, where F^n is any nonbounding n -dimensional manifold. This completely solves the Pergher problem when F has one component. Note that, if n is odd (that is, $p = 0$), then $\mathcal{M}(n) = n + 1$; this special small bound had been obtained by D.C. Royster in [18], when classifying, up to equivariant cobordism, involutions fixing two real projective spaces.

With the cases $F = F^n$ and $F = F^n \cup \{point\}$ completed, the next natural step is the case $F = F^n \cup F^j$, $0 < j < n$. Concerning this more general case, one has the following results, which show the relevance of $\mathcal{M}(n)$: for $j = 1$, $m \leq \mathcal{M}(n - 1) + 1$ if n is odd and $m \leq \mathcal{M}(n - 1) + 2$ if n is even, the two bounds being best possible (see [10] and [11]). For $j = 2$, one has the best possible bound $m \leq \mathcal{M}(n - 2) + 4$ (see [7,6] and [9]). For $j = n - 1$, $m \leq 2n$, which is best possible [8]. For every $2 \leq j < n$ not of the form $j = 2^t - 1$, if F^j is indecomposable, then $m \leq \mathcal{M}(n - j) + 2j + 1$, and there is an example with $m = \mathcal{M}(n - j) + 2j$ [16]. Also, in [5,15] and [14], we find related results.

In this paper, we contribute to this problem, dealing with the case $j = 3$. As mentioned in the abstract, in [1] it was shown that in this case $m \leq n + 4$ if n is even, and this bound is best possible. We will prove the following

Theorem 1.1. *Let M be a closed smooth m -dimensional manifold equipped with a smooth involution having fixed point set of the form $F^n \cup F^3$, where $3 < n < m$ is odd and with the normal bundles over F^n and F^3 being nonbounding. Then $m \leq \mathcal{M}(n - 3) + 6$, and there is an example with $m = \mathcal{M}(n - 3) + 5$.*

Remark. For example, take $n = 2^p + 3$, where $p \geq 1$. Then $\mathcal{M}(n - 3) + 6 = 2^{p+1} + 2^{p-1} + 6$, which is equal to one less than the Boardman bound associated to n ($= 2^{p+1} + 2^{p-1} + 7$).

To obtain the bound in question, we introduce some special cohomology classes associated to line bundles over closed smooth manifolds, by using the *splitting principle*, and mix them with some special polynomials in the characteristic classes of total spaces of projective space bundles, introduced by R. Stong and P. Pergher in [17]; the basic theoretical support is the equivariant cobordism theory of Conner and Floyd of [4]. As will be seen, this will require more sophisticated computations than the case n even.

2. Preliminaries

If (M, T) is an involution pair as discussed above, we call $\eta \rightarrow F$ the *fixed-data* of (M, T) when F is the fixed point set of T and $\eta \rightarrow F$ is the normal bundle of F in M . Let η be a general k -dimensional vector bundle over a closed smooth n -dimensional manifold F . Write $W(\eta) = 1 + w_1(\eta) + w_2(\eta) + \dots + w_k(\eta) \in H^*(F, \mathbb{Z}_2)$ for the *Stiefel–Whitney class* of η , and $W(F) = 1 + w_1(F) + w_2(F) + \dots + w_n(F)$ for the *Stiefel–Whitney class* of the tangent bundle of F . From [4], one has an algebraic scheme to determine the cobordism class of η , given by the set of *Whitney numbers* (or *characteristic numbers*) of η ; such modulo 2 numbers are obtained by evaluating n -dimensional \mathbb{Z}_2 -cohomology classes of the form

$$w_{i_1}(F)w_{i_2}(F)\dots w_{i_r}(F)w_{j_1}(\eta)w_{j_2}(\eta)\dots w_{j_s}(\eta) \in H^n(F, \mathbb{Z}_2)$$

(that is, with $i_1 + i_2 + \dots + i_r + j_1 + j_2 + \dots + j_s = n$) on the fundamental homology class $[F] \in H_n(F, \mathbb{Z}_2)$. For example, suppose F three-dimensional. In this case, we can denote $W(F) = 1 + w_1 + w_2 + w_3$ and $W(\eta) = 1 + v_1 + v_2 + v_3$. Then the set of Whitney numbers comes from the ten three-dimensional cohomology classes $w_3, w_1w_2, w_1^3, v_3, v_1v_2, v_1^3, w_1v_1^2, w_1v_2, w_1^2v_1$ and w_2v_1 . However, this number can be reduced. Indeed, from [1] one has the following

Lemma 2.1. *Let η a vector bundle over a three-dimensional manifold as above. Then $w_3 = w_1w_2 = w_1^3 = v_1^2w_1 = 0, w_2v_1 = w_1^2v_1$ and $v_2w_1 = v_1v_2 + v_3$. So any cobordism class of a bundle over a three-dimensional manifold is determined by the numbers coming from the three-dimensional classes v_1^3, v_3, v_2w_1 and $v_1w_1^2$. Further, any nonempty subset of this set of cohomology classes is realized by a stable cobordism class, in the sense that there is a bundle over a three-dimensional manifold whose set of nonzero Whitney numbers comes from the subset in question (which means that one has fifteen nonzero stable cobordism classes of bundles over closed three-dimensional manifolds).*

3. The bound $m \leq \mathcal{M}(n - 3) + 6$

This section will be devoted to the proof of the part “ $m \leq \mathcal{M}(n - 3) + 6$ ” of [Theorem 1.1](#). One then has an involution pair (M, T) with fixed set of the form $F^n \cup F^3$, where $3 < n < m$ is odd and with the normal bundles over F^n and F^3 being nonbounding, and wants to prove the bound in question, where $m = \dim(M)$. Let $(\mu \mapsto F^n) \cup (\eta \mapsto F^3)$ be the fixed-data of (M, T) and, as in [Lemma 2.1](#), write $W(F^3) = 1 + w_1 + w_2 + w_3$ and $W(\eta) = 1 + v_1 + v_2 + v_3$. The following lemma is crucial:

Lemma 3.1. *If $m > \mathcal{M}(n - 3) + 6$, then $v_1^3 = v_1 w_1^2$, $v_3 = 0$ and $v_2 w_1 = v_1 w_1^2$.*

Taking into account the nonempty subsets of $\{v_1^3, v_3, v_2 w_1, v_1 w_1^2\}$ mentioned in Lemma 2.1, we note that the unique nonzero stable cobordism class over a three-manifold which satisfies the relations of Lemma 3.1 is the one whose nonzero Whitney numbers come from $v_1 w_1^2$, v_1^3 and $v_2 w_1$; call this class β . Thus this lemma will reduce our task to the following:

Theorem 3.1. *In the statement of Theorem 1.1, suppose that $\eta \mapsto F^3$ represents β . Then $m \leq \mathcal{M}(n - 3) + 6$.*

The following basic fact, which follows from the Conner and Floyd exact sequence of [4], will be necessary for the proof of Lemma 3.1 (and Theorem 3.1): if E_μ and E_η denote the total spaces of the projective space bundles $\mathbb{R}P(\mu)$ and $\mathbb{R}P(\eta)$, respectively, and $\lambda_\mu \mapsto E_\mu$ and $\lambda_\eta \mapsto E_\eta$ denote the line bundles of the double covers $S(\mu) \rightarrow E_\mu$ and $S(\eta) \rightarrow E_\eta$, $S(\)$ meaning sphere bundles, then $\lambda_\mu \mapsto E_\mu$ and $\lambda_\eta \mapsto E_\eta$ are cobordant as elements of the cobordism group $\mathcal{N}_{m-1}(BO(1))$, that is, the cobordism group of 1-dimensional real vector bundles over $(m - 1)$ -dimensional closed smooth manifolds. Therefore any cohomology class of dimension $m - 1$, given by a product of the classes $w_i(E_\mu)$ and $w_1(\lambda_\mu)$, evaluated on the fundamental homology class $[E_\mu]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(E_\eta)$ and $w_1(\lambda_\eta)$, evaluated on $[E_\eta]$. With this tool in hand, our strategy will be: first, we will use a very special class, denoted by X , introduced by Pergher and Stong in [17]. X is associated to E_μ and, as above required, is a product of the classes $w_i(E_\mu)$ and $w_1(\lambda_\mu)$; further, X has dimension $\mathcal{M}(n - 3)$. Second, by using the splitting principle and the partitions of 3, ω_1, ω_2 and ω_3 , we introduce three special cohomology classes of dimension 6 associated to line bundles λ over closed smooth s -dimensional manifolds B^s , denoted by $f_{\omega_1}(\lambda)$, $f_{\omega_2}(\lambda)$ and $f_{\omega_3}(\lambda)$, and which are special polynomials in the characteristic classes of λ and B^s . Write Y for the cohomology class of E_η which corresponds to X . Then, if $m > \mathcal{M}(n - 3) + 6$, $m - 1 \geq \mathcal{M}(n - 3) + 6$ and we can form a modulo 2 system of equations

$$\begin{cases} X \cdot f_{\omega_1}(\lambda_\mu) \cdot w_1(\lambda_\mu)^{m - (\mathcal{M}(n-3)+6)} [E_\mu] = Y \cdot f_{\omega_1}(\lambda_\eta) \cdot w_1(\lambda_\eta)^{m - (\mathcal{M}(n-3)+6)} [E_\eta] \\ X \cdot f_{\omega_2}(\lambda_\mu) \cdot w_1(\lambda_\mu)^{m - (\mathcal{M}(n-3)+6)} [E_\mu] = Y \cdot f_{\omega_2}(\lambda_\eta) \cdot w_1(\lambda_\eta)^{m - (\mathcal{M}(n-3)+6)} [E_\eta] \\ X \cdot f_{\omega_3}(\lambda_\mu) \cdot w_1(\lambda_\mu)^{m - (\mathcal{M}(n-3)+6)} [E_\mu] = Y \cdot f_{\omega_3}(\lambda_\eta) \cdot w_1(\lambda_\eta)^{m - (\mathcal{M}(n-3)+6)} [E_\eta]. \end{cases}$$

The solution of this system will be given by the relations of Lemma 3.1, thus providing the proof.

Next, we detail the technical steps regarding this strategy, and the first thing to do is to describe the class X of Stong and Pergher of [17]. Set $k = m - n$, and write $W(F^n) = 1 + \theta_1 + \theta_2 + \dots + \theta_n$, $W(\mu) = 1 + u_1 + u_2 + \dots + u_k$ and $W(\lambda_\mu) = 1 + w_1(\lambda_\mu) = 1 + c$ for the Stiefel–Whitney classes of F^n , μ and λ_μ , respectively. One has (see [3]):

$$W(E_\mu) = (1 + \theta_1 + \theta_2 + \dots + \theta_n) \{ (1 + c)^k + (1 + c)^{k-1} u_1 + \dots + u_k \},$$

where here we are suppressing bundle maps. First, for any integer r , Stong and Pergher introduced the following variant of $W(E_\mu)$:

$$W[r] = \frac{W(E_\mu)}{(1 + c)^{k-r}},$$

noting that each class $W[r]_j$ is still a polynomial in the classes $w_i(E_\mu)$ and c . Next, for $n \geq 5$, write $n - 3 = 2^p q$, where $p \geq 1$ ($n - 3$ is even) and q is odd; X is built in terms of p and q . Specifically, suppose first that $p < q + 1$. Then, in this case, X is

$$X = W[2^p - 1]_{2^{p+1-1}}^{q+1-p} \cdot W[r_1]_{2r_1} \cdot W[r_2]_{2r_2} \cdots W[r_p]_{2r_p},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq p$.

If $p \geq q + 1$, X is

$$X = W[r_1]_{2r_1} \cdot W[r_2]_{2r_2} \cdots W[r_{q+1}]_{2r_{q+1}},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq q + 1$. Stong and Pergher proved that X has the following two crucial properties:

(i) $\text{dimension}(X) = \mathcal{M}(2^p q) = \mathcal{M}(n - 3)$;

(ii) X has the form $X = A_t \cdot c^{\mathcal{M}(2^p q)-t} + \text{terms with smaller } c \text{ powers}$, where A_t is a cohomology class of dimension $t \geq 2^p q + 1$ and comes from the cohomology of F^n . In our case, $X = A_t \cdot c^{\mathcal{M}(n-3)-t} + \text{terms with smaller } c \text{ powers}$, where A_t is a cohomology class of dimension $t \geq n - 2$ coming from the cohomology of F^n .

The next technical step is, as above announced, to introduce the three special 6-dimensional cohomology classes $f_{\omega_i}(\lambda)$, $i = 1, 2, 3$, where λ is a line bundle over a smooth closed s -dimensional manifold B^s . Using the *splitting principle*, write $W(B^s) = (1+x_1) \cdot (1+x_2) \cdots (1+x_s)$ and $W(\lambda) = 1 + c$. We then consider the following symmetric polynomials in the variables x_1, x_2, \dots, x_s, c , of degree 6 and related to the partitions of 3, $\omega_1 = (1, 1, 1)$, $\omega_2 = (2, 1)$ and $\omega_3 = (3)$:

$$f_{\omega_1} = \sum_{i < j < m} x_i(c + x_i)x_j(c + x_j)x_m(c + x_m),$$

$$f_{\omega_2} = \sum_{i \neq j} x_i(c + x_i)x_j^2(c + x_j)^2 \quad \text{and}$$

$$f_{\omega_3} = \sum_i x_i^3(c + x_i)^3.$$

Following the pattern procedure, f_{ω_1} , f_{ω_2} and f_{ω_3} determine polynomials of dimension 6 in the classes $w_i(B^s)$ and $w_1(\lambda) = c$, which we call $f_{\omega_i}(\lambda)$, $i = 1, 2, 3$. Specializing for $\lambda_\mu \mapsto E_\mu$, write

$$W(F^n) = (1 + x_1) \cdot (1 + x_2) \cdots (1 + x_n) \quad \text{and}$$

$$W(\mu) = (1 + y_1) \cdot (1 + y_2) \cdots (1 + y_k).$$

Then

$$W(E_\mu) = (1 + x_1) \cdots (1 + x_n) \cdot (1 + c + y_1) \cdots (1 + c + y_k).$$

It follows that

$$f_{\omega_1}(\lambda_\mu) = \left(\left(\sum_{i < j < m} x_i x_j x_m \right) + \left(\sum_{i < j < m} y_i y_j y_m \right) + \left(\sum_{\substack{i, j, m \\ j < m}} x_i y_j y_m \right) + \left(\sum_{\substack{i, j, m \\ i < j}} x_i x_j y_m \right) \right) c^3 + \text{terms with smaller } c \text{ powers},$$

$$f_{\omega_2}(\lambda_\mu) = \left(\left(\sum_{i \neq j} x_i x_j^2 \right) + \left(\sum_{i \neq j} y_i y_j^2 \right) + \left(\sum_{i, j} x_i^2 y_j \right) + \left(\sum_{i, j} x_i y_j^2 \right) \right) c^3 + \text{terms with smaller } c \text{ powers}$$

and

$$f_{\omega_3}(\lambda_\mu) = \left(\left(\sum_i x_i^3 \right) + \left(\sum_j y_j^3 \right) \right) c^3 + \text{terms with smaller } c \text{ powers.}$$

Therefore, every term of $f_{\omega_1}(\lambda_\mu)$, $f_{\omega_2}(\lambda_\mu)$ and $f_{\omega_3}(\lambda_\mu)$ has a factor of dimension at least 3 from the cohomology of F^n . On the other hand, as before cited, each term of our previous class X has a factor of dimension at least $n - 2$ from the cohomology of F^n , which means that, for $i = 1, 2, 3$, $X \cdot f_{\omega_i}(\lambda_\mu)$ is a class in $H^{\mathcal{M}(n-3)+6}(E_\mu, \mathbb{Z}_2)$ with each one of its terms having a factor of dimension at least $n + 1$ from F^n . Thus $X \cdot f_{\omega_i}(\lambda_\mu) = 0$, which means that the characteristic number $X \cdot f_{\omega_i}(\lambda_\mu) \cdot c^{m-1-(\mathcal{M}(n-3)+6)}[E_\mu]$ is zero. Hence, the left side of our system of equations is zero, and thus the next task is to analyze the right side of it. To do that, first we study $f_{\omega_i}(\lambda_\eta)$, $i = 1, 2, 3$. Set $W(\lambda_\eta) = 1 + d$. Write $W(F^3) = (1 + x_1)(1 + x_2)(1 + x_3)$, $W(\eta) = (1 + y_1)(1 + y_2)(1 + y_3)$, and denote by σ_i the i th elementary symmetric function in the variables x_1, x_2, x_3, y_1, y_2 and y_3 ; that is

$$(1 + x_1)(1 + x_2)(1 + x_3)(1 + y_1)(1 + y_2)(1 + y_3) = 1 + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6.$$

This is the factored form of the Whitney sum $\tau \oplus \eta$, where τ is the tangent bundle over F^3 . One has

$$W(E_\eta) = (1 + w_1 + w_2 + w_3) \left((1 + d)^{n+k-3} + (1 + d)^{n+k-4} v_1 + (1 + d)^{n+k-5} v_2 + (1 + d)^{n+k-6} v_3 \right).$$

Rewriting,

$$\begin{aligned} W(E_\eta) &= (1 + d)^{n+k-6} \left((1 + w_1 + w_2 + w_3) \left((1 + d)^3 + (1 + d)^2 v_1 + (1 + d)^1 v_2 + v_3 \right) \right) \\ &= (1 + d)^{n+k-6} \cdot (1 + x_1) \cdot (1 + x_2) \cdot (1 + x_3) \cdot (1 + d + y_1) \cdot (1 + d + y_2) \cdot (1 + d + y_3). \end{aligned}$$

Since $d + d = 0$, the part $(1 + d)^{n+k-6}$ does not contribute to $f_{\omega_i}(\lambda_\eta)$. Then a straightforward calculation shows that $f_{\omega_1}(\lambda_\eta) = \sigma_3 d^3 + \text{terms with smaller } c \text{ powers}$, $f_{\omega_2}(\lambda_\eta) = (\sigma_1 \sigma_2 + \sigma_3) d^3 + \text{terms with smaller } c \text{ powers}$ and $f_{\omega_3}(\lambda_\eta) = (\sigma_1^3 + \sigma_1 \sigma_2 + \sigma_3) d^3 + \text{terms with smaller } c \text{ powers}$. In other words, setting $W(\tau \oplus \eta) = 1 + V_1 + V_2 + V_3$ and noting that if a term (with dimension 6) has a power of d less than 3, it necessarily has a factor of dimension greater than 3 from the cohomology of F^3 , one then has $f_{\omega_1}(\lambda_\eta) = V_3 d^3$, $f_{\omega_2}(\lambda_\eta) = (V_1 V_2 + V_3) d^3$ and $f_{\omega_3}(\lambda_\eta) = (V_1^3 + V_1 V_2 + V_3) d^3$.

Next we analyze the class Y , which is obtained from X by replacing each $W[r]_i$ by $W[n + r - 3]_i$. Denoting by \mathcal{I} the ideal of $H^*(E_\eta, \mathbb{Z}_2)$ generated by the classes coming from F^3 and with positive dimension, one has by dimensional reasons that $f_{\omega_i}(\lambda_\eta) \cdot A = 0$ for each $A \in \mathcal{I}$ and $i = 1, 2, 3$. Thus, in the computation of Y , one needs to consider only that

$$W(E_\eta) \equiv (1 + d)^{n+k-3} \pmod{\mathcal{I}}$$

and, for each integer l ,

$$W[l] \equiv (1 + d)^l \pmod{\mathcal{I}}.$$

For $r_i = 2^p - 2^{p-i}$, $i = 1, 2, \dots, p$, set $l_i = n + r_i - 3 = 2^p q + 3 + 2^p - 2^{p-i} - 3 = 2^p q + 2^p - 2^{p-i}$. Then

$$W[l_i]_{2r_i} \equiv \binom{2^p q + 2^p - 2^{p-i}}{2^{p+1} - 2^{p-i+1}} d^{2r_i} \pmod{\mathcal{I}}.$$

Also, if $r = 2^p - 1$, $l = n + r - 3 = 2^p q + 2^p - 1$ and

$$W[l]_{2r+1} \equiv \binom{2^p q + 2^p - 1}{2^{p+1} - 1} d^{2r+1} \pmod{\mathcal{I}}.$$

The lesser term of the 2-adic expansion of $2^p q + 2^p$ is 2^{p+1} . Using the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of b is a subset of the 2-adic expansion of a , we conclude that the above binomial coefficients are nonzero modulo 2. It follows that all classes $W[r]_i$ occurring in Y satisfy $W[r]_i \equiv d^i \pmod{\mathcal{I}}$, which implies that $Y \equiv d^{\mathcal{M}(n-3)} \pmod{\mathcal{I}}$. Thus, if $b \in H^3(E_\eta, \mathbb{Z}_2)$ is a cohomology class coming from $H^3(F^3, \mathbb{Z}_2)$, since $H^*(E_\eta, \mathbb{Z}_2)$ is the free $H^*(F^3, \mathbb{Z}_2)$ -module on $1, d, d^2, \dots, d^{n+k-4}$ [see 3], we have that

$$b \cdot d^3 \cdot Y \cdot d^{m-1-(\mathcal{M}(n-3)+6)} [E_\eta] = d^{m-4} \cdot b[E_\eta] = b[F^3],$$

and $b[F^3] = 0$ if and only if $b = 0$. So our system of equations becomes the cohomological system of equations

$$\begin{cases} 0 = V_3 \\ 0 = V_1 V_2 + V_3 \\ 0 = V_1^3 + V_1 V_2 + V_3. \end{cases}$$

Thus $V_3 = 0$, $V_1 V_2 = 0$ and $V_1^3 = 0$. Since $V_1 = v_1 + w_1$, $V_1^3 = v_1^3 + v_1 w_1^2 + v_1^2 w_1 + w_1^3$. One knows that the first Wu class of F^3 is $u_1 = w_1$. So, if Sq denotes the Steenrod operation, one has $v_1^2 w_1 = Sq^1(v_1^2) = Sq^1(v_1 v_1 + v_1 Sq^1(v_1)) = 0$. Also, since F^3 bounds, $w_1^3 = 0$. We conclude that $v_1^3 = v_1 w_1^2$, the first relation of the lemma. We have $Sq(1 + u_1) = 1 + u_1 + u_1^2 = 1 + w_1 + w_2$, that is, $w_1^2 = w_2$. Then $0 = V_1 V_2 = (v_1 + w_1)(v_2 + w_2 + v_1 w_1) = v_1 v_2 + v_1 w_2 + v_1^2 w_1 + w_1 v_2 + w_1 w_2 + v_1 w_1^2 = v_1 v_2 + w_1 v_2$. By the Wu Formula, $w_1 v_2 = Sq^1(v_2) = v_1 v_2 + v_3$. Hence $v_3 = 0$, the second relation of the lemma. Finally, $0 = V_3 = v_3 + w_3 + v_2 w_1 + v_1 w_2 = v_2 w_1 + v_1 w_1^2$, the third relation, and [Lemma 3.1](#) is proved.

Now we prove [Theorem 3.1](#). Let $\mathbb{R}P^n$ be the n -dimensional real projective space, and $\xi_n \mapsto \mathbb{R}P^n$ be the canonical line bundle. If X is a space, $j\mathbb{R} \mapsto X$ will denote the j -dimensional trivial vector bundle over X . An easy calculation shows that $\eta \mapsto F^3 = \xi_1 \oplus \xi_2 \oplus (m - 5)\mathbb{R} \mapsto \mathbb{R}P^1 \times \mathbb{R}P^2$ is a representative for β , where here we are omitting pullback notations. We maintain the previous notations for the characteristic classes referring to the component F^n , and repeat the notations $\lambda_\eta \mapsto E_\eta$ and $W(\lambda_\eta) = 1 + d$ on the component F^3 . Our strategy will consist in showing that, if $m > \mathcal{M}(n - 3) + 6$, then it is possible to find special polynomials in the characteristic classes so that the corresponding characteristics numbers are zero on F^n and nonzero on F^3 , thus giving the contradiction. Taking into account our previous knowledge on the behavior of the class X of Stong and Pergher on F^n , the additional key point will be a subtle (but routine) calculation based on the structure of the cohomology ring of E_η , described as follows: let $\alpha \in H^1(\mathbb{R}P^1, \mathbb{Z}_2)$ and $\beta \in H^1(\mathbb{R}P^2, \mathbb{Z}_2)$ be the respective generators. Then $H^*(E_\eta, \mathbb{Z}_2)$ is the free $H^*(F^3, \mathbb{Z}_2)$ -module on $1, d, d^2, \dots, d^{m-4}$, subject to the relation $d^{m-3} = d^{m-4}(\alpha + \beta) + d^{m-5}\alpha\beta$. From this relation, we obtain $d^{m-1} = d^{m-2}(\alpha + \beta) + d^{m-3}\alpha\beta$,

$d^{m-2}\alpha = d^{m-3}\alpha\beta$, $d^{m-2}\beta = d^{m-3}\alpha\beta + d^{m-3}\beta^2 + d^{m-4}\alpha\beta^2$, $d^{m-3}\beta^2 = d^{m-4}\alpha\beta^2$ and $d^{m-3}\alpha\beta = d^{m-4}\alpha\beta^2$. Combining these relations, we obtain $d^{m-1} = d^{m-2}\beta = d^{m-3}\alpha\beta = d^{m-4}\alpha\beta^2$, which is the (top-dimensional) generator of $H^{m-1}(E_\eta, \mathbb{Z}_2)$.

Write $n - 3 = 2^p q$, where $p > 0$ and q is odd, and first suppose $p > 1$. On F^n one takes the same class X considered before; that is, $X \in H^{\mathcal{M}(n-3)}(E_\mu, \mathbb{Z}_2)$ and each term of X has a factor of dimension at least $n - 2$ from the cohomology of F^n . Note that, on F^n , $W[0]_2 = u_1c + u_2 + \theta_1u_1 + \theta_2$. Hence every term of $W[0]_2^3 = (u_1^2c^2 + u_2^2 + \theta_1^2u_1^2 + \theta_2^2)(u_1c + u_2 + \theta_1u_1 + \theta_2)$ has a factor of dimension at least 3 from F^n . If $m > \mathcal{M}(n-3)+6$, one then has the zero characteristic number

$$X \cdot W[0]_2^3 \cdot c^{m-1-(\mathcal{M}(n-3)+6)}[E_\mu].$$

Our next task will be to show that, on F^3 , the corresponding characteristic number

$$Y \cdot W[n - 3]_2^3 \cdot d^{m-1-(\mathcal{M}(n-3)+6)}[E_\eta]$$

is nonzero.

One has

$$W(E_\eta) = (1 + \beta + \beta^2)\left((1 + d)^{n+k-3} + (1 + d)^{n+k-4}(\alpha + \beta) + (1 + d)^{n+k-5}(\alpha\beta)\right).$$

Then $W[n - 3]_2 = \binom{n-3}{2} d^2 + d(\alpha + \beta)$. Since $n - 3 = 2^p q$ with $p > 1$ and q odd, one has that 2 does not belong to the 2-adic expansion of $n - 3$, and thus $W[n - 3]_2 = d(\alpha + \beta)$. Hence, $W[n - 3]_2^3 = d^3\alpha\beta^2$. Concerning the class Y , which is obtained from X by replacing each $W[r]_i$ by $W[n + r - 3]_i$, we are exactly in the same situation of [Lemma 3.1](#). In fact, $W[n - 3]_2^3 \cdot A = d^3\alpha\beta^2 \cdot A = 0$ for each $A \in \mathcal{I}$, the ideal of $H^*(E_\eta, \mathbb{Z}_2)$ generated by the classes coming from F^3 and with positive dimension. Then, as in [Lemma 3.1](#), in the computation of Y , one needs to consider only that

$$W(E_\eta) \equiv (1 + d)^{n+k-3} \pmod{\mathcal{I}}$$

and, for each integer l ,

$$W[l] \equiv (1 + d)^l \pmod{\mathcal{I}}.$$

In this way, similarly we conclude that $Y \equiv d^{\mathcal{M}(n-3)} \pmod{\mathcal{I}}$. It follows that $Y \cdot W[n - 3]_2^3 \cdot d^{m-1-(\mathcal{M}(n-3)+6)}[E_\eta] = d^{m-4} \cdot \alpha\beta^2[E_\eta] = \alpha\beta^2[F^3] = 1$, which proves [Theorem 3.1](#) when $p > 1$.

Now suppose $p = 1$. On $\mathbb{R}P(\mu)$ we have

$$W[1] = (1 + \theta_1 + \dots + \theta_n)\left((1 + c) + u_1 + \frac{u_2}{(1 + c)} + \dots + \frac{u_k}{(1 + c)^{k-1}}\right).$$

Then

$$W[1]_3 = u_2c + u_3 + \theta_1u_2 + \theta_2c + \theta_2u_1 + \theta_3.$$

Hence every term of $W[1]_3^2$ has a factor of dimension at least 4 from $H^*(F^n, \mathbb{Z}_2)$. Since, as before, each term of X has a factor of dimension at least $n - 2$ from the cohomology of F^n , if $m > \mathcal{M}(n - 3) + 6$, we have the zero characteristic number

$$X \cdot W[1]_3^2 \cdot c^{m-1-(\mathcal{M}(n-3)+6)}[\mathbb{R}P(\mu)].$$

So, the next and final task will be to show that, on F^3 , the corresponding characteristic number

$$Y \cdot W[n - 2]_3^2 \cdot d^{m-1-(\mathcal{M}(n-3)+6)}[E_\eta]$$

is nonzero. One has

$$W[n - 2] = (1 + \beta + \beta^2) \left((1 + d)^{n-2} + (1 + d)^{n-3}(\alpha + \beta) + (1 + d)^{n-4}\alpha\beta \right).$$

Thus

$$\begin{aligned} W[n - 2]_3 &= \binom{n-2}{3} d^3 + \binom{n-3}{2} d^2(\alpha + \beta) + \binom{n-4}{1} d\alpha\beta \\ &+ \binom{n-2}{2} d^2\beta + \binom{n-3}{1} d(\alpha\beta + \beta^2) + \binom{n-4}{0} \alpha\beta^2 \\ &+ \binom{n-2}{1} d\beta^2 + \binom{n-3}{0} \alpha\beta^2. \end{aligned}$$

An easy inspection of 2-adic expansions shows that 1 and 2 belong to the 2-adic expansion of $n - 2 = 2q + 1$, and 2 belongs to the 2-adic expansion of $n - 3 = 2q$. We conclude that $W[n - 2]_3 = d^3 + d^2(\alpha + \beta) + d\alpha\beta + d^2\beta + d\beta^2 = d^3 + d^2\alpha + d\alpha\beta + d\beta^2$ and $W[n - 2]_3^2 = d^6$.

Now, since $p = 1$, the class X (on E_μ) is $W[1]_2 \cdot (W[1]_3)^q$. Therefore $Y = W[n - 2]_2 \cdot (W[n - 2]_3)^q$. One has

$$\begin{aligned} W[n - 2]_2 &= \binom{n-2}{2} d^2 + \binom{n-3}{1} d(\alpha + \beta) + \binom{n-4}{0} \alpha\beta \\ &+ \binom{n-2}{1} d\beta + \binom{n-3}{0} (\alpha\beta + \beta^2) + \binom{n-2}{0} \beta^2 \\ &= \binom{n-2}{2} d^2 + d\beta \\ &= d^2 + d\beta, \end{aligned}$$

and, as seen above, $W[n - 2]_3 = d^3 + d^2\alpha + d\alpha\beta + d\beta^2$.

Thus,

$$\begin{aligned} Y &= (d^3 + d^2\alpha + d(\alpha\beta + \beta^2))^q (d^2 + d\beta) \\ &= \left(\sum_{i=0}^q \binom{q}{i} (d^3 + d^2\alpha)^{q-i} (d(\alpha\beta + \beta^2))^i \right) (d^2 + d\beta) \\ &= \left((d^3 + d^2\alpha)^q + \left((d^3 + d^2\alpha)^{q-1} \right) (d(\alpha\beta + \beta^2)) \right) (d^2 + d\beta), \end{aligned}$$

since $(d(\alpha\beta + \beta^2))^j = 0$ if $j \geq 2$.

But

$$\begin{aligned} (d^3 + d^2\alpha)^q &= \sum_{i=0}^q \binom{q}{i} (d^3)^{q-i} + (d^2\alpha)^i \\ &= d^{3q} + d^{3q-1}\alpha \end{aligned}$$

and

$$(d^3 + d^2\alpha)^{q-1} = \sum_{i=0}^{q-1} \binom{q-1}{i} (d^3)^{q-1-i} + (d^2\alpha)^i = d^{3q-1}$$

because q is odd and $\alpha^j = 0$ if $j > 1$.

Therefore,

$$\begin{aligned} Y &= (d^{3q} + d^{3q-1}\alpha + d^{3q-2}(\alpha\beta + \beta^2))(d^2 + d\beta) \\ &= d^{3q+2} + d^{3q+1}(\alpha + \beta) + d^{3q}\beta^2 + d^{3q-1}\alpha\beta^2 \\ &= d^t + d^{t-1}(\alpha + \beta) + d^{t-2}\beta^2 + d^{t-3}\alpha\beta^2, \end{aligned}$$

where $t = 3q + 2 = \mathcal{M}(n - 3)$.

Thus

$$\begin{aligned} Y \cdot W[n - 2]_3 \cdot d^{m-1-(t+6)} &= (d^t + d^{t-1}(\alpha + \beta) + d^{t-2}\beta^2 + d^{t-3}\alpha\beta^2) \cdot d^6 \\ &\quad \cdot d^{m-1-(t+6)} \\ &= d^{m-1} + d^{m-2}(\alpha + \beta) + d^{m-3}\beta^2 + d^{m-4}\alpha\beta^2. \end{aligned}$$

From the relation $d^{m-3} = d^{m-4}(\alpha + \beta) + d^{m-5}\alpha\beta$ one obtains $d^{m-1} = d^{m-2}\alpha + d^{m-2}\beta + d^{m-3}\alpha\beta$; replacing in the above expression, we get

$$Y \cdot W[n - 2]_3 \cdot d^{m-1-(t+6)} = d^{m-3}\alpha\beta + d^{m-3}\beta^2 + d^{m-4}\alpha\beta^2.$$

Again, from $d^{m-3} = d^{m-4}(\alpha + \beta) + d^{m-5}\alpha\beta$, one obtains $d^{m-2}\beta = d^{m-3}\alpha\beta + d^{m-3}\beta^2 + d^{m-4}\alpha\beta^2$. But, as seen before, $d^{m-2}\beta$, $d^{m-3}\alpha\beta$ and $d^{m-4}\alpha\beta^2$ are the top-dimensional generator, which means that $d^{m-3}\beta^2$ also is. It follows that

$$Y \cdot W[n - 2]_3 \cdot d^{m-1-(t+6)} = d^{m-3}\alpha\beta + d^{m-3}\beta^2 + d^{m-4}\alpha\beta^2 = d^{m-4}\alpha\beta^2,$$

which ends the proof.

4. An example with $m = \mathcal{M}(n - 3) + 5$

In this section we construct the example announced in the abstract, with $m = \mathcal{M}(n - 3) + 5$. Consider the vector bundle $\tau \otimes \xi_1 \mapsto \mathbb{R}P^2 \times \mathbb{R}P^1$, where τ is the tangent bundle over $\mathbb{R}P^2$ (again we are omitting pullback notations). Maintaining the notations of the previous section, one has

$$W(\tau \otimes \xi_1) = (1 + \alpha)^2 + (1 + \alpha)\beta + \beta^2,$$

which gives $w_1(\tau \otimes \xi_1) = \beta$ and $w_2(\tau \otimes \xi_1) = \alpha\beta + \beta^2$. Then $w_1(\mathbb{R}P^2 \times \mathbb{R}P^1).w_2(\tau \otimes \xi_1) = \alpha.\beta^2 \neq 0$, which means that $\tau \otimes \xi_1$ does not bound. Let E be the total space of the projective space bundle $\mathbb{R}P(\tau \otimes \xi_1)$ and $\lambda \mapsto E$ the usual line bundle. From the Conner–Floyd exact sequence of [4], $\tau \otimes \xi_1$ is the fixed-data of some involution (V^5, S) if, and only if, $\lambda \mapsto E$ bounds. Set $W(\lambda) = 1 + c$. One has

$$W(E) = (1 + \beta + \beta^2)((1 + c)^2 + (1 + c)\beta + \alpha\beta + \beta^2),$$

subject to the relation $c^2 = c\beta + \alpha\beta + \beta^2$. An easy calculation then shows that $W(E) = 1$. Thus the only relevant characteristic number of λ comes from c^4 . From the relation $c^2 = c\beta + \alpha\beta + \beta^2$, we get $c^4 = c^3\beta + c^2\alpha\beta + c^2\beta^2$, $c^2\beta^2 = 0$, $c^3\beta = c^2\beta^2 + c\alpha\beta^2$ and

$c^2\alpha\beta = \alpha\beta^2$. Since $\alpha\beta^2 \in H^4(E, \mathbb{Z}_2)$ is the (top-dimensional) generator, $c^3\beta$ and $c^2\alpha\beta$ also are. Then $c^4 = 0$ and $\lambda \mapsto E$ bounds, which then gives an involution (V^5, S) with nonbounding fixed-data $\tau \otimes \xi_1 \mapsto \mathbb{R}P^2 \times \mathbb{R}P^1$. For $n \geq 5$, now take the maximal involution $(M^{\mathcal{M}(n-3)}, T)$ of Stong and Pergher, with fixed point set of the form $F^{n-3} \cup \{point\}$. The product involution $(N = M^{\mathcal{M}(n-3)} \times V^5, T \times S)$ has $\dim(N) = \mathcal{M}(n-3) + 5$ and fixes the disjoint union $(F^{n-3} \times \mathbb{R}P^2 \times \mathbb{R}P^1) \cup (\mathbb{R}P^2 \times \mathbb{R}P^1)$. The normal bundle of $\mathbb{R}P^2 \times \mathbb{R}P^1$ in N is $(\tau \otimes \xi_1) \oplus \mathcal{M}(n-3)\mathbb{R}$, which does not bound. We assert that the normal bundle of $F^{n-3} \times \mathbb{R}P^2 \times \mathbb{R}P^1$ in N does not bound. Otherwise, after an equivariant surgery, it can be removed to give an involution (W, L) , where $\dim(W) = \mathcal{M}(n-3) + 5$ and the fixed-data is $(\tau \otimes \xi_1) \oplus \mathcal{M}(n-3)\mathbb{R} \mapsto \mathbb{R}P^2 \times \mathbb{R}P^1$. For $n \geq 5$ odd, the lesser value of $\mathcal{M}(n-3) + 5$ is 10, corresponding to $n = 5$. Then (W, L) has fixed set $\mathbb{R}P^2 \times \mathbb{R}P^1$ of constant dimension 3 and $\dim(W) \geq 10 > 6 = 2\dim(\mathbb{R}P^2 \times \mathbb{R}P^1)$. From the Kosniowski–Stong theorem cited in the first remark of Section 1, one then has that (W, L) bounds equivariantly, contradicting the fact that its fixed-data does not bound. Therefore $(N, T \times S)$ is the required example.

Remark. Theorem 1.1 leaves open the question of either to construct a maximal example, that is, with $m = \mathcal{M}(n-3) + 6$, or to improve the bound $m \leq \mathcal{M}(n-3) + 6$ to $m \leq \mathcal{M}(n-3) + 5$. Regarding to the first alternative, since the desired dimension $\mathcal{M}(n-3) + 6$ involves the Stong–Pergher number, it is difficult to try anything other than the procedure used above to get our almost maximal example, that is, an example of the form $(M^{\mathcal{M}(n-3)} \times V^6, T \times S)$, where S is a nonbounding involution defined on a 6-dimensional manifold V^6 fixing a three-dimensional manifold P^3 ; for example, in the case $F = F^n \cup F^2$, the best possible bound is $m \leq \mathcal{M}(n-2) + 4$ and a (simpler) maximal example is $(M^{\mathcal{M}(n-2)}, T) \times (\mathbb{R}P^2 \times \mathbb{R}P^2, twist)$ (see [7]). However, any involution (V^6, S) fixing some P^3 bounds equivariantly: again, this follows from the Kosniowski–Stong theorem cited in the first remark of Section 1. In fact, in this case (V^6, S) is equivariantly cobordant to $(P^3 \times P^3, twist)$, whose fixed-data is the tangent bundle $\tau^3 \rightarrow P^3$. Since any three-dimensional manifold bounds, $\tau^3 \rightarrow P^3$ bounds as a bundle, and thus (V^6, S) bounds. Therefore, we believe it is more plausible to try the second alternative; unfortunately, all efforts made in this direction have been unsuccessful so far.

Acknowledgments

We would like to thank the reviewers for suggestions that improved the content of this paper. The authors were partially supported by CNPq, FAPESP and CAPES.

References

- [1] E.M. Barbaresco, P.E. Desideri, P.L.Q. Pergher, Involutions whose fixed set has three or four components: a small codimension phenomenon, *Math. Scand.* 110 (2) (2012) 223–234.
- [2] J.M. Boardman, On manifolds with involution, *Bull. Amer. Math. Soc.* 73 (1967) 136–138.
- [3] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces, I, *Amer. J. Math.* 80 (1958) 458–538.
- [4] P.E. Conner, E.E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, Berlin, 1964.
- [5] P.E. Desideri, P.L.Q. Pergher, Improvements of the Five Halves Theorem of J. Boardman with respect to the decomposability degree, *Asian J. Math.* 18 (3) (2014) 427–438.
- [6] F.G. Figueira, P.L.Q. Pergher, Dimensions of fixed point sets of involutions, *Arch. Math. (Basel)* 87 (3) (2006) 280–288.
- [7] F.G. Figueira, P.L.Q. Pergher, Involutions fixing $F^n \cup F^2$, *Topology Appl.* 153 (14) (2006) 2499–2507.
- [8] F.G. Figueira, P.L.Q. Pergher, Two commuting involutions fixing $F^n \cup F^{n-1}$, *Geom. Dedicata* 117 (2006) 181–193.
- [9] F.G. Figueira, P.L.Q. Pergher, Bounds on the dimension of manifolds with involution fixing $F^n \cup F^2$, *Glasg. Math. J.* 50 (2008) 595–604.

- [10] S.M. Kelton, Involutions fixing $\mathbb{R}P^j \cup F^n$, *Topology Appl.* 142 (2004) 197–203.
- [11] S.M. Kelton, Involutions fixing $\mathbb{R}P^j \cup F^n$, II, *Topology Appl.* 149 (1–3) (2005) 217–226.
- [12] C. Kosniowski, R.E. Stong, Involutions and characteristic numbers, *Topology* 17 (1978) 309–330.
- [13] P.L.Q. Pergher, Bounds on the dimension of manifolds with certain Z_2 fixed sets, *Mat. Contemp.* 13 (1996) 269–275.
- [14] P.L.Q. Pergher, Involutions whose top dimensional component of the fixed point set is indecomposable, *Geom. Dedicata* 146 (1) (2010) 1–7.
- [15] P.L.Q. Pergher, An improvement of the Five Halves Theorem of J. Boardman, *Israel J. Math.* 11 (2011) 1–8.
- [16] P.L.Q. Pergher, Involutions fixing $F^n \cup \{Indecomposable\}$, *Canad. Math. Bull.* 55 (1) (2012) 164–171.
- [17] P.L.Q. Pergher, R.E. Stong, Involutions fixing $\{point\} \cup F^n$, *Transform. Groups* 6 (2001) 78–85.
- [18] D.C. Royster, Involutions fixing the disjoint union of two projective spaces, *Indiana Univ. Math. J.* 29 (2) (1980) 267–276.