JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **126**, 483–493 (1987)

Some Theorems Concerning Holomorphic Fourier Transforms

R. A. ZALIK

Department of Algebra, Combinatorics, and Analysis, Division of Mathematics, 120 Mathematics Annex, Auburn University, Alabama 36849-3501

AND

T. ABUABARA SAAD*

Department of Mathematics, UNESP at Rio Claro, 13500 Rio Claro, Sao Paulo, Brazil

Submitted by R. P. Boas

Received July 21, 1986

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In the sequel t, x, and y will denote real numbers, and $z = x + iy$. Given a function f, by $\hat{f}$ we shall denote its Fourier transform. Thus, if f is in $L_1(-\infty, \infty)$, $\hat{f}(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) \exp(ixt)\, dt$. Given p and α, q, and β will denote their conjugates (i.e., $p^{-1} + q^{-1} = 1$, $\alpha^{-1} + \beta^{-1} = 1$), and $K(\beta, x) = \beta^{-1}(\alpha x)^{-\beta/\alpha}$. By $\|f\|_p$ we shall denote the $L_p(-\infty, \infty)$ norm of $f(x)$, and $\|f(x+iy)\|_{x,p}$ will stand for the $L_p(-\infty, \infty)$ norm of $f(x+iy)$ with respect to the variable x (i.e., $\|f(x+iy)\|_{x,p}$ is a function of y). We shall always assume that $\alpha > 1$ and $p \geqslant 1$.

In [1, Theorem 1] we proved a proposition which in the equivalent form can be stated as follows:

THEOREM. *Let $a, b > 0$. Then the following assertions are equivalent*:

(a) *$f(z)$ is an entire function and for every $\varepsilon > 0$ but for no $\varepsilon < 0$, $\|\exp[-(-b+\varepsilon)|t|^{\alpha}]\, f(t)\|_{\infty} < \infty$, and $\|\exp[-(a+\varepsilon)|y|^{\alpha}]\, f(x+iy)\|_{x,\infty}$ is uniformly bounded in y.*

(b) *$\hat{f}(z)$ is an entire function and for every $\varepsilon > 0$ but for no $\varepsilon > 0$, $\|\exp[-(-K(\beta, a)+\varepsilon)|t|^{\beta}]\, \hat{f}(t)\|_{\infty} < \infty$,*

and $\|\exp[-(K(\beta, b)+\varepsilon)|y|^{\beta}]\, \hat{f}(x+iy)\|_{x,\infty}$ is uniformly bounded in y.

* Supported by a scholarship from FAPESP, Sao Paulo, Brazil.

0022-247X/87 $3.00

This theorem is a refinement of previous work by Gel'fand and Šilov [2].

The purpose of this paper is to obtain results of the same type involving norms other than that of $L_\infty(-\infty, \infty)$ and to give some applications of these results to the study of the properties of sequences of translates and sequences of weighted exponentials. We start with

THEOREM 1. *Let* $1 \leqslant p \leqslant 2$, $a > 0$, *and assume that* $f(t)\exp(a|t|^\alpha)$ *is in* $L_p(-\infty, \infty)$. *Then* $\hat{f}$ *is an entire function and*

$$|y|^{(\beta/2)-1}\exp[-pK(\beta, a)|y|^\beta](\|\hat{f}(x+iy)\|_{x,q})^p \qquad \text{is in } L_1(-\infty, \infty).$$

Conversely we have

THEOREM 2. *Let* $f(z)$ *be an entire function*, $2 \leqslant p < \infty$, $a > 0$, *and assume that*

$$|y|^{(\beta/2)-1}\exp[-pK(\beta, a)|y|^\beta](\|f(x+iy\|_{x,q})^p \qquad \textit{is in } L_1(-\infty, \infty),$$

Then $\exp(a|t|^\alpha)\hat{f}(t)$ *is in* $L_p(-\infty, \infty)$.

For $p = 2$, Theorems 1 and 2 yield a necessary and sufficient condition:

THEOREM 3. *Let* $a > 0$. *Then the following propositions are equivalent*:

(a) $f(t)\exp(a|t|^\alpha)$ *is in* $L_2(-\infty, \infty)$

(b) $\hat{f}$ *is an entire function and*

$$|y|^{(\beta/2)-1}\exp[-2K(\beta, a)|y|^\beta](\|\hat{f}(x+iy)\|_{x,2})^2 \textit{ is in } L_1(-\infty, \infty).$$

We now turn to the applications of Theorem 3. A sequence of elements of a given topological vector space is called "minimal" if each element of the sequence lies outside the closure of the linear span of the others, and it is called "fundamental" (or "complete") if its linear span is dense in the given space (cf. [3]). We have

THEOREM 4. *Let* $a > 0$ *be given, let* $\{c_k;\ k = 0, 1, 2,...\}$ *be a sequence of complex numbers, and assume that both* $f(t)\exp(a|t|^\alpha)$ *and* $1/(f(t)\exp(a|t|^\alpha))$ *are essentially bounded on* $(-\infty, \infty)$. *Let S denote either of the sequences* $\{f(t)\exp(ic_k t)\}$ *or (if* $\{c_k\}$ *is a real sequence)* $\{\hat{f}(t+c_k)\}$. *Then S is minimal in* $L_2(-\infty, \infty)$ *if and only if there is an entire function* $h(z)$, *not identically, zero such that* $h(c_k) = 0$ *for all k and*

$$|y|^{(\beta/2)-1}\exp[-2K(\beta, a)|y|^\beta](\|(1+|z|)^{-1}h(x+iy)\|_{x,2})^2 \textit{ is in } L_1(-\infty, \infty).$$

Theorem 4 is similar to a result of Paley and Wiener for sequences of exponentials on bounded intervals (cf. [4, p. 419]) and has an analogous proof. It is well known that the sequence $\{\exp(int)\}$ is both fundamental and minimal in, say, $L_2[-\pi, \pi]$ (cf., [3, p. 113, Proposition 1]). The question thus naturally arises as to whether there is a sequence of weighted exponentials having a similar property on $L_2(-\infty, \infty)$. Before answering this question, we need to introduce some additional notation.

Let $c=(1+i)(2\pi)^{1/2}$, $c_n=c(n-1)^{1/2}$, $n=2, 3, 4,\ldots$, $c_n=-c_{-n}$, $n=-2, -3, -4,\ldots$, $S_0=\{c_n, n=\pm 2, \pm 3, \pm 4,\ldots\}$, and assume that a is a nonzero complex number such that neither a nor $-ia$ are in S_0. Let $c_1=a$, $c_{-1}=-a$, and let S denote the set of all points of the form c_n, $n=\pm 1, \pm 2, \pm 3,\ldots$, or ic_n, $n=\pm 2, \pm 3, \pm 4,\ldots$. Finally, $w(t)=\exp(-t^2/2)$, and for a given function g, $S(g)=\{g(t)\,w(t)\exp(idt), d\in S\}$. We shall use Theorem 4 to prove part (b) of

THEOREM 5. *If g is an essentially bounded even function, a.e. different from zero, then* (a) *$S(g)$ is fundamental in $L_2(-\infty, \infty)$,* (b) *$S(g)$ is minimal in $L_2(-\infty, \infty)$.*

Although the proof of Theorem 6 below does not use Theorem 4, the method employed is similar to the one used in the proof of Theorem 5(a), and this justifies its inclusion here.

If $c_n=(2\pi)^{1/2}\operatorname{sign}(n)|n|^{1/2}$, $S'=\{c_n\}$, and $S'(g)=\{g(t)\,w(t)\exp(dti), d\in S'\}$, we have

THEOREM 6. *If g is an essentially bounded function, a.e. different from zero, then $S'(g)$ is fundamental in $L_2(-\infty, \infty)$.*

For other results concerning the fundamentality of sequences of weighted exponentials and of sequences of translates see, e.g., [5–7], and references therein.

2. AUXILIARY PROPOSITIONS

If $\lim_{t\to\infty} f(t)/g(t)=1$, we shall write $f(t)\sim g(t)$, $t\to\infty$. With this notation we have

LEMMA 1. *Let $\gamma>-1$ and*

$$I(t)=\int_0^\infty y^\gamma \exp[p(yt-at^\alpha-K(\beta, a)\,y^\beta)]\,dy.$$

Then

$$I(t) \sim (\alpha/\beta)^{1/2}(2\pi/p)^{(1/2)(\beta c)^{-(\alpha/\beta)(\gamma+1/2)}} t^{(\alpha/\beta)(\gamma+1)-(\alpha/2)}, \qquad t \to +\infty,$$

where $c = K(\beta, a)$.

Proof. If $c = K(\beta, a)$, then $a = K(\alpha, c)$ and

$$I(t) = \int_0^\infty y^\gamma \exp[p(yt - K(\alpha, c)\, t^\alpha - cy^\beta)]\, dy.$$

Let $y = (\beta c)^{-(\alpha/\beta)} t^{\alpha/\beta} s$. Since $\alpha = (\alpha/\beta) + 1$, we see that

$$\begin{aligned} yt &= (\beta c)^{-(\alpha/\beta)} t^\alpha s \\ &= \beta^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha (\beta s) \end{aligned} \tag{1}$$

and

$$\begin{aligned} -cy^\beta &= -c(\beta c)^{-\alpha} t^\alpha s^\beta \\ &= -\beta^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha s^\beta. \end{aligned} \tag{2}$$

Moreover, since

$$(\beta/\alpha) = \beta - 1, \tag{3}$$

$$\begin{aligned} -K(\alpha, c)\, t^\alpha &= -\alpha^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha \\ &= -\beta^{-1}(\beta - 1)(\beta c)^{-(\alpha/\beta)} t^\alpha \\ &= \beta^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha (1 - \beta), \end{aligned} \tag{4}$$

combining (1), (2), and (4) we therefore see that

$$p(-cy^\beta - K(\alpha, c)\, t^\alpha + yt) = -p\beta^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha u(s),$$

where $u(s) = s^\beta - \beta s + \beta - 1$. Since clearly $dy/ds = (\beta c)^{-(\alpha/\beta)} t^{\alpha/\beta}$, we readily deduce that

$$\begin{aligned} I(t) &= (\beta c)^{-(\alpha/\beta)(\gamma+1)} t^{(\alpha/\beta)(\gamma+1)} \int_0^\infty s^\gamma \exp[-p\beta^{-1}(\beta c)^{-(\alpha/\beta)} t^\alpha u(s)]\, ds \\ &= (\beta c)^{-(\alpha/\beta)(\gamma+1)} t^{(\alpha/\beta)(\gamma+1)} q(t^\alpha), \end{aligned}$$

where

$$
\begin{aligned}
q(t) &= \int_0^\infty s^\gamma \exp[-p\beta^{-1}(\beta c)^{-(\alpha/\beta)} t u(s)]\, ds \\
&= \int_1^\infty s^\gamma \exp[-p\beta^{-1}(\beta c)^{-(\alpha/\beta)} t u(s)]\, ds \\
&\quad + \int_0^1 s^\gamma \exp[-p\beta^{-1}(\beta c)^{-(\alpha/\beta)} t u(s)]\, ds.
\end{aligned}
$$

Applying the method of Laplace to each of the integrals on the right-hand member of the preceding identity (cf., e.g., Widder [8, p. 278, Theorem 2b. and p. 279, Corollary 2b.2]), and applying (3), we readily infer that

$$q(t) \sim (\alpha/\beta)^{1/2}(2\pi/p)^{1/2}(\beta c)^{\alpha/(2\beta)} t^{-(1/2)}, \qquad t \to +\infty,$$

whence the conclusion follows. Q.E.D.

LEMMA 2. *Assume that* $1 < q \leqslant 2$ *and that* $f(z)$ *is an entire function, and let* $\|f(x+iy)\|_{x,q} < \infty$ *for almost every real* y. *Then* $\|\exp(yt)\hat{f}(t)\|_{t,p} \leqslant (2\pi)^{1/2} \|f(x+iy)\|_{x,q}$.

Proof. Assume first that $q = 2$. It is then clear that $\exp(yt)\hat{f}(t)$ is the Fourier transform of $f(x+iy)$. (To see this, note that $f(x+iy)$ is the Fourier transform of $\exp(yt)\hat{f}(t)$ evaluated at $-x$, and apply the inversion theorem.) Since the Fourier transform is an isometry in $L_2(-\infty, \infty)$, we see that $\|\exp(yt)\hat{f}(t)\|_{t,2} = \|f(x+iy)\|_{x,2}$, and the conclusion follows.

To prove the assertion in the general case, let $c > 0$, $g_c(t) = c\exp(-c^2t^2)$ and $f_c = g_c * f$ (where "$*$" denotes the convolution product). Since for any s in $[1, \infty]$, g_c is in $L_s(-\infty, \infty)$, setting in particular $s = (2q)/(3q-2)$ (whence $1/q + 1/s - 1 = \frac{1}{2}$) and applying a theorem of W. H. Young (cf., e.g., [9, p. 178, (1.1) or 10, p. 414, (21.56)]) we infer that $f_c(x+iy)$ is in $L_2(-\infty, \infty)$ for any real y, and therefore that $\exp(yt)\hat{f}_c(t)$ is the Fourier transform of $f_c(x+iy)$. Since g_c is also in $L_1(-\infty, \infty)$, we conclude in addition that $f_c(x+iy)$ is in $L_q(-\infty, \infty)$, for any real y. Applying the Hausdorff–Young inequality [9] we see that $\|\exp(yt)\hat{f}_c(t)\|_{t,p} \leqslant \|f_c(x+iy)\|_{x,q}$. Since $\hat{g}_c(t) = 2^{-1/2}\exp(-t^2/4c)$ (cf. [9, p. 6]), a second application of Young's theorem shows that: $2^{-1/2}\|\exp(yt)\exp(-t^2/4c)\hat{f}(t)\|_{t,p} \leqslant \|f_c(x+iy)\|_{x,q} \leqslant \|f(x+iy)\|_{x,q}$ $\|g_c(x)\|_1 = \pi^{1/2}\|f(x+iy)\|_{x,q}$. Making $c \to \infty$, the conclusion readily follows from Fatou's lemma. Q.E.D.

3. Proofs of the Main Results

Proof of Theorem 1. That $\hat{f}(z)$ is an entire function follows from a standard application of the theorems of Morera and Fubini, as has been done, e.g., in [11, pp. 403–404]. Clearly $\hat{f}(x+iy)$ is the Fourier transform of $f(t)\exp(-yt)$ evaluated at x. Applying the inequality of Hausdorff and Young, as in Lemma 2, we have:

$$\begin{aligned}
&(\|\hat{f}(x+iy)\|_{x,q})^p \\
&\quad\leqslant (\|f(t)\exp(-yt)\|_{t,p})^p \\
&\quad\leqslant \int_{-\infty}^{\infty} \exp[p(|yt| - a|t|^\alpha)]\,|\exp(a|t|^\alpha)|\,f(t)|^p\,dt \\
&\quad= \int_0^{\infty} \exp[p(|y|\,t - at^\alpha)]\exp(pat^\alpha)(|f(t)|^p + |f(-t)|^p)\,dt \\
&\quad= \int_0^{\infty} \exp[p(|y|\,t - at^\alpha)]\,g(t)\,dt,
\end{aligned}$$

where $g(t) = \exp(pat^\alpha)(|f(t)|^p + |f(-t)|^p)$. Hence, we have that

$$\begin{aligned}
&\int_{-\infty}^{\infty} |y|^{(\beta/2)-1}\exp[-pK(\beta,a)\,|y|^\beta](\|\hat{f}(x+iy)\|_{x,q})^p\,dy \\
&\quad\leqslant \int_{-\infty}^{\infty}\int_0^{\infty} |y|^{(\beta/2)-1}\exp[p(|y|\,t - at^\alpha - K(\beta,a)\,|y|^\beta)]\,g(t)\,dt\,dy \\
&\quad= 2\int_0^{\infty}\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt - at^\alpha - K(\beta,a)\,y^\beta)]\,g(t)\,dt\,dy.
\end{aligned}$$

Now, by the Fubini and Tonelli theorems (cf., e.g., [12 or 13]), the last expression is identical to

$$2\int_0^{\infty}\left(\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt - at^\alpha - K(\beta,a)\,y^\beta)]\,dy\right)g(t)\,dt,$$

i.e.,

$$\begin{aligned}
&\int_{-\infty}^{\infty} |y|^{(\beta/2)-1}\exp[-pK(\beta,a)\,|y|^\beta](\|\hat{f}(x+iy)\|_{x,q})^p\,dy \\
&\qquad\leqslant 2\int_0^{\infty} J(t)\,g(t)\,dt,
\end{aligned}$$

where

$$J(t)=\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt-at^{\alpha}-K(\beta, a)\,y^{\beta})]\,dy.$$

Applying Lemma 1 with $\gamma=(\beta/2)-1$ we have that

$$\lim_{t\to\infty} J(t)=(\alpha/\beta)^{1/2}(2\pi/p)^{1/2}(\beta c)^{-1/2}.$$

Hence $J(t)$ is bounded on $[0, \infty)$, and the conclusion follows. Q.E.D.

Proof of Theorem 2. From Lemma 2 we know that $\|\exp(yt)\hat{f}(t)\|_{t,p}\leqslant (2\pi)^{1/2}\|f(x+iy)\|_{x,q}$. Raising both sides of the preceding inequality to the power p and applying the hypotheses, we readily deduce that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y|^{(\beta/2)-1}\exp[-pK(\beta, a)|y|^{\beta}]\exp(p|yt|)|\hat{f}(t)|^{p}\,dt\,dy$$
$$\leqslant(2\pi)^{p/2}\int_{-\infty}^{\infty} |y|^{(\beta/2)-1}\exp[-pK(\beta, a)|y|^{\beta}](\|f(x+iy)\|_{x,q})^{p}\,dy<\infty,$$

and therefore, setting $g(t)=\exp(a|t|^{\alpha})(|\hat{f}(t)|^{p}+|\hat{f}(-t)|^{p})$,

$$\int_0^{\infty}\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt-at^{\alpha}-K(\beta, a)\,y^{\beta})]\,g(t)\,dt\,dy<\infty.$$

By the theorems of Fubini and Tonelli we have that

$$\int_0^{\infty}\left(\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt-at^{\alpha}-K(\beta, a)\,y^{\beta})]\,dy\right)g(t)\,dt<\infty.$$

i.e., $\int_0^{\infty} J(t)\,g(t)\,dt<\infty$, where $J(t)=\int_0^{\infty} y^{(\beta/2)-1}\exp[p(yt-at^{\alpha}-K(\beta, a)\,y^{\beta})]\,dy$. The proof will be complete if we can show that $J(t)$ is bounded away from zero on $(0, +\infty)$. But, by Lemma 1 with $\gamma=(\beta/2)-1$ we have that $\lim_{t\to+\infty} J(t)=(\alpha/\beta)^{1/2}(2\pi/p)^{1/2}(\beta c)^{-(1/2)}\neq 0$, and the conclusion readily follows. Q.E.D.

Proof of Theorem 4. Since $\hat{f}(x+c_k)$ is the Fourier transform of $\exp(ic_k t)f(t)$, and every function in $L_2(-\infty, \infty)$ is the Fourier transform of a function in $L_2(-\infty, \infty)$, we readily infer from the isometric character of the Fourier transform that only the sequence $\{f(t)\exp(ic_k t)\}$ need be considered.

If the system is minimal there is a bounded linear functional on $L_2(-\infty, \infty)$ that equals 1 for $f(t)\exp(ic_0 t)$ and vanishes for all the remaining functions $f(t)\exp(ic_k t)$. Thus there is a square-integrable function $g(t)$ such that

$$\int_R \exp(ic_k t) f(t)\, g(t)\, dt = \delta_{0k}, \qquad k = 0, 1, 2, \ldots,$$

where δ_{0k} is Kronecker's delta. Let $q(z)$ denote the Fourier transform of $f(t)\,g(t)$. Since $\exp(a|t|^\alpha) f(t)\, g(t)$ is in $L_2(-\infty, \infty)$, from Theorem 3 we deduce that $|y|^{(\beta/2)-1} \exp[-2K(\beta, a)|y|^\beta](\|q(x+iy)\|_{x,2})^2$ is in $L_1(-\infty, \infty)$, and setting $h(z) = (z - c_0)\, q(z)$ the conclusion readily follows.

Conversely, assume there is a function $h(z)$ that satisfies the hypotheses of the theorem. Since $h(z)$ does not vanish identically, all its zeros are of finite order. Let k be given, let $m \geqslant 1$ be the order of the zero c_k of $h(z)$, and define $G_k(z) = m!h(z)/[(z-c_k)^m h^{(m)}(c_k)]$ if $z \neq c_k$, and $G_k(c_k) = 1$. It is readily verified that $G_k(z)$ is an entire function.

Since $m \geqslant 1$, for $|y|$ sufficiently large we have that

$$(\|G_k(x+iy)\|_{x,2})^2 \leqslant c(\|(1+|z|)^{-1} h(x+iy)\|_{x,2})^2$$

for some constant $c > 0$, and as $G_k(z)$ is continuous is every neighborhood of the point c_k, from the hypothesis we readily infer that

$$|y|^{(\beta/2)-1} \exp[-2K(\beta, a)|y|^\beta](\|G_k(x+iy)\|_{x,2})^2 \qquad \text{is in } L_1(R).$$

Thus, if g_k denotes the inverse Fourier transform of G_k, we infer from Theorem 3(b) that $g_k(t)\exp(a|t|^\alpha)$ is in $L_2(-\infty, \infty)$. Setting $h_k(t) = g_k(t)/f(t) = [g_k(t)\exp(a|t|^\alpha)][\exp(-a|t|^\alpha)/f(t)]$ we readily see from the hypotheses on f that $h_k(t)$ is in $L_2(-\infty, \infty)$. Since $g_k(t) = f(t)\, h_k(t)$ we have

$$\int_R \exp(ic_j t) f(t)\, h_k(t)\, dt = \int_R \exp(ic_j t)\, g_k(t)\, dt = G_k(c_j) = \delta_{kj}.$$

We have therefore shown that for any k there is a bounded linear functional on $L_2(-\infty, \infty)$ that equals 1 for $f(t)\exp(ic_k t)$ and vanishes for all the remaining functions $f(t)\exp(ic_j t)$, and the conclusion follows.

Q.E.D.

Proof of Theorem 5. (a) Assume that $S(g)$ is not fundamental in $L_2(-\infty, \infty)$. Then there is a non-zero bounded linear functional on $L_2(-\infty, \infty)$ that vanishes for all the functions $\exp(idt)\, g(t)\, w(t)$, $d \in S$; thus there is a function $r(t)$ in $L_2(-\infty, \infty)$, not equivalent to zero, such that

$$\int_R \exp(idt)\, g(t)\, w(t)\, r(t)\, dt = 0, \qquad d \in S.$$

Let $H(z)$ be the Fourier transform of $g(t)\,w(t)\,r(t)$. Applying the Cauchy–Schwarz inequality we have

$$\begin{aligned}|H(z)|^2 &\leqslant (2\pi)^{-1}(\|r\|_2)^2 \int_R \exp(-2yt-t^2)\,|g(t)|^2\,dt\\ &= (2\pi)^{-1}(\|r\|_2)^2 \exp(y^2) \int_R \exp[-(y+t)^2]\,|g(t)|^2\,dt\\ &\leqslant A_1 \exp(y^2).\end{aligned}$$

Thus $|H(z)| \leqslant A_2 \exp(y^2/2)$. By an application of the Morera and Fubini theorems we see that $H(z)$ is an entire function. Since $g(t)\,w(t)\,r(t)$ is not equivalent to zero, it is clear that $H(z)$ does not vanish identically. Since $H(a)=H(-a)=0$, we conclude that $H_1(z)=(z^2-a^2)^{-1}z^2H(z)$ is entire and vanishes at zero and at all points of S except, perhaps, at $\pm a$. It is also clear that $|H_1(z)| \leqslant A_3 \exp(y^2/2)$. Thus, since the imaginary part of $(1+i)(2\pi)^{1/2}z$ is $(2\pi)^{1/2}(x+y)$, $|H_1[(1+i)(2\pi)^{1/2}z]| \leqslant A_3 \exp[\pi(x+y)^2]$. Let $Q(z)=\exp(i\pi z^2)\,H_1[(1+i)(2\pi)^{1/2}z]$. In view of the preceding remarks we see that $Q(z)$ is an entire function, $|Q(z)| \leqslant A_3 \exp(\pi|z|^2)$, and

$$Q(n^{1/2})=Q(in^{1/2})=0, \qquad n=1, 2,\ldots. \tag{5}$$

We shall consider two cases, according to whether $r(t)$ is (essentially) odd or not. Assume first that $r(t)$ is odd; we then see that $Q(z)$ is odd. Thus, if $\{z_n; n=0, 1, 2,\ldots\}$ is the set of zeros of $Q(z)$ that have a positive real part, or are purely imaginary and have a positive imaginary part, then every zero of $Q(z)$ is either at the origin, or equals z_n for some n, or equals $-z_n$ for some n. From (5), we also know that $\{z_n^2; n=0, 1, 2,\ldots]$ contains the set of all non-vanishing integers. Applying Hadamard's factorization theorem we readily infer that

$$Q(z)=Bz^{2m-1}\exp(az^2)\,\Pi[(1-z^2/z_n^2)\exp(z^2/z_n^2)],$$

where $m>0$ is an integer and $B\neq 0$; thus,

$$R(z)=z\exp(az)\,\Pi[(1-z/z_n^2)\exp(z/z_n^2)]$$

is an entire function that vanishes on the integers; moreover, since $|R(z^2)| \leqslant |B|^{-1}|z|\,|Q(z)|$ if $|z| \geqslant 1$, we readily deduce that $|R(z)| \leqslant C|z|^{1/2}\exp(\pi|z|)$. From a theorem of Valiron and Pólya (cf., e.g., [14, 9.4.2]), $R(z)=A\sin(\pi z)$; thus $Q(z)=K_1 z^{2m-3}\sin(\pi z^2)$, whence $H_1(z)=K_2 z^{2m-3}\exp(-z^2/4)\sinh(z^2/4)$, and therefore $H(z)=K_2 z^{2m-5}(z^2-a^2)\exp(-z^2/4)\sinh(z^2/4)$. Since $H(x)$ is the Fourier transform of a function in $L_1(-\infty,\infty)$, it must vanish at infinity, and therefore $m=1$. This implies that $H(z)$ has a singularity at $z=0$. Since $H(z)$ is an entire function, we have obtained a contradiction.

Assume now that $r(t)$ is not an odd function; since $r(t)$ is not equivalent to zero, we also conclude that $r(t)+r(-t)$ is not equivalent to zero. Since $d \in S$ also implies $-d \in S$, substituting $r(t)+r(-t)$ for $r(t)$ if necessary, we can assume without any loss of generality that $r(t)$ is even. This also implies that $Q(z)$ is even, and Hadamard's factorization theorem yields: $Q(z)=Bz^{2m}\exp(az^2)\,\Pi[(1-z^2/z_n^2)\exp(z^2/z_n^2)]$, $m>0$. Applying the Valiron–Pólya theorem in this case we readily conclude that $H(z)=K_2 z^{2m-4}(z^2-a^2)\exp(-z^2/4)\sinh(z^2/4)$. This implies that $H(x)$ does not vanish at infinity. Since $H(x)$ is the Fourier transform of a function in $L_1(-\infty,\infty)$, we have obtained a contradiction.

(b) Let $h(z)=z^{-2}(z^2-a^2)\exp(-z^2/4)\sinh(z^2/4)$, and $q(z)=2\exp(-z^2/4)\sinh(z^2/4)$. Then $|h(z)| \leqslant c\,|q(z)|$. Since $q^2(z) = 1 - 2\exp(-z^2/2)+\exp(-z^2)$, we infer that $\exp(-y^2)\,|q^2(z)| \leqslant \exp(-y^2)+2\exp[-(x^2+y^2)/2]+\exp(-x^2)$. It is therefore easy to see that $\exp(-y^2)(\|(1+|z|)^{-1}\,h(x+iy)\|_{x,2})^2$ is in $L_1(-\infty,\infty)$, and the conclusion follows from Theorem 4. Q.E.D.

Proof of Theorem 6. Assume that $S'(g)$ is not fundamental in $L_2(-\infty,\infty)$. Proceeding as in the proof of Theorem 5 we readily infer that there is a function $H(z)$, entire and not identically zero, such that $H(d)=0$ if $d \in S'$, and $|H(z)| \leqslant A\exp(y^2/2)$.

Setting $H_1(z)=H(z)\,H(iz)$, $Q(z)=H_1[(2\pi)^{1/2}z]$, and applying Hadamard's factorization theorem and the Valiron–Pólya theorem, as was done in the proof of Theorem 5, we easily see that $Q(z)=Bz^{-1}\sin(\pi z^2)$. This implies that all of the zeros of $H_1(z)$ must be of order 1. Since $Q(0)=0$, we see that $0=H_1(0)=[H(0)]^2$, and we have a contradiction.

Q.E.D.

References

1. R. A. Zalik, Remarks on a paper of Gel'fand and Šilov on Fourier transforms, *J. Math. Anal. Appl.* **102** (1984), 102–112.
2. I. M. Gel'fand and G. E. Šilov, Fourier transforms of rapidly increasing functions and questions of the uniqueness of the solution of Cauchy's problem, *Amer. Math. Soc. Transl. (2)* **5** (1957), 221–274. (Translation of *Uspehi Mat. Nauk* **8** (1953), 3–54.)
3. R. M. Young, "An Introduction to Nonharmonic Fourier Series," Academic Press, New York, 1980.
4. B. Ja. Levin, "Distribution of Zeros of Entire Functions," Translations of Math. Monographs Vol. 5, Amer. Math. Soc., Providence, RI, 1964.
5. B. Faxén, On approximation by translates and related problems in function theory, *Ark. Mat.* **19** (1981), 271–289.
6. R. A. Zalik, The fundamentality of sequences of translates, *in* "Approximation Theory," Vol. III (E. Cheney, Ed.), pp. 927–932, Academic Press, New York, 1980.
7. R. A. Zalik, The Müntz–Szász theorem and the closure of translates, *J. Math. Anal. Appl.* **82** (1981), 361–369.

8. D. V. Widder, "The Laplace Transform," Princeton Univ. Press, Princeton, NJ, 1946.
9. E. M. Stein and G. Weiss, "Introduction the Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1975.
10. E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Springer-Verlag, New York, 1965.
11. W. Rudin, "Real and Complex Analysis," 2nd ed., McGraw–Hill, New York, 1974.
12. N. Dunford and J. T. Schwartz, "Linear Operators," Part I, Interscience, New York, 1957.
13. M. Cotlar and R. Cignoli, "An Introduction to Functional Analysis," North–Holland/American Elsevier, New York, 1974.
14. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.