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Lefschetz coincidence class for several maps

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To Professor Carlos Biasi

Abstract. The aim of this paper is to define a Lefschetz coincidence class for several maps. More specifically, for maps $f_1, \dots, f_k : X \to N$ from a topological space X into a connected closed n-manifold (even nonorientable) N, a cohomological class

$$L(f_1, \dots, f_k) \in H^{n(k-1)}\big(X; (f_1, \dots, f_k)^*(R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)\big)$$

is defined in such a way that $L(f_1, \dots, f_k) \neq 0$ implies that the set of coincidences

$$\mathrm{Coin}(f_1, \dots, f_k) = \{x \in X \mid f_1(x) = \cdots = f_k(x)\}$$

is nonempty.

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1. Introduction

In [1], a Lefschetz coincidence class is defined for continuous functions,

$$f_1, \dots, f_k : X \to N,$$

from a topological space into a closed connected oriented n-manifold, where $k \geq 2$. Such class, $\mathcal{L}(f_1, \dots, f_k)$, lives in $H^{n(k-1)}(X; \mathbb{Z})$ and if $\mathcal{L}(f_1, \dots, f_k) \neq 0$, then there is $x \in X$ such that $f_1(x) = f_2(x) = \cdots = f_k(x)$. Accurately,

$$\begin{aligned}&\mathcal{L}(f_1, \dots, f_k)\\ &\quad = (f_1, f_2)^*(j^*(\mu)) \smile (f_2, f_3)^*(j^*(\mu)) \smile \cdots \smile (f_{k-1}, f_k)^*(j^*(\mu)),\end{aligned}$$

where $\mu \in H^n(N \times N, N \times N \setminus \Delta; \mathbb{Z})$ is the Thom class of the oriented manifold N and $j : N \times N \hookrightarrow (N \times N, N \times N \setminus \Delta)$ is the inclusion. In [3], the authors considered a Lefschetz coincidence number for maps $f_1, f_2 : M \to N$ between closed manifolds of the same dimension, not necessarily orientable, using twisted coefficients and assuming f_2 orientation true, that is, a loop α in M preserves local orientation if and only if the loop $f_2 \circ \alpha$ preserves local orientation. In this work, using twisted coefficients, we present an extension of the definition of $\mathcal{L}(f_1, \ldots, f_k)$ given in [1] to the case where N is nonorientable. In order to construct our Lefschetz class, which we denote by $L(f_1, \ldots, f_k)$, we consider the composition

$$X \xrightarrow{(f_1,\ldots,f_k)} N^k \xrightarrow{\quad i \quad} (N^k, N^k \setminus \Delta_k(N)),$$

where

$$\Delta_k(N) = \big\{(x, \ldots, x) \in N^k \mid x \in N\big\}$$

is the diagonal in N^k and

$$i : N^k \to (N^k,\ N^k \setminus \Delta_k(N))$$

is the inclusion. Let $\xi_k(N)$ be the fiber bundle pair given by

$$(N^k, N^k \setminus \Delta_k(N)) \xrightarrow{\pi_1} N,$$

where π_1 is the projection onto the first factor of N^k. Thus, the fiber over $x \in N$ is

$$F_x = \{x\} \times \big(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}\big).$$

In [6] it was proved that $\xi_2(N)$ has a unique Thom class

$$\mu \in H^n\big(N \times N, N \times N \setminus \Delta(N); R \times \Gamma_N^*\big),$$

where R is a principal ideal domain and Γ_N is the orientation system (over R) of N. Similarly, one can prove that $\xi_k(N)$ has a unique Thom class

$$\mu_k \in H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

for each $k \geq 2$. We define

$$L(f_1, \ldots, f_k) := (f_1, \ldots, f_k)^*(i^*(\mu_k))$$

which is an element of $H^{n(k-1)}(X; (f_1, \ldots, f_k)^*(R \times \Gamma_N^* \times \cdots \times \Gamma_N^*))$. In Section 4 we prove that the above class is given by the cup product

$$L(f_1, \ldots, f_k) = L(f_1, f_2) \smile L(f_1, f_3) \smile \cdots \smile L(f_1, f_k).$$

We also show that whenever N is R-oriented, our definition coincides with that found in [1]. In Section 5 we focus on the case where N is the real projective n-space, n even. We prove that, in such case, $L(f_1, f_2, \ldots, f_k)$ does not depend on f_1.

For products in cohomology, we are following [5].

2. System of orientation

Throughout this paper, R denotes a principal ideal domain.

An n-manifold means a paracompact Hausdorff space having an open covering of coordinate neighborhoods each homeomorphic to $\mathbb{R}^n$.

For definitions of local system and of the homology and cohomology with coefficients in a local system see [6] or [9].

Given a local system Γ on a topological space X, we denote by Γ^* the local system $\mathrm{Hom}(\Gamma, R)$ on X. Given a local system Γ on X and a local system Γ' on Y, we denote by $\Gamma \times \Gamma'$ the local system on $X \times Y$ defined by

$$(\Gamma \times \Gamma')(x, y) = \Gamma(x) \otimes \Gamma(y)$$

for $(x, y) \in X \times Y$ and

$$(\Gamma \times \Gamma')(\omega_1, \omega_2) = \Gamma(\omega_1) \otimes \Gamma'(\omega_2)$$

for a path (ω_1, ω_2) in $X \times Y$.

Let N be an n-manifold. A small cell of N is defined to be a subset C having an open neighborhood V such that (V, C) is homeomorphic to $(\mathbb{R}^n, E^n)$, where $E^n = \{z \in \mathbb{R}^n \mid \|z\| \leq 1\}$.

For our purposes, we will consider Γ_N the *orientation system (over R)* of N. In such system, for each $x \in N$, $\Gamma_N(x) = H^n(N, N \setminus x; R)$ and if ω is a path in N, the definition of $\Gamma_N(\omega)$ is given by the following: Let $\{C\}$ be a family of small cells of N whose interiors cover N and such that if $C, C' \in \{C\}$ and $C \cap C' \neq \emptyset$, then $C \cup C'$ is contained in some small cell of N. Given a path $\omega : I \to N$, let $0 = t_0 < t_1 < \cdots < t_m = 1$ be points of I such that for $1 \leq i \leq m$ there is some $C_i \in \{C\}$ with $\omega([t_{i-1}, t_i]) \subset C_i$. Then the composite isomorphism

$$\begin{aligned} H^n(N, N \setminus \omega(0); R) &\xrightarrow{\approx} H^n(N, N \setminus C_1; R) \xleftarrow{\approx} H^n(N, N \setminus \omega(t_1); R) \\ &\xrightarrow{\approx} \cdots \xrightarrow{\approx} H^n(N, N \setminus C_m; R) \xleftarrow{\approx} H^n(N, N \setminus \omega(1); R) \end{aligned}$$

is independent of the choice of the points $\{t_i\}$ and the collection $\{C\}$ and is defined to be $\Gamma_N(\omega)$. When $R = \mathbb{Z}$ we will use the notation $\mathcal{O}_N$ instead of Γ_N.

Another way to define $\Gamma_N(\omega)$ is the following.

Lemma 2.1. *Let $\omega : [0, 1] \to N$ be a path and $F : N \times I \to N$ an isotopy such that $F(x, 0) = x$ for all $x \in N$ and $F(\omega(0), t) = \omega(t)$ for all $t \in [0, 1]$. Then $\Gamma_N(\omega) = (F(\,\cdot\,, 1)^*)^{-1}$.*

Proof. Let C be a small cell in N such that $\omega(0) \in \mathrm{int}\, C$. We can find a partition

$$0 = s_0 < s_1 < \cdots < s_K = 1$$

of $[0, 1]$ such that for all $k \in \{0, 1, \ldots, K - 1\}$ we have

(a) $\omega([s_k, s_{k+1}]) \subset C_k := F(C, s_k)$,
(b) $\omega(s_{k+1}) \in F(C, t)$ for each $t \in [s_k, s_{k+1}]$.

For each $s \in [0, 1]$, let $G^s : N \times [0, 1] \to N$ be the isotopy defined by

$$G^s(x, t) = F(F_s^{-1}(x), t),$$

where $F_s : N \to N$ is the homeomorphism given by $F_s(x) = F(x, s)$. We have

(c) $G^s(x, s) = x$ for all $x \in N$,
(d) $G^s(F_s(x), t) = F(x, t)$ for all $(x, t) \in N \times [0, 1]$.

For each $k \in \{0, 1, \ldots, K-1\}$, from (b) it follows that G^{s_k} defines a homotopy

$$(N, N \setminus C_k) \times [s_k, s_{k+1}] \to (N, N \setminus \omega(s_{k+1})).$$

Note that this homotopy connects the maps $i^k_{s_{k+1}}$ and $G^{s_k}(\,\cdot\,, s_{k+1}) \circ i^k_{s_k}$, where

$$G^{s_k}(\,\cdot\,, s_{k+1}) : (N, N \setminus \omega(s_k)) \to (N, N \setminus \omega(s_{k+1})),$$

and $i^k_{s_k} : (N, N \backslash C_k) \hookrightarrow (N, N \backslash \omega(s_k))$, $i^k_{s_{k+1}} : (N, N \backslash C_k) \hookrightarrow (N, N \backslash \omega(s_{k+1}))$ are the inclusions. Hence,

$$\left(G^{s_k}(\,\cdot\,, s_{k+1})^*\right)^{-1} = \left((i^k_{s_{k+1}})^*\right)^{-1} \circ \left(i^k_{s_k}\right)^*.$$

Thus

$$\begin{aligned}
\Gamma_N(\omega) &= \left[\left(\left(i^{K-1}_{s_K}\right)^*\right)^{-1} \circ \left(i^{K-1}_{s_{K-1}}\right)^*\right] \circ \cdots \circ \left[\left(\left(i^0_{s_1}\right)^*\right)^{-1} \circ \left(i^0_{s_0}\right)^*\right] \\
&= \left(G^{s_{K-1}}(\,\cdot\,, s_K)^*\right)^{-1} \circ \cdots \circ \left(G^{s_0}(\,\cdot\,, s_1)^*\right)^{-1} \\
&= \left(G^{s_0}(\,\cdot\,, s_1)^* \circ \cdots \circ G^{s_{K-1}}(\,\cdot\,, s_K)^*\right)^{-1} \\
&= \left((G^{s_{K-1}}(\,\cdot\,, s_K) \circ \cdots \circ G^{s_0}(\,\cdot\,, s_1))^*\right)^{-1}.
\end{aligned}$$

Now, note that

$$G^{s_{K-1}}(\,\cdot\,, s_K) \circ \cdots \circ G^{s_0}(\,\cdot\,, s_1) = F(\,\cdot\,, s_K) = F(\,\cdot\,, 1).$$

Therefore,

$$\Gamma_N(\omega) = (F(\,\cdot\,, 1)^*)^{-1}.$$

□

The next lemma shows the existence of an isotopy such that $F(x, 0) = x$ for all $x \in N$ and $F(\omega(0), t) = \omega(t)$. Its statement and proof are adaptations of [8, Lemma 6.4, p. 150].

Lemma 2.2. *Let $\omega : I \to N$ be a path in N. Then, there is an isotopy $F : N \times I \to N$ such that $F(x, 0) = x$ for all $x \in N$ and $F(\omega(0), t) = \omega(t)$ for all $t \in I$.*

Proof. First, consider the case where $\omega(I)$ is contained in a euclidean neighborhood U. Let $h : U \to E^n \backslash S^{n-1}$ be a homeomorphism. Let $g : E^n \backslash S^{n-1} \to \mathbb{R}^n$ be the homeomorphism given by

$$g(z) = \frac{z}{1 - |z|},$$

whose inverse map is given by

$$g^{-1}(y) = \frac{y}{1 + |y|}.$$

Let $\beta : I \to \mathbb{R}^n$ be the path $\beta(t) = g(h(\omega(t)))$ between $g(h(\omega(0)))$ and $g(h(\omega(1)))$. Let $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ be the homotopy given by

$$F(y, t) = f_t(y) := \beta(t) + y - g(h(\omega(0)))$$

Note that F is an isotopy between the identity map and the translation

$$y \mapsto y + (g(h(\omega(1))) - g(h(\omega(0)))).$$

Thus, for each $t \in I$ we have a homeomorphism

$$g^{-1} \circ f_t \circ g : E^n \setminus S^{n-1} \to E^n \setminus S^{n-1}$$

with $g^{-1} \circ f_0 \circ g = \mathrm{id}$ and $g^{-1} \circ f_t \circ g(h(\omega(0))) = h(\omega(t))$. Note that, for each t, the homeomorphism $g^{-1} \circ f_t \circ g : E^n \setminus S^{n-1} \to E^n \setminus S^{n-1}$ can be extended to a homeomorphism from E^n over E^n defining such extension as being the identity map over the boundary. Now, define the isotopy $h_t : N \to N$ by

$$h_t(x) = \begin{cases} x & \text{if } x \in N \setminus U, \\ h^{-1} \circ g^{-1} \circ f_t \circ g(x) & \text{if } x \in U. \end{cases}$$

Such isotopy h_t satisfies the required conditions.

Now, consider $\omega(I)$ covered by the euclidean neighborhoods $U_1, \dots, U_k$ and let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of the interval I such that $\omega([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \dots, k$. Suppose, by induction, that it is defined an isotopy $F : N \times [0, t_{k-1}] \to N$ such that

$$\begin{aligned} F(x, 0) &= x && \text{for all } x \in N, \\ F(\omega(0), t) &= \omega(t) && \text{for all } t \in [0, t_{k-1}], \\ F(x, t) &= x && \text{if } x \notin U_1 \cup \cdots \cup U_{k-1}. \end{aligned}$$

From the previous step, there is an isotopy $H : N \times [t_{k-1}, 1] \to N$ with

$$\begin{aligned} H(x, t_{k-1}) &= x && \text{for all } x \in N, \\ H(\omega(t_{k-1}), t) &= \omega(t) && \text{for all } t \in [t_{k-1}, 1], \\ H(x, t) &= x && \text{if } x \notin U_k. \end{aligned}$$

Define $G : N \times I \to N$ by

$$G(x, t) = \begin{cases} F(x, t) & \text{if } t \in [0, t_{k-1}], \\ H(F(x, t_{k-1}), t) & \text{if } t \in [t_{k-1}, 1]. \end{cases}$$

The map G is an isotopy such that

$$\begin{aligned} G(x, 0) &= x && \text{for all } x \in N, \\ G(\omega(0), t) &= \omega(t) && \text{for all } t \in I, \end{aligned}$$

and if $x \notin U_1 \cup \cdots \cup U_{k-1} \cup U_k$, then $G(x, t) = x$. □

For any $x \in N$ there is the canonical generator $z_{x,N}$ of $H_n(N, N \setminus x; \Gamma_N)$ (cf. [3, p. 5]) induced by the relative cycle $g_\sigma \sigma$ defined by

(a) $\sigma : \Delta^n \to N$ is an embedding with $x = \sigma(p)$, $p \in \mathrm{int}\,\Delta^{\mathrm{n}}$,
(b) $g_\sigma \in \Gamma_N(\sigma)$ is the section such that $g_\sigma(p)$ is the generator of

$$\Gamma_N(x) = H^n(N, N \setminus x; \mathbb{R})$$

induced by the relative singular cocycle dual to the relative singular cycle 1σ, where $1 \in R$.

(Here, we use the description of singular homology with local coefficients given in [7].) Note that $z_{x,N}$ does not depend on the choice of σ.

Lemma 2.3 (See [3, Lemma 3.1]**).** *For any compact set $A \subset N$ there exists a unique element $z_{A,N} \in H_n(N, N \setminus A; \Gamma_N)$ such that for any $x \in A$ the natural homomorphism $H_n(N, N \setminus A; \Gamma_N) \to H_n(N, N \setminus x; \Gamma_N)$ sends $z_{A,N}$ to $z_{x,N}$.*

Corollary 2.4 (Existence of fundamental class). *If N is compact, then there is a unique element $z_N \in H_n(N; \Gamma_N)$, called* fundamental class*, such that for any $x \in N$ the natural homomorphism $H_n(N; \Gamma_N) \to H_n(N, N \setminus x; \Gamma_N)$ sends z_N to $z_{x,N}$.*

Let R_N be an arbitrary local coefficient system over a closed n-manifold N with typical group R. Then the cap product with the fundamental class z_N give us the *Poincaré duality*

$$H^j(N; R_N) \xrightarrow{\simeq} H_{n-j}(N; \mathcal{O}_N \otimes R_N) \tag{2.1}$$

(see [6, Theorem 6.1, p. 107] or [2, Theorem 9.3, p. 330]).

Let us consider the fiber bundle pair $\xi_k(N)$ given by

$$(N^k, N^k \setminus \Delta_k(N)) \xrightarrow{\pi_1} N,$$

where π_1 is the projection onto the first factor of N^k,

$$\Delta_k(N) = \{(x_1, \ldots, x_k) \in N^k \mid x_1 = \cdots = x_k\}$$

is the *kth diagonal* of N^k and the fiber over $x \in N$ is

$$F_x = \{x\} \times \left(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}\right).$$

A *Thom class of the bundle* $\xi_k(N)$ is an element

$$\mu \in H^{n(k-1)}\left(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\right)$$

such that for all $x \in N$, the restriction

$$\mu|F_x \in H^{n(k-1)}\left(F_x; R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\right)$$

is dual to the generator $z_{x^{k-1},N^{k-1}} \in H_{n(k-1)}(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}; \Gamma_{N^{k-1}})$, that is,

$$\mu|F_x/\ (z_{x^{k-1},N^{k-1}}) = 1 \in H^0(x; R)$$

for all $x \in N$, where / denotes the slant product.

In [6] it was proved that $\xi_2(N)$ has a unique Thom class. Similarly, one can prove that $\xi_k(N)$ has a unique Thom class for all $k \geq 2$.

3. Properties of the Thom class of $\xi_k(N)$

The fiber bundle pair $\xi_k(N)$ is said to be orientable over R if there exists an element

$$U \in H^{n(k-1)}\left(N^k, N^k \setminus \Delta_k(N); R\right)$$

such that for all $x \in N$, the restriction

$$U|\{x\} \times \left(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}\right)$$

is a generator of $H^{n(k-1)}(\{x\} \times (N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}); R)$. Such a cohomology class U is called an orientation of $\xi_k(N)$ over R. For $R = \mathbb{Z}$ we simply say that $\xi_k(N)$ is orientable instead of orientable over $\mathbb{Z}$.

Let $\omega : I \to N$ be a path. For each $x \in N$, let

$$F_x = \{x\} \times \big(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}\big)$$

be the fiber of $\xi_k(N)$ over x. There is a map

$$G : F_{\omega(0)} \times I \to (N^k, N^k - \Delta_k(N))$$

such that $\pi_1(G(x,t)) = \omega(t)$ and $G(x,0) = x$ for $x \in \omega(0) \times N^{k-1}$ and $t \in I$. Indeed, we can consider an isotopy $F : N \times I \to N$ as in Lemma 2.2 and we define

$$G\big(\omega(0), x_2, \dots, x_k, t\big) = \big(F(\omega(0), t), F(x_2, t), \dots, F(x_k, t)\big).$$

Let us consider the map

$$g := G(\,\cdot\,, 1) : F_{\omega(0)} \to F_{\omega(1)}.$$

Let $[g] \in [F_{\omega(0)}, F_{\omega(1)}]$ be the homotopy class of g. The association of the path class $[\omega]$ with the homotopy class $[g]$ is a well-defined correspondence (see [5, Theorem 12, p. 101]). Let $h^k[\omega] = [g]$ and let $h^k[\omega]^*$ denote the homomorphism g^* induced by g, from $H^{n(k-1)}(F_{\omega(1)}; R)$ into $H^{n(k-1)}(F_{\omega(0)}; R)$. From [5], we have the following theorem.

Theorem 3.1 (cf. [5, Theorem 19, p. 263]**).** *The fiber bundle pair $\xi_k(N)$ is orientable over R if and only if*

$$h^k[\omega]^* : H^{n(k-1)}(F_{\omega(0)}; R) \to H^{n(k-1)}(F_{\omega(0)}; R)$$

is the identity homomorphism for every closed path ω in N.

Remark 3.2. A connected n-manifold X is said to be orientable (over R) if there exists an element $U \in H^n(X \times X, X \times X \setminus \Delta(X); R)$ such that for all $x \in X$, $U|\{x\} \times (X, X \setminus x)$ is a generator of $H^n(\{x\} \times (X, X \setminus x))$. Such a cohomology class U is called an orientation of X (see [5, p. 294]). Thus, saying the manifold X is orientable (over R) is the same as saying the fiber bundle pair $\xi_2(X)$ is orientable (over R). Moreover, X is orientable (over R) if and only if the orientation system Γ_N (over R) is constant.

Now, we are able to prove the following theorem.

Theorem 3.3. *The fiber bundle pair $\xi_k(N)$ satisfies the following conditions:*

(a) *for k odd, $\xi_k(N)$ is orientable (over arbitrary R);*
(b) *for k even, $\xi_k(N)$ is orientable (over R) if and only if N is orientable (over R).*

Proof. Following Theorem 3.1, we need to analyze the homomorphisms

$$h^k[\omega]^* : H^{n(k-1)}(F_{\omega(0)}; R) \to H^{n(k-1)}(F_{\omega(0)}; R)$$

for every closed path $\omega : I \to N$.

For each $x \in N$, the fiber of $\xi_k(N)$ over x is given by

$$\begin{aligned} F_x &= \{x\} \times \big(N^{k-1}, N^{k-1} \setminus \{x\}^{k-1}\big) \\ &= \{x\} \times \underbrace{(N, N \setminus x) \times (N, N \setminus x) \times \cdots \times (N, N \setminus x)}_{(k-1)\text{ times}}. \end{aligned}$$

By the Kunneth formula,

$$H^{n(k-1)}(F_x;R) = H^0(x;R) \otimes \underbrace{H^n(N, N\setminus x;R) \otimes \cdots \otimes H^n(N, N\setminus x;R)}_{(k-1)\text{ times}}.$$

With this, we can see that, for each path $\omega : I \to N$,

$$\begin{aligned} h^k[\omega]^* &= \mathrm{id} \otimes \underbrace{F(\,\cdot\,,1)^* \otimes \cdots \otimes F(\,\cdot\,,1)^*}_{(k-1)\text{ times}} \\ &= \mathrm{id} \otimes \underbrace{\Gamma_N(\omega)^{-1} \otimes \cdots \otimes \Gamma_N(\omega)^{-1}}_{(k-1)\text{ times}}. \end{aligned}$$

Since

$$h^k[\omega]^* = \mathrm{id} \otimes \underbrace{\Gamma_N(\omega)^{-1} \otimes \cdots \otimes \Gamma_N(\omega)^{-1}}_{(k-1)\text{ times}}$$

and $\Gamma_N(\omega) = \pm\,\mathrm{id}$, for k odd, $h^k[\omega]^*$ is always the identity homomorphism. Therefore, for k odd, $\xi_k(N)$ is orientable (over arbitrary R).

If N is orientable over R, then $\Gamma_N(\omega)$ is the identity homomorphism for every closed path ω in N. It follows that $h^k[\omega]^*$ is the identity homomorphism for every closed path ω in N. Therefore, if N is orientable over R, then $\xi_k(N)$ is orientable over R for arbitrary k.

If N is nonorientable over R, then there is a closed path ω in N such that $\Gamma_N(\omega) = -\,\mathrm{id}$. It follows that for k even, $h^k[\omega]^* = -\,\mathrm{id}$. Therefore, if N is nonorientable over R, then $\xi_k(N)$ is nonorientable over R for every k even. ☐

Lemma 3.4. *Let N be a compact manifold. Then there exists a neighborhood V of $\Delta_k(N)$ in N^k such that the projections $\pi_1|V, \ldots, \pi_k|V : V \to N$ are homotopic relatively to $\Delta_k(N)$.*

Proof. The proof is analogous to that of [8, Lemma 6.15, p. 164]. ☐

Theorem 3.5. *Let N be a closed n-manifold and suppose that $\xi_k(N)$ is orientable over R. Let*

$$U \in H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R\big)$$

be an orientation. Then, there is an isomorphism between

$$H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R\big)$$

and

$$H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

which maps U onto the Thom class

$$\mu \in H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

of $\xi_k(N)$.

Proof. We will show that restriction of $\Gamma := R \times \Gamma_N^* \times \cdots \times \Gamma_N^*$ to a neighborhood of $\Delta_k(N)$ is a constant system. We can suppose N connected. Let V be a neighborhood (connected) of $\Delta_k(N)$ in N^k such that the projections $\pi_1|V, \dots, \pi_k|V : V \to N$ are homotopic relatively to $\Delta_k(N)$. By excision, we have the isomorphism

$$H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); \Gamma\big) \xrightarrow{\approx} H^{n(k-1)}\big(V, V \setminus \Delta_k(N); \Gamma|_V\big).$$

In order to know the behavior of the local system Γ on V, we just need to know the action of the fundamental group $\pi_1(V; (x_1, \dots, x_k))$ over R with respect to such local system for a point $(x_1, \dots, x_k) \in V$ (see [9, Theorems 1.11 and 1.12, p. 263]. Thus, let us consider a point $(x, \dots, x) \in \Delta_k(N) \subset V$. By Lemma 3.4, each closed path α in V with base point in $\Delta_k(N)$ is homotopic, relatively to the end points, to a closed path in $\Delta_k(N)$. Let $\alpha = (\beta, \dots, \beta)$ be a closed path based on $(x, \dots, x)$. Since $\xi_k(N)$ is orientable over R, by Theorem 3.1, $h^k[\beta]^* = \mathrm{id}$. Moreover, in the proof of Theorem 3.3, we saw that

$$h^k[\beta]^* = \mathrm{id} \otimes \underbrace{\Gamma_N(\beta)^{-1} \otimes \cdots \otimes \Gamma_N(\beta)^{-1}}_{(k-1)\text{ times}}.$$

On the other hand, by definition,

$$\Gamma(\alpha) = \mathrm{id} \otimes \underbrace{\Gamma_N^*(\beta) \otimes \cdots \otimes \Gamma_N^*(\beta)}_{(k-1)\text{ times}}.$$

Since $\Gamma_N(\beta) = \pm\,\mathrm{id}$ and $\Gamma_N^*(\beta) = \mathrm{Hom}(\Gamma_N(\beta), R)$, it follows that $\Gamma(\alpha)$ is the identity isomorphism.

We conclude that the action of the fundamental group $\pi_1(V; (x, \dots, x))$ over R with respect to the local system Γ is trivial. Hence, there is an isomorphism between

$$H^{n(k-1)}(V, V \setminus \Delta_k(N); \Gamma|_V)$$

and

$$H^{n(k-1)}(V, V \setminus \Delta_k(N); R).$$

It follows that there is an isomorphism between

$$H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

and

$$H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R\big),$$

and we can take such isomorphism sending the Thom class μ of the bundle $\xi_k(N)$ onto the element U. □

Corollary 3.6. *If k is odd, then $H^{n(k-1)}(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)$ is isomorphic to $H^{n(k-1)}(N^k, N^k \setminus \Delta_k(N); R)$.*

Proof. It is an immediate consequence of Theorems 3.5 and 3.3. □

Let $k, l \geq 2$ and let

$$e : \big(N^{k+l-1}, N^{k+l-1} \setminus \Delta_{k+l-1}(N)\big) \to \big(N^k, N^k \setminus \Delta_k(N)\big) \times \big(N^l, N^l \setminus \Delta_l(N)\big)$$

be the map defined by

$$e(x_1, \dots, x_{k+l-1}) = \big((x_1, \dots, x_k), (x_1, x_{k+1}, \dots, x_{k+l-1})\big).$$

Note that the local system

$$e^*\Bigg((R \times \underbrace{\Gamma_N^* \times \cdots \times \Gamma_N^*}_{(k-1)\text{ times}}) \times (R \times \underbrace{\Gamma_N^* \times \cdots \times \Gamma_N^*}_{(l-1)\text{ times}})\Bigg)$$

is isomorphic to the local system

$$R \times \underbrace{\Gamma_N^* \times \cdots \times \Gamma_N^*}_{(k+l-2)\text{ times}}$$

over $(N^{k+l-1}, N^{k+l-1} \setminus \Delta_{k+l-1})$. We have the following result.

Proposition 3.7. *If* $\mu_k \in H^{n(k-1)}(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)$ *is the Thom class of* $\xi_k(N)$ *and* $\mu_l \in H^{n(l-1)}(N^l, N^l \setminus \Delta_l(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)$ *is the Thom class of* $\xi_l(N)$*, then*

$$e^*(\mu_k \times \mu_l) \in H^{n(k+l-2)}\big(N^{k+l-1}, N^{k+l-1} \setminus \Delta_{k+l-1}(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

is the Thom class of $\xi_{k+l-1}(N)$.

Proof. Let $x_1 \in N$ be arbitrary. We need to show that the image of $e^*(\mu_k \times \mu_l)$ in $H^{n(k+l-2)}(x_1 \times (N^{k+l-2}, N^{k+l-2} \setminus \{x_1\}^{k+l-2}); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)$ is dual to z_{k+l-2}. In order to show that, consider the homeomorphism between

$$x_1 \times \big(N^{k+l-2}, N^{k+l-2} \setminus \{x_1\}^{k+l-2}\big)$$

and

$$\big(x_1 \times \big(N^{k-1}, N^{k-1} \setminus \{x_1\}^{k-1}\big)\big) \times \big(x_1 \times \big(N^{l-1}, N^{l-1} \setminus \{x_1\}^{l-1}\big)\big)$$

given by

$$(x_1, x_2, \dots, x_{k+l-1}) \mapsto \big((x_1, \dots, x_k), (x_1, x_{k+1}, \dots, x_{k+l-1})\big).$$

Then, the result follows from the commutativity of the diagram

$$\begin{array}{ccc}
\big(N^{k+l-1}, N^{k+l-1} \setminus \Delta_{k+l-1}\big) & \xrightarrow{\ e\ } & \big(N^k, N^k \setminus \Delta_k\big) \times \big(N^l, N^l \setminus \Delta_l\big) \\
\uparrow & & \uparrow \\
\big(N^{k_1}, N^{k_1} \setminus \{x_1\}^{k_1}\big) & \longrightarrow & \big(N^{k_2}, N^{k_2} \setminus \{x_1\}^{k_2}\big) \times \big(N^{k_3}, N^{k_3} \setminus \{x_1\}^{k_3}\big)
\end{array}$$

where $k_1 = k + l - 2$, $k_2 = k - 1$ and $k_3 = l - 1$, and the vertical arrows are the inclusions. $\square$

Corollary 3.8. *Let*

$$e' : \big(N^k, N^k \setminus \Delta_k(N)\big) \to \underbrace{(N^2, N^2 \setminus \Delta_2(N)) \times \cdots \times (N^2, N^2 \setminus \Delta_2(N))}_{(k-1)\ \textit{times}}$$

be defined by

$$e'(x_1,\dots,x_k) = \big((x_1,x_2),(x_1,x_3),\dots,(x_1,x_k)\big).$$

If $\mu \in H^n(N^2, N^2 \setminus \Delta(N); R \times \Gamma_N^*)$ *is the Thom class of* $\xi_2(N)$, *then* $e'^*(\mu \times \cdots \times \mu) \in H^{n(k-1)}(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)$ *is the Thom class of* $\xi_k(N)$.

4. The Lefschetz coincidence class

Let N be a closed connected manifold of dimension n. Let

$$\mu_k \in H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

be the Thom class of $\xi_k(N)$. Then, given k maps $f_1,\dots,f_k : X \to N$ from a topological space X into the manifold N, the Lefschetz coincidence class $L(f_1,\dots,f_k)$ is defined by

$$L(f_1,\dots,f_k) = (f_1,\dots,f_k)^*(i^*(\mu_k)),$$

where $i : N^k \to (N^k, N^k \setminus \Delta_k(N))$ is the inclusion. Thus, $L(f_1,\dots,f_k)$ is an element of

$$H^{n(k-1)}\big(X; (f_1,\dots,f_k)^*(R \times \Gamma_N^* \times \cdots \times \Gamma_N^*)\big).$$

Theorem 4.1. *If* $L(f_1,\dots,f_k) \neq 0$, *then the set of coincidences*

$$\mathrm{Coin}(f_1,f_2,\dots,f_k) = \{x \in X \mid f_1(x) = f_2(x) = \cdots = f_k(x)\}$$

is nonempty.

Proof. If there is no $x \in X$ such that $f_1(x) = \cdots = f_k(x)$, then we have the factorization

$$\begin{tikzcd}
X \arrow[r, "(f_1,\dots,f_k)"] \arrow[d] & N^k \arrow[r, "i"] & \big(N^k,\ N^k \setminus \Delta_k(N)\big) \\
N^k \setminus \Delta_k(N) \arrow[rr] & & \big(N^k \setminus \Delta_k(N),\ N^k \setminus \Delta_k(N)\big) \arrow[u]
\end{tikzcd}$$

which implies $L(f_1,\dots,f_k) = 0$. □

In Corollary 3.8 we proved that if

$$e' : (N^k, N^k \setminus \Delta_k(N)) \to \underbrace{(N^2, N^2 \setminus \Delta_2(N)) \times \cdots \times (N^2, N^2 \setminus \Delta_2(N))}_{(k-1)\text{ times}}$$

is the map defined by

$$e'(x_1,\dots,x_k) = \big((x_1,x_2),(x_1,x_3),\dots,(x_1,x_k)\big)$$

and $\mu \in H^n(N^2, N^2 \setminus \Delta(N); R \times \Gamma_N^*)$ is the Thom class of $\xi_2(N)$, then

$$e'^*(\mu \times \cdots \times \mu) \in H^{n(k-1)}\big(N^k, N^k \setminus \Delta_k(N); R \times \Gamma_N^* \times \cdots \times \Gamma_N^*\big)$$

is the Thom class of $\xi_k(N)$.

Now, denote by $j : N^2 \to (N^2, N^2 \setminus \Delta_2(N))$ the inclusion and consider the map

$$h : X \to \underbrace{N^2 \times \cdots \times N^2}_{(k-1) \text{ times}},$$

where

$$h = \big((f_1, f_2), (f_1, f_3), \ldots, (f_1, f_k)\big).$$

We have that

$$e' \circ i \circ (f_1, \ldots, f_k) = (\underbrace{j \times \cdots \times j}_{(k-1) \text{ times}}) \circ h.$$

Thus

$$\begin{aligned} L(f_1, \ldots, f_k) &= (f_1, \ldots, f_k)^*(i^*(\mu_k)) \\ &= (f_1, \ldots, f_k)^*\big(i^*\big(e'^*(\mu \times \cdots \times \mu)\big)\big) \\ &= \big((f_1, f_2), (f_1, f_3), \ldots, (f_1, f_k)\big)^*\big((j \times \cdots \times j)^*(\mu \times \cdots \times \mu)\big) \\ &= (f_1, f_2)^*(j^*(\mu)) \smile (f_1, f_3)^*(j^*(\mu)) \smile \cdots \smile (f_1, f_k)^*(j^*(\mu)) \\ &= L(f_1, f_2) \smile L(f_1, f_3) \smile \cdots \smile L(f_1, f_k). \end{aligned}$$

Theorem 4.2. $L(f_1, \ldots, f_k) = L(f_1, f_2) \smile L(f_1, f_3) \smile \cdots \smile L(f_1, f_k)$.

Theorem 4.2 tells us that the Lefschetz class is almost symmetric, in the following sense.

Corollary 4.3. *For each permutation $\sigma \in S_k$ satisfying $\sigma(1) = 1$,*

$$L(f_1, f_2, \ldots, f_k) = \operatorname{sign}(\sigma)^n L\big(f_1, f_{\sigma(2)}, \ldots, f_{\sigma(k)}\big).$$

Remark 4.4. The R-oriented case presents a stronger form of symmetricity. Namely, for each permutation $\sigma \in S_k$,

$$L(f_1, \ldots, f_k) = \pm L\big(f_{\sigma(1)}, \ldots, f_{\sigma(k)}\big).$$

Indeed, analogously to [8, Lemma 5.16], if

$$t_\sigma : (N^k, N^k \setminus \Delta_k(N)) \to (N^k, N^k \setminus \Delta_k(N))$$

is the map defined by

$$t_\sigma(x_1, \ldots, x_k) = \big(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\big),$$

then for any orientation U of $\xi_k(N)$, $t^*(U) = U$ if the permutation σ is even, and $t^*(U) = (-1)^n U$ otherwise. Since

$$t_\sigma \circ i \circ (f_1, \ldots, f_k) = i \circ \big(f_{\sigma(1)}, \ldots, f_{\sigma(k)}\big),$$

it follows that

$$L(f_1,\dots,f_k)=\begin{cases} L(f_{\sigma(1)},\dots,f_{\sigma(k)}) & \text{if } \sigma \text{ is even,}\\ (-1)^n L(f_{\sigma(1)},\dots,f_{\sigma(k)}) & \text{if } \sigma \text{ is odd.}\end{cases}$$

Remark 4.5. In [1] a Lefschetz class is defined as follows. First of all, it is requested that the closed connected manifold N be R-orientable, i.e., orientable over R. Then, denoting by $U \in H^n(N^2, N^2 \setminus \Delta; R)$ the orientation class (also called Thom class), the Lefschetz class of the given maps $f_1,\dots,f_k : X \to N$ is defined by

$$\begin{aligned}
&\mathcal{L}(f_1,\dots,f_k)\\
&\quad = ((f_1,f_2),\dots,(f_{k-1},f_k))^*(j^*(U)\times\cdots\times j^*(U))\\
&\quad = (f_1,f_2)^*(j^*(U)) \smile (f_2,f_3)^*(j^*(U)) \smile \cdots \smile (f_{k-1},f_k)^*(j^*(U))\\
&\quad = \mathcal{L}(f_1,f_2) \smile \mathcal{L}(f_2,f_3) \smile \cdots \smile \mathcal{L}(f_{k-1},f_k) \in H^{n\cdot(k-1)}(X;R),
\end{aligned}$$

where $j : N^2 \to (N^2, N^2 \setminus \Delta)$ is the inclusion.

We observe that the formula presented in Theorem 4.2 is slightly different than the formula established in [1]. Despite such difference, we shall show, by induction on the number of maps, that in R-oriented case our definition coincides with the class defined in [1]. For two maps the result is obvious. Suppose that the statement is true for k maps. Then, applying Theorem 4.2, the symmetricity of the Lefschetz class in the R-oriented case and the induction hypothesis, we have

$$\begin{aligned}
L(f_1,f_2\dots,f_{k-1},f_k,f_{k+1}) &= (-1)^n L(f_k,f_2,\dots,f_{k-1},f_1,f_{k+1})\\
&= (-1)^n L(f_k,f_2,\dots,f_{k-1},f_1) \smile L(f_k,f_{k+1})\\
&= L(f_1,f_2,\dots,f_{k-1},f_k) \smile L(f_k,f_{k+1})\\
&= \mathcal{L}(f_1,f_2,\dots,f_{k-1},f_k) \smile \mathcal{L}(f_k,f_{k+1})\\
&= \mathcal{L}(f_1,f_2\dots,f_{k-1},f_k,f_{k+1}).
\end{aligned}$$

5. Examples

Let us now consider the case where R is a field. Let $y_i \in H^*(N,R)$ and $y_i' \in H^*(N;\Gamma_N)$ be bases such that $\langle y_i', D(y_i)\rangle = 1$, where

$$D : H^j(N;R) \to H_{n-j}(N;\Gamma_N)$$

denotes the Poincaré isomorphism. Then we have the following result.

Proposition 5.1. *With the above notation, the image of the Thom class μ of $\xi_2(N)$ is given by*

$$j^*(\mu) = \sum_i (-1)^{|y_i|}\, y_i \times y_i',$$

where $|y_i|$ denotes the dimension of y_i, i.e., $y_i \in H^{|y_i|}(N;R)$.

Proof. The proof is analogous to that of [4, Proposition 30.18, p. 288]. □

Example 5.2. Consider $N = \mathbb{R}P^2$ the projective plane and $R = \mathbb{Q}$. Then

$$\begin{aligned} H^0(\mathbb{R}P^2;\mathbb{Q}) &= \mathbb{Q}, \\ H^q(\mathbb{R}P^2;\mathbb{Q}) &= 0 \quad \text{for } q > 0, \\ H^2(\mathbb{R}P^2;\Gamma_{\mathbb{R}P^2}) &= H_0(\mathbb{R}P^2;\mathbb{Q}) = \mathbb{Q}, \\ H^q(\mathbb{R}P^2;\Gamma_{\mathbb{R}P^2}) &= H_{2-q}(\mathbb{R}P^2;\mathbb{Q}) = 0 \quad \text{for } q \neq 2. \end{aligned}$$

Thus,

$$j^*(\mu) = 1 \times e,$$

where the element $1 \in H^0(\mathbb{R}P^2;\mathbb{Q})$ is the identity of the ring $H^*(\mathbb{R}P^2;\mathbb{Q})$ and the element $e \in H^2(\mathbb{R}P^2;\Gamma_{\mathbb{R}P^2})$ is a generator.

It follows that, given maps $f_1, f_2 : X \to \mathbb{R}P^2$, the Lefschetz class is given by

$$L(f_1, f_2) = (f_1, f_2)^*(1 \times e) = f_1^*(1) \smile f_2^*(e) = f_2^*(e).$$

This shows that, in general,

$$L(f_1, f_2;\Gamma_{\mathbb{R}P^2}) \neq \pm L(f_2, f_1;\Gamma_{\mathbb{R}P^2}).$$

In view of Example 5.2, below we will discuss the general case where the target space is the projective space $\mathbb{R}P^n$, n even.

5.1. The Lefschetz class for the target space $\mathbb{R}P^n$, n even

Consider the projective space $\mathbb{R}P^n$, where n is an even number. As in Example 5.2,

$$\begin{aligned} H^0(\mathbb{R}P^n;\mathbb{Q}) &= \mathbb{Q}, \\ H^q(\mathbb{R}P^n;\mathbb{Q}) &= 0 \quad \text{for } q > 0, \\ H^n(\mathbb{R}P^n;\Gamma_{\mathbb{R}P^n}) &= H_0(\mathbb{R}P^n;\mathbb{Q}) = \mathbb{Q}, \\ H^q(\mathbb{R}P^n;\Gamma_{\mathbb{R}P^n}) &= H_{n-q}(\mathbb{R}P^n;\mathbb{Q}) = 0 \quad \text{for } q \neq n. \end{aligned}$$

Thus, the Thom class μ of $\mathbb{R}P^n$ is given by

$$j^*(\mu) = 1 \times e,$$

where the element $1 \in H^0(\mathbb{R}P^n;\mathbb{Q})$ is the identity of the ring $H^*(\mathbb{R}P^n;\mathbb{Q})$ and the element $e \in H^n(\mathbb{R}P^n;\Gamma_{\mathbb{R}P^n})$ is a generator. It follows that, given maps $f_1, \ldots, f_k : X \to \mathbb{R}P^n$, the Lefschetz class is given by

$$\begin{aligned} L(f_1, \ldots, f_k) &= L(f_1, f_2) \smile L(f_1, f_3) \smile \cdots \smile L(f_1, f_k) \\ &= f_2^*(e) \smile \cdots \smile f_k^*(e) \\ &= (f_2, \ldots, f_k)^*(e \times \cdots \times e). \end{aligned} \tag{5.1}$$

The above formula does not depend on f_1. Consider the particular case where $X = \mathbb{R}P^n$, $f_1 : \mathbb{R}P^n \to \mathbb{R}P^n$ is an arbitrary self-map and f_2 is the identity map. Then we obtain the well-known fact that $\mathbb{R}P^n$ has the fixed point property if n is even, since $L(f_1, \mathrm{id}) = \mathrm{id}^*(e) = e \neq 0$.

A map $f : M \to N$ between manifolds is called *orientation true* if each $\alpha \in \pi_1(M)$ preserves local orientation of M if and only if $f\alpha \in \pi_1(N)$ preserves local orientation of N. If $\dim M = \dim N$, then the *degree* of f is defined as being the natural number k satisfying $f_*(z_M) = k \cdot z_N$.

If X is a closed connected manifold of dimension $n(k-1)$ and $f_2, \ldots, f_k$ are orientation true, it is well defined the degree of

$$(f_2, \ldots, f_k) : X \to (\mathbb{R}P^n)^{k-1}.$$

From (5.1), $L(f_1, \ldots, f_k) \neq 0$ if and only if $\deg(f_2, \ldots, f_k) \neq 0$.

Theorem 5.3. *Let X be a closed connected manifold of dimension $n(k-1)$ and $f_1, \ldots, f_k : X \to \mathbb{R}P^n$ orientation true. If, for some $1 \leq i \leq k$, $\deg(\hat{f}_i) \neq 0$, then there is $x \in X$ such that $f_1(x) = f_2(x) = \cdots = f_k(x)$, where $\hat{f}_i$ denotes the map $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k) : X \to (\mathbb{R}P^n)^{k-1}$.*

Proof. Suppose $i \in \{1, \ldots, k\}$ such that $\deg(\hat{f}_i) \neq 0$, where

$$\hat{f}_i = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k) : X \to (\mathbb{R}P^n)^{k-1}.$$

From (5.1),

$$L(f_i, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k) = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k)^*(e \times \cdots \times e).$$

Since $\deg(\hat{f}_i) \neq 0$, $L(f_i, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k) \neq 0$. Therefore, by Theorem 4.1, there is $x \in X$ such that $f_i(x) = f_1(x) = \cdots = f_k(x)$. $\square$

Lemma 5.4 (See [3, Lemma 4.11]**).** *If $p : \tilde{M} \to M$ is a k-fold covering, then $\deg(p) = k$.*

Example 5.5. Consider the maps $c, f, g : S^2 \times S^2 \to \mathbb{R}P^2$, where c is a constant map, $f(x, y) = \{x, -x\}$ and $g(x, y) = \{y, -y\}$. Then,

$$(f, g) : S^2 \times S^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$$

is a 4-fold covering. It follows from Lemma 5.4 that $\deg(f, g) = 4$. Therefore, by the above theorem, the Lefschetz class $L(c, f, g)$ is nontrivial. On the other hand, considering (co)-homology with coefficients in $\mathbb{Z}_2$ we have

$$L(c, f, g; \mathbb{Z}_2) = \deg_2(f, g) = 0.$$

Here, $\deg_2$ denotes the degree that we obtain when we consider homology with coefficients in $\mathbb{Z}_2$.

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